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A MODEL OF STRATEGIC SUSTAINABLE INVESTMENT

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ABSTRACT. We study a problem of optimal irreversible investment and emission reduction formulated as a nonzero-sum dynamic game between an investor with environmental preferences and a firm. The game is set in continuous time on an infinite-time horizon. The firm generates profits with a stochastic dynamics and may spend part of its revenues towards emission reduction (e.g., renovating the infrastructure). The firm's objective is to maximize the discounted expectation of a function of its profits. The investor participates in the profits and may decide to invest to support the firm's production capacity. The investor uses a profit function which accounts for both financial and environmental factors. Nash equilibria of the game are obtained via a system of variational inequalities. We formulate a general verification theorem for this system in a diffusive setup and construct an explicit solution in the zero-noise limit. Our explicit results and numerical approximations show that both the investor's and the firm's optimal actions are triggered by moving boundaries that increase with the total amount of emission abatement.

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JEL Classification: C73, Q52.
MSC 2020 Classification: 93E20, 91A15, 49N90, 65K15.

1. INTRODUCTION

As acknowledged in the 2015 Paris Agreement, making finance flows consistent with a pathway towards low greenhouse gas emissions is a key step towards decarbonizing the economy with the aim of limiting the global warming well below 2° C compared to pre-industrial levels. Sustainable investors, who care not only about the financial performance but also about the environmental performance of their assets, play a key role, as they finance the green companies and create incentives for brown companies to reduce their emissions. However, the interests of sustainable investors may be misaligned with those of the company management, especially if executive compensation schemes do not account for environmental aspects. The process of investment can thus be seen as a nonzero-sum game, where the investors provide capital to companies, aiming to maximize both environmental and financial performance, and company managers determine mitigation strategies, aiming to maximize financial performance, but taking into account the capital provision by the sustainable investors.

In this paper, we develop a model for green investment and emission reduction, framing it as a dynamic game between a representative investor and a representative privately owned company. The investor continuously provides capital directly to the company, and the company determines its emission abatement strategy. The optimization criterion of the investor is concave increasing in both financial and environmental performance of the company and decreasing in the cost of capital provided to the company. Instead, the company maximizes the discounted expectation of its future financial performance. We formalize market equilibrium as the Nash equilibrium of this stochastic game, and characterize the value functions of the company and of the investor as the solution of a system of variational inequalities, via a verification theorem.

We formulate a general verification theorem in a diffusive set-up. When the diffusion coefficient is set to zero the game becomes deterministic and we are able to produce an explicit solution of the variational problem and an explicit equilibrium for the game. In the general stochastic game we design a numerical algorithm to solve the problem using a variant of policy iteration. Remarkably, the structure of the equilibrium we obtain from the algorithm is qualitatively the same as the one for the equilibrium we obtain explicitly in the deterministic case. That shows that our explicit solution may be used as a proxy for the solution to the general problem.

Our results show that both the investor's and the firm's optimal actions are triggered by moving boundaries that increase with the total amount of emission abatement performed by the firm. More precisely, denoting by X_t the firm's production capacity at time $t \geq 0$ and by R_t the total abatement performed up to time $t \geq 0$ we determine two functions $r \mapsto a(r)$ and $r \mapsto b(r)$, with $a(r) < b(r)$, that fully characterize the equilibrium strategies of the investor and of the firm, respectively. In particular, when the production capacity is high and thus the financial performance of the firm is very good, relatively to the current abatement level R_t (i.e., for $X_t > b(R_t)$) neither the firm nor the investor act. However, when the current production capacity is more modest, the firm finds it convenient to face abatement costs in order to attract future investments. This translates into the fact that when $X_t \leq b(R_t)$ the firm invests in pollution abatement at the maximum allowed rate, shifting the dynamics of the pair (X_t, R_t) closer to the investor's optimal investment boundary $a(R_t)$. In turn, the investor guarantees to the firm an optimal level of investment to keep the cashflow dynamics above the increasing threshold $t \mapsto a(R_t)$. This structure of the solution is in line with the findings of the literature on impact investing, which argues that sustainable investors achieve greatest impact when they allocate capital to small impactful companies with high growth potential rather than large established ones [23].

It is worth mentioning that the form of our equilibrium contrasts sharply with the results normally found in the literature on nonzero-sum games of irreversible investment (cf. next section). In that literature it is often the case that the firm acts when profits are high and the investor acts when profits are low. So the two players' action sets are normally separated by an inaction region where neither player acts. Here instead we see that for large profits neither player acts and that firm's and investor's action regions overlap (the firm's action set contains the investor's one). Then the economic conclusions we can draw are different from the usual ones, showing that our model captures new (or at least uncommon) phenomena in the nonzero-sum games' literature.

Our mathematical contribution in the deterministic economy is also novel. We establish a correspondence between an equilibrium in our game and a solution to a system of first-order, fully degenerate, semilinear, partial differential equations in the domain $(0, \infty)^2$ (cf. Corollary 4). Standard PDE methods do not seem to be directly applicable in our setup to determine existence (and uniqueness) of a solution to the PDE and so we rely on different methods rooted in the theory of stochastic control. Besides, we go beyond existence of a solution by explicitly characterizing the free boundaries for both the investor and the firm (up to numerical root-finding). That enables explicit construction of optimal strategies for both players. The question of uniqueness for the solution of the PDE system remains open. This is not surprising, because it relates to uniqueness of equilibria in our nonzero-sum game, which is not to be expected in general.

1.1. **Review of literature.**

From the applied point of view, our paper contributes to the burgeoning financial literature on sustainable investment. Sustainable investors are motivated either by the non-pecuniary benefits they get from holding green stocks (warm glow), as in, e.g., [27, 28] or by their concern for the provision of public goods (e.g., climate change mitigation) by the companies they invest in as, e.g., in [7,26]. This concern for public good provision may be driven by the genuine concern of investors for the environment, but also by the purely financial concern that companies with bad environmental performance are exposed to higher risk, as demonstrated, e.g., in [6]. Our paper belongs to the second category: as in [11] we jointly model the behavior of investors and the mitigation strategies of companies as a dynamic game. The vast majority of papers in the sustainable investment literature, including [11], focus on publicly owned companies. Differently from those papers, we do not set up a financial market, and consider a representative investor who provides capital directly to a representative privately owned company.

We also contribute to the literature on irreversible investment. The vast majority of papers in this domain adopts the point of view of one firm (or several firms) which looks for an optimal investment level to maximize uncertain future profits [12, 14, 17, 22, 29]. The firm is considered as a single economic entity and it is assumed that the interests of the investors are aligned with those of the managers of the firm. However, in the presence of environmental concerns this may no longer be the case: as evidenced by the vast literature on shareholder activism [3, 18, 19], the investors may care for the environmental impact of the company much more than the managers. We therefore frame our irreversible investment problem as a game between a representative company and a representative investor, who pursue different objectives.

From the more theoretical viewpoint, we contribute to the literature on nonzero-sum dynamic games with singular controls [8,9,24]. Those games arise in, e.g., irreversible investment problems [2] and constitute the class of two-player games that is perhaps the most difficult to analyze, with few contributions going beyond a verification theorem or numerical illustrations [1, 15]. The reference [15], which describes a game of pollution control between the government and a representative firm is, perhaps, closest in spirit to our work. However, in [15] the emissions are directly linked to the production process, the only decision the firm takes is to expand production, and the state variable is one-dimensional. Instead we consider separately the financial and the environmental performance of the firm, leading to a state process with two components, both of which are controlled by the agents. Moreover, in our model the investor uses singular controls and the firm uses classical controls in order to guarantee well-posedness of the game (cf. Remarks 1 and 7).

1.2. **Structure of the paper.** In Section 2 we formulate our model and introduce the notion of Nash equilibrium in our game. In Section 3 we state a general verification theorem (Theorem 1) with some ancillary considerations in Corollary 1 and Remark 3. In Section 4 we construct an explicit equilibrium in our game when there is no diffusive component. The construction relies partially upon the use of Theorem 1 and partially upon the use of Corollary 1 and Remark 3. In particular, the investor's optimal strategy and equilibrium payoff are obtained without using PDE arguments. Instead, the firm's payoff and optimal strategy are obtained by solving an Hamilton-Jacobi-Bellman equation. After we obtain both the firm's and the investor's equilibrium payoffs, we are able to show that also the investor's payoff solves an Hamilton-Jacobi-Bellman equation in a slightly relaxed form (cf. Section 4.3). In Section 5 we devise a numerical algorithm for the solution of the variational problem associated to our game as presented in the verification theorem. In Section 6 we illustrate our numerical findings for the general stochastic game. A summary of the paper and some conclusions are provided in Section 7.

2. Model formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtered probability space. Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion and let $\mathbb{E}[\cdot]$ denote the expectation under $\mathbb{P}$. We use the standard notations $\mathbb{R} \triangleq (-\infty, \infty)$ and $\mathbb{R}_+ \triangleq (0, \infty)$ and set $(z)^+ = [z]^+ \triangleq \max(0, z)$. Finally, given a set $A \subseteq [0, \infty) \times \mathbb{R}_+$ we denote by A^c its complement, by $\overline{A}$ its closure relative to $[0, \infty) \times \mathbb{R}_+$ and by $\partial A = \overline{A} \cap \overline{A^c}$ its boundary.

In our model, we consider a representative firm, which produces a single good and has a production capacity denoted as $(X_t)_{t\geq 0}$. We assume that the firm generates a continuous stream of profits, which is proportional to its production capacity. Without loss of generality, we assume the proportionality factor to be equal to one, that is, we shall interchangeably refer to X_t both as the production capacity and the instantaneous profit of the firm.

The firm is privately owned by a pool of investors, described in our model as a single representative investor, who can inject funds directly into the company, increasing its production capacity and hence, its profit stream. The investor's decisions are guided by financial and environmental considerations. The financial performance of the firm is measured by a profit function applied to the production capacity process $(X_t)_{t\geq 0}$. The environmental performance is measured by the total expenditure $(R_t)_{t\geq 0}$, allocated by the firm towards emission reduction activities.

The problem is formulated as a stochastic differential game of optimal controls. In the absence of any intervention from either the firm or the investor, the dynamics of the production capacity process $(X_t)_{t\geq 0}$ is given by

$$X^0_t = X_0 + \int_0^t \mu X^0_s \mathrm{d}s + \int_0^t \sigma X^0_s \mathrm{d}B_s, \quad X_0 = x > 0, \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ are model parameters. The investor picks a control from the class

$$\mathcal{A}_I \triangleq \left\{ \nu : \begin{array}{c} (\nu_t)_{t\geq 0} \text{ is càdlàg, nondecreasing, } \mathbb{F}\text{-adapted,} \\ \text{with } \nu_0 \geq 0 \text{ and } \mathbb{E}\Big[\int_{[0,\infty)} \mathrm{e}^{-\rho t}\mathrm{d}\nu_t\Big] < \infty \end{array} \right\}. \tag{2}$$

For $\nu \in \mathcal{A}_I$, the random variable ν_t corresponds to the total amount of investment that the investor has provided to the firm over the time interval $[0, t]$, where ν_0 corresponds to the initial investment. We notice that investment may arrive with lump-sum payments, i.e., it may be $\Delta\nu_t \triangleq \nu_t - \nu_{t-} > 0$ for $t > 0$, and there is no cap on the investment rate (this corresponds to the situation of so-called *singular controls*). Any admissible control $\nu \in \mathcal{A}_I$ admits a decomposition $\nu_t = \nu^c_t + \sum_{s\leq t} \Delta\nu_s$ in a continuous part plus a sum of jumps.

The dynamics for the total expenditure allocated by the firm towards emission reduction reads:

$$R^\eta_t = R_0 + \int_0^t \eta_s \mathrm{d}s, \quad R_0 = r \geq 0, \tag{3}$$

where the process $(\eta_t)_{t\geq 0}$ is the control chosen by the firm and it belongs to the class, for some $\eta_{\max} > 0$,

$$\mathcal{A}_F \triangleq \big\{\eta : (\eta_t)_{t\geq 0} \text{ is progressively measurable, with } 0 \leq \eta_t \leq \eta_{\max}\big\}. \tag{4}$$

Any pair $(\nu_t, \eta_t)_{t\geq 0}$ describes the investment and emission reduction policies of the two agents. For a fixed choice of $(\nu_t, \eta_t)_{t\geq 0}$, the dynamics of the production capacity reads:

$$X^{\nu,\eta}_t = X_0 + \int_0^t \mu X^{\nu,\eta}_s \mathrm{d}s + \int_0^t \sigma X^{\nu,\eta}_s \mathrm{d}B_s + \nu_t - \int_0^t \eta_s \mathrm{d}s, \tag{5}$$

with initial condition $X_0 = x$ before a possible lump-sum investment ν_0 at time zero. Investment increases the production capacity of the company whereas abatement decreases it: to reduce emissions, the company switches to a greener but costlier technology or modifies its energy mix by using more expensive clean energy. Note that for simplicity and without loss of generality, both investment and abatement are measured in units of production capacity. By an application of Itô's formula it is readily verified that the dynamics of $X^{\nu,\eta}$ can be written more explicitly as

$$X^{\nu,\eta}_t = X^0_t \Big(x + \int_{[0,t]} \frac{1}{X^0_s} \big(\mathrm{d}\nu_s - \eta_s \mathrm{d}s\big)\Big). \tag{6}$$

The firm's optimization criterion is given in terms of expected discounted future profits. Mathematically we express it as

$$\mathcal{J}^F_{r,x}(\eta, \nu) \triangleq \mathbb{E}_{r,x}\left[\int_0^\infty \mathrm{e}^{-\bar{\rho} t} \pi(X^{\nu,\eta}_t)\mathrm{d}t\right], \tag{7}$$

where $\bar{\rho} > 0$ is the discount rate of the firm, $\mathbb{E}_{r,x}$ stands for the conditional expectation given $(R_0, X_0) = (r, x)$ and $\pi : [0,\infty) \to [0,\infty)$ is a continuous profit function, which we interpret as the compensation plan of the company's management. Its specific form will be further detailed below.

The investor's optimization criterion is also expressed in terms of discounted future profits, but these are computed differently. To take into account the investor's preference for green assets, we assume that their profit function depends on both the firm's production capacity and its environmental performance: the profits from a green firm are valued higher than the profits from a brown firm. This is in line with the literature on sustainable investment, where the utility of investors also depends both on their wealth and on the environmental performance of the firms they invest in [27]. We write the investor's optimization functional as follows:

$$\mathcal{J}^I_{r,x}(\eta,\nu) \triangleq \mathbb{E}_{r,x}\left[\int_0^\infty \mathrm{e}^{-\rho t}\Pi(R^\eta_t, X^{\nu,\eta}_t)\mathrm{d}t - \alpha\int_{[0,\infty)} \mathrm{e}^{-\rho t}\mathrm{d}\nu_t\right], \tag{8}$$

where $\rho > 0$ is the investor's discount rate, $\Pi : [0,\infty)^2 \to [0,\infty)$ is a continuous function which is further specified below and $\alpha > 0$ is a weighting factor, which quantifies the cost for the investor of increasing the firm's production capacity by one unit. The function Π measures, in monetary units, the profits of the investor adjusted for the environmental performance of the firm. The parameter α can also quantify the investment frictions such as the fund manager's fees if the investment is managed through a fund.

For a fixed investment strategy $\nu \in \mathcal{A}_I$ selected by the investor, the firm aims to maximize its profits. Therefore, the firm's problem is formulated as

$$\bar{w}(r,x;\nu) \triangleq \sup_{\eta\in\mathcal{A}_F} \mathcal{J}^F_{r,x}(\eta,\nu). \tag{9}$$

Conversely, when the firm selects an emission reduction strategy $\eta \in \mathcal{A}_F$, the investor's objective is to maximize the inter-temporal optimization functional (8). Therefore the investor's problem reads

$$\bar{v}(r,x;\eta) \triangleq \sup_{\nu\in\mathcal{A}_I} \mathcal{J}^I_{r,x}(\eta,\nu). \tag{10}$$

We are interested in obtaining a pair of strategies (η^*,ν^*), that constitutes a Nash equilibrium for the game according to the following definition:

Definition 1. *A pair $(\eta^*,\nu^*) \in \mathcal{A}_F \times \mathcal{A}_I$ is a Nash equilibrium for the game starting at (r,x) if*

$$\mathcal{J}^F_{r,x}(\eta^*,\nu^*) \geq \mathcal{J}^F_{r,x}(\eta,\nu^*) \quad \text{and} \quad \mathcal{J}^I_{r,x}(\eta^*,\nu^*) \geq \mathcal{J}^I_{r,x}(\eta^*,\nu), \tag{11}$$

for any other pair $(\eta,\nu) \in \mathcal{A}_F \times \mathcal{A}_I$. Then we say that $w(r,x) \triangleq \bar{w}(r,x;\nu^*)$ and $v(r,x) \triangleq \bar{v}(r,x;\eta^*)$ are the equilibrium payoffs (or values) for the firm and the investor, respectively.

Remark 1. *We notice that the admissible control classes for the investor and the firm are different. The firm's maximum investment rate is capped by $\eta_{\max} > 0$ whereas the investor's one is uncapped. There are two reasons for this choice. From a modeling perspective, $(R^\eta_t)_{t\geq 0}$ may be interpreted as a continuous flow of spending towards emission reduction/compensation, for example, by purchasing bio-fuel instead of fossil-fuel, green energy certificates or carbon offsets. Of course one could also interpret $(R^\eta_t)_{t\geq 0}$ as spending on large emission reduction projects, such as refurbishing a steel-making plant to use the electric arc furnace technology instead of a more carbon-intensive blast furnace. In the latter case, perhaps, discontinuous $(R^\eta_t)_{t\geq 0}$ would make more sense. However, from a mathematical perspective, it is not clear that our stochastic game is well-posed when both the firm and the investor are allowed to use singular controls, even under rather innocuous specifications of functions π and Π. We are going to illustrate this issue with an example during the solution of the zero-noise limit of our game (cf. Remark 7).*

Remark 2. *The definition of equilibrium is formulated on the product space $\mathcal{A}_F \times \mathcal{A}_I$ of pairs (η,ν). A note of caution is necessary here, because our game is dynamic and we expect equilibrium controls (η^*,ν^*) in feedback form (i.e., as functionals of the path of the controlled dynamics). It is not obvious that any choice of (η,ν) in feedback form would yield a well-posed dynamics of the pair $(X^{\nu,\eta}, R^\eta)$ (i.e., a unique strong solution of the resulting SDE). This is a common feature in stochastic games in continuous time and we tacitly adopt the convention that a pair (η,ν) which does not yield a well-posed dynamics is associated with payoffs $\mathcal{J}^I_{r,x}(\eta,\nu) = \mathcal{J}^F_{r,x}(\eta,\nu) = -\infty$. Then, additionally to the conditions in Definition 1, a pair (η,ν) is an equilibrium if it also yields a well-posed dynamics of the system. For the purpose of this paper we do not need to dig deeper in this direction but we* *point the interested reader to [30, Sec. 2] for an extended discussion in the context of zero-sum games and to [9, Sec. 3] for rigorous game-theoretic formulations of nonzero-sum stochastic games of singular control (which our setting is a special case of).*

3. Variational problem

By standard considerations based on dynamic programming, we expect that a pair of equilibrium payoffs should be obtained by solving a system of variational inequalities. First, we state the variational problem and then we formally connect it with the game via a so-called *verification theorem* (cf. Theorem 1). In what follows, given a function φ we denote by φ_x, φ_{xx} and φ_r its first and second order derivatives with respect to x and its first order derivative with respect to r, respectively.

The infinitesimal generator of the uncontrolled process X^0 is defined via its action on sufficiently smooth functions φ as

$$(\mathcal{L}\varphi)(r,x) \triangleq \frac{\sigma^2 x^2}{2}\varphi_{xx}(r,x) + \mu x \varphi_x(r,x).$$

The Hamiltonian of the firm's problem is defined as

$$\mathcal{H}(r,x;\varphi) \triangleq \sup_{0\le\eta\le\eta_{\max}} \big(\varphi_r(r,x) - \varphi_x(r,x)\big)\eta, \tag{12}$$

for smooth φ. Then, we expect a pair (w,v) of equilibrium payoffs (as in Definition 1) to be solution, in a sense to be specified later, of the following system: let

$$\mathcal{M} \triangleq \{(r,x) : v_x(r,x) = \alpha\}, \quad \mathcal{I} \triangleq \mathcal{M}^c = \big([0,\infty)\times\mathbb{R}_+\big)\setminus\mathcal{M} \quad \text{and} \quad \partial\mathcal{M} = \overline{\mathcal{M}}\cap\overline{\mathcal{I}};$$

the function w solves

$$\begin{cases} (\mathcal{L}w - \bar\rho w)(r,x) + \mathcal{H}(r,x;w) + \pi(x) = 0, & (r,x)\in\mathcal{I},\\ w_x(r,x) = 0, & (r,x)\in\partial\mathcal{M}; \end{cases} \tag{13}$$

letting $\eta^*(r,x) \triangleq \eta_{\max}1_{\{w_r>w_x\}}(r,x)$, the function v solves

$$\max\left\{(\mathcal{L}v-\rho v)(r,x) + \left(v_r(r,x)-v_x(r,x)\right)\eta^*(r,x) + \Pi(r,x), v_x(r,x)-\alpha\right\} = 0, \tag{14}$$

for $(r,x)\in[0,\infty)\times\mathbb{R}_+$.

For now we may assume that all solutions are understood in the classical sense (i.e., with continuous derivatives). Later, in Section 4.3, we will adopt a notion of *strong* solution. Suitable growth conditions should also be specified. These will be encoded in the so-called transversality conditions of our verification theorem. It is immediate to check that

$$\eta^*(r,x) = \operatorname{argmax}_{0\le\eta\le\eta_{\max}}\big\{\big(w_r(r,x) - w_x(x,r)\big)\eta\big\}.$$

In the next theorem, we show that if a sufficiently smooth solution pair (v,w) of the above problem exists, then indeed it corresponds to a pair of equilibrium payoffs for the game. It is convenient to also define the following compact notation: for $q\ge 0$

$$\mathcal{G}^q[\varphi,\eta](r,x) \triangleq \left(\mathcal{L}\varphi - q\varphi + (\varphi_r-\varphi_x)\eta\right)(r,x), \tag{15}$$

for any pair of sufficiently smooth functions (φ,η).

Theorem 1 (Verification). *Assume there is a pair (v,w) of non-negative, continuous functions on $[0,\infty)^2$ that solves (13)–(14) with $w\in C^{1,2}(\overline{\mathcal{I}})$ and $v\in C^{1,2}((0,\infty)^2)$. Assume further that there* *is a control* $\nu^*\in\mathcal{A}_I$ *such that for any* $\eta\in\mathcal{A}_F$, *the pair* $(R^\eta_t, X^{\nu^*,\eta}_t)_{t\geq 0}$ *is well-posed and such that* $\mathbb{P}_{r,x}$*-a.s.*

$$
\begin{aligned}
&(R^\eta_t, X^{\nu^*,\eta}_t)\in\overline{\mathcal{I}}, \quad \textit{for } t\geq 0,\\
&1_{\{(r,x)\in\mathcal{I}\}}\,\nu^*_0+\int_{(0,T]}1_{\{(R^\eta_t,X^{\nu^*,\eta}_{t-})\in\mathcal{I}\}}\mathrm{d}\nu^*_t=0, \quad \textit{for any } T>0,\\
&(R^\eta_t, X^{\nu^*,\eta}_{t-}+\Delta\nu^*_t)\in\partial\mathcal{M}, \quad \textit{for any } t>0.
\end{aligned}
\tag{16}
$$

Finally, assume that the process $(R^{\eta^*}_t, X^{\nu^*,\eta^*}_t)_{t\geq 0}$ *with* $\eta^*_t=\eta^*(R^{\eta^*}_t, X^{\nu^*,\eta^*}_t)$ *is well-defined and the transversality conditions*

$$
\lim_{n\to\infty}\mathbb{E}_{r,x}\Big[\mathrm{e}^{-\bar\rho\theta^*_n}w\big(R^{\eta^*}_{\theta^*_n},X^{\nu^*,\eta^*}_{\theta^*_n}\big)\Big]=0 \quad \textit{and} \quad \lim_{n\to\infty}\mathbb{E}_{r,x}\Big[\mathrm{e}^{-\rho\theta^*_n}v\big(R^{\eta^*}_{\theta^*_n},X^{\nu^*,\eta^*}_{\theta^*_n}\big)\Big]=0,
\tag{17}
$$

hold with $\theta^*_n\triangleq\inf\{t\geq 0:(R^{\eta^*}_t, X^{\nu^*,\eta^*}_t)\notin[0,n)^2\}$. *Then* (v,w) *are a pair of equilibrium payoffs as in Definition 1 and the pair* $(\eta^*_t,\nu^*_t)_{t\geq0}$ *is a pair of optimal strategies.*

Proof. The proof follows familiar arguments from stochastic control theory and so we only outline it here. For $(r,x)\in\overline{\mathcal{I}}$ and any $\eta\in\mathcal{A}_F$, setting $\theta_n\triangleq\inf\{t\geq0:(R^\eta_t,X^{\nu^*,\eta}_t)\notin[0,n)^2\}$, by an application of Itô's formula we obtain

$$
\begin{aligned}
&\mathrm{e}^{-\bar\rho(t\wedge\theta_n)}w\big(R^\eta_{t\wedge\theta_n},X^{\nu^*,\eta}_{t\wedge\theta_n}\big)\\
&=w(r,x+\nu^*_0)+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\mathcal{G}^{\bar\rho}[w,\eta_s]\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\mathrm{d}s\\
&\quad+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}w_x\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\mathrm{d}\nu^{*,c}_s+\sum_{0<s\leq t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\big[w\big(R^\eta_s,X^{\nu^*,\eta}_s\big)-w\big(R^\eta_s,X^{\nu^*,\eta}_{s-}\big)\big]\\
&\quad+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}w_x\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\sigma X^{\nu^*,\eta}_s\mathrm{d}B_s\\
&\leq w(r,x)+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\big(\mathcal{L}w-\bar\rho w+\mathcal{H}(\cdot,w)\big)\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\mathrm{d}s\\
&\quad+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}w_x\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\sigma X^{\nu^*,\eta}_s\mathrm{d}B_s\\
&=w(r,x)-\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\pi\big(X^{\nu^*,\eta}_s\big)\mathrm{d}s+\int_0^{t}\mathrm{e}^{-\bar\rho s}w_x\big(R^\eta_s,X^{\nu^*,\eta}_s\big)\sigma X^{\nu^*,\eta}_s\mathrm{d}B_s,
\end{aligned}
$$

where the inequality is by definition of the Hamiltonian (12) and we use $w_x(R^\eta_s,X^{\nu^*,\eta}_s)\mathrm{d}\nu^{*,c}_s=0$, $w\big(R^\eta_s,X^{\nu^*,\eta}_s\big)-w\big(R^\eta_s,X^{\nu^*,\eta}_{s-}\big)=0$ and $w(r,x+\nu^*_0)-w(r,x)=0$ by (16) and the second equation in (13). Now, taking expectations and rearranging terms we obtain

$$
\begin{aligned}
w(r,x)&\geq\mathbb{E}_{r,x}\Big[\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\pi\big(X^{\nu^*,\eta}_s\big)\mathrm{d}s+\mathrm{e}^{-\bar\rho(t\wedge\theta_n)}w\big(R^\eta_{t\wedge\theta_n},X^{\nu^*,\eta}_{t\wedge\theta_n}\big)\Big]\\
&\geq\mathbb{E}_{r,x}\Big[\int_0^{t\wedge\theta_n}\mathrm{e}^{-\bar\rho s}\pi\big(X^{\nu^*,\eta}_s\big)\mathrm{d}s\Big],
\end{aligned}
$$

where the inequality holds because $w\geq0$. Since also $\pi\geq0$, letting $t\to\infty$ and $n\to\infty$ and using Monotone Convergence yields

$$
w(r,x)\geq\mathbb{E}_{r,x}\Big[\int_0^{\infty}\mathrm{e}^{-\bar\rho s}\pi\big(X^{\nu^*,\eta}_s\big)\mathrm{d}s\Big]=\mathcal{J}^F_{r,x}(\nu^*,\eta).
$$

Repeating the arguments above with $\eta=\eta^*$ yields

$$w(r,x)=\mathbb{E}_{r,x}\Big[\int_0^{t\wedge\theta_n^*}\mathrm{e}^{-\bar\rho s}\pi\big(X_s^{\nu^*,\eta^*}\big)\mathrm{d}s+\mathrm{e}^{-\bar\rho(t\wedge\theta_n^*)}w\big(R^{\eta^*}_{t\wedge\theta_n^*},X^{\nu^*,\eta^*}_{t\wedge\theta_n^*}\big)\Big].$$

First we let $t\to\infty$ and use dominated convergence. Then we let $n\to\infty$ and use Monotone Convergence and the transversality condition to obtain $w(r,x)=\mathcal{J}^F_{r,x}(\nu^*,\eta^*)$. This shows that η^* is a best response to ν^*.

Now, let us look at the investor's payoff. Take $(r,x)\in[0,\infty)\times\mathbb{R}_+$ and arbitrary $\nu\in\mathcal{A}_I$. Assume that the dynamics $(R_t^{\eta^*},X_t^{\nu,\eta^*})$ is well-posed and redefine $\theta_n\triangleq\inf\{t\ge0:(R_t^{\eta^*},X_t^{\nu,\eta^*})\notin[0,n)^2\}$ with a slight abuse of notation. By Itô's formula we have

$$\begin{aligned}
&\mathrm{e}^{-\rho(t\wedge\theta_n)}v\big(R^{\eta^*}_{t\wedge\theta_n},X^{\nu,\eta^*}_{t\wedge\theta_n}\big)\\
&=v(r,x+\nu_0)+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\rho s}\mathcal{G}^\rho[v,\eta_s^*]\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)\mathrm{d}s+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\rho s}v_x\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)\mathrm{d}\nu_s^{*,c}\\
&\quad+\sum_{0<s\le t\wedge\theta_n}\mathrm{e}^{-\rho s}\big[v\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)-v\big(R_s^{\eta^*},X_{s-}^{\nu,\eta^*}\big)\big]+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\rho s}v_x\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)\sigma X_s^{\nu,\eta^*}\mathrm{d}B_s\\
&\le v(r,x)-\int_0^{t\wedge\theta_n}\mathrm{e}^{-\rho s}\Pi\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)\mathrm{d}s+\int_{[0,t\wedge\theta_n]}\mathrm{e}^{-\rho s}\alpha\mathrm{d}\nu_s^*\\
&\quad+\int_0^{t\wedge\theta_n}\mathrm{e}^{-\rho s}v_x\big(R_s^{\eta^*},X_s^{\nu,\eta^*}\big)\sigma X_s^{\nu,\eta^*}\mathrm{d}B_s,
\end{aligned}$$

where the inequality holds by (14). Taking expectations, rearranging terms and passing to the limit in t and n yields $v(r,x)\ge\mathcal{J}^I_{r,x}(\eta^*,\nu)$, upon using also that $v\ge0$. Repeating the argument with $\nu=\nu^*$ the inequality becomes equality and we obtain

$$v(r,x)=\mathbb{E}_{r,x}\Big[\int_0^{t\wedge\theta_n^*}\mathrm{e}^{-\rho s}\Pi\big(R_s^{\eta^*},X_s^{\nu^*,\eta^*}\big)\mathrm{d}s+\mathrm{e}^{-\rho(t\wedge\theta_n)}v\big(R^{\eta^*}_{t\wedge\theta_n^*},X^{\nu^*,\eta^*}_{t\wedge\theta_n^*}\big)\Big].$$

Letting $t\to\infty$ and $n\to\infty$ and using dominated and monotone convergence along with the transversality condition we arrive at $v(r,x)=\mathcal{J}^I_{r,x}(\eta^*,\nu^*)$. This shows that ν^* is the best response to η^* and it concludes the proof. □

There is a straightforward corollary of the verification theorem which can be interpreted as a verification theorem only for the firm. This assumes that the optimal strategy for the investor is known and it is essentially characterized by a set $\mathcal{M}$. However, the corollary does not assume that the investor's equilibrium payoff be a smooth solution of the variational problem (14) nor that the set $\mathcal{M}$ be of the form specified before (13). The proof is omitted because it is a repetition, line by line, of the first part of the proof of Theorem 1.

Corollary 1. *Assume that there is a set $\mathcal{M}\subset[0,\infty)\times\mathbb{R}_+$ whose complement we denote by $\mathcal{I}\triangleq\mathcal{M}^c$ and a continuous function w such that the following conditions hold:*

(i) *The function $w\in C^{1,2}(\overline{\mathcal{I}})$ solves (13);*
(ii) *For any $\eta\in\mathcal{A}_F$ there is $\nu^*=\nu^*(\eta)\in\mathcal{A}_I$ which is optimal for the investor in the sense that $\bar v(r,x;\eta)=\mathcal{J}^I_{r,x}(\eta,\nu^*(\eta))$ for all $(r,x)\in[0,\infty)\times\mathbb{R}_+$;*
(iii) *For any $\eta\in\mathcal{A}_F$, the dynamics $(R_t^\eta,X_t^{\nu^*,\eta})_{t\ge0}$ with $\nu^*(\eta)$ as in (ii) is well-posed and it satisfies (16);*
(iv) *The process $(R_t^{\eta^*},X_t^{\nu^*,\eta^*})_{t\ge0}$ with $\eta^*_t=\eta^*(R_t^{\eta^*},X_t^{\nu^*,\eta^*})$ is well-posed and the transversality condition for the function w in (17) holds.*

Then, the pair $(\eta_t^*,\nu_t^*)_{t\ge0}$ is a Nash equilibrium for the game with equilibrium payoffs v and w, where $v(r,x)\triangleq\bar v(r,x;\eta^*)$ as in (10).

Remark 3 (Verification with deterministic dynamics). *In the special case of no diffusive component, i.e.,* $\sigma = 0$*, the problem becomes deterministic. In this case the infinitesimal generator simplifies to* $(\mathcal{L}\varphi)(r, x) = \mu x \varphi_x(r, x)$*. The verification theorem above continues to hold under the less restrictive assumptions that* v *and* w *be continuous on* $[0, \infty)^2$ *with* $w \in C^1(\overline{\mathcal{I}})$ *and* $v \in C^1((0, \infty)^2)$ *(for Corollary 1 only* $w \in C^1(\overline{\mathcal{I}})$ *is needed).*

Remark 4. *It seems unlikely that one could construct smooth solutions to the system of variational inequalities* (13)–(14)*, either with* $\sigma > 0$ *or* $\sigma = 0$*. Therefore weaker notions of solution may be needed. In particular, we will see in the next section (especially in subsection 4.3) that a notion of* strong *solution is needed for the investor's payoff when* $\sigma = 0$*.*

It is clear that Corollary 1 imposes very restrictive assumptions, but we are able to use it in order to obtain an equilibrium in the deterministic game in the next section.

4. Equilibrium in the zero-noise limit

In the special case of a deterministic dynamics of the firm's profits (i.e., in the zero-noise limit $\sigma = 0$), with decreasing profit stream (i.e., $\mu \le 0$) we are able to obtain an explicit equilibrium. The assumption $\mu \le 0$ is in line with standard economic modeling, where the profitability of a firm, in the absence of investment, must decrease over time (e.g., due to ageing of manufacturing machines, etc.). Instead the assumption $\sigma = 0$ is harder to justify from an economic perspective but we can think of equilibria in this setting as a zero-order approximation of equilibria in stochastic setups with small noise. Indeed, we will observe numerically in the subsequent sections that equilibria in the stochastic framework maintain the same qualitative structure as the one we obtain here.

The benchmark example that we have in mind is when firm's profit function is linear (i.e., the firm is risk-neutral) and the investor adopts a Cobb-Douglas profit function. In that case we have

$$\pi(x) = x \quad \text{and} \quad \Pi(r, x) = x^\beta r^\gamma, \quad \text{for } (r, x) \in [0, \infty) \times \mathbb{R}_+, \tag{18}$$

with $\beta, \gamma \in (0, 1)$ and $\gamma + \beta \ge 1$. The latter condition ensures that item (iii) in the assumption below holds, because in this case the function $a(r)$ specified therein is proportional to $r^{\frac{\gamma}{1-\beta}}$. However, in the interest of mathematical generality we solve the problem under the following assumption, which is enforced throughout the section.

Assumption 1. *We have* $\sigma = 0$*,* $\mu \le 0$ *and the following properties hold:*

(i) *The function* Π *is non-decreasing in both variables;* $\Pi(r, \cdot)$ *is strictly concave,* Π_x *and* Π_{xx} *exist and are continuous on* $(0, \infty)^2$*, with derivative* $\Pi_x(r, \cdot)$ *satisfying the Inada conditions*

$$\lim_{x \to 0} \Pi_x(r, x) = +\infty, \qquad \lim_{x \to +\infty} \Pi_x(r, x) = 0.$$

Therefore $\Pi_x(r, \cdot)$ *admits an inverse* $G(r, z) \triangleq (\Pi_x(r, \cdot))^{-1}(z)$*. We assume that for* $\delta \triangleq \rho - \mu \ge 0$*, the mapping* $r \mapsto a(r) \triangleq G(r, \alpha\delta)$ *is strictly increasing and continuously differentiable on* $(0, \infty)$*, with*

$$\int_0^\infty \mathrm{e}^{-\rho t} a(\eta_{\max} t) \mathrm{d}t < \infty.$$

(ii) *The function* π *is continuously differentiable and strictly increasing on* $(0, \infty)$*; the derivative* $\dot{\pi}$ *is non-decreasing on* $(0, \infty)$*; we extend* π *to* $(-\infty, 0]$ *as* $\pi(x) = \pi([x]^+)$*; finally,*

$$\int_0^\infty \mathrm{e}^{-\bar{\rho} t} \pi(a(\eta_{\max} t)) \mathrm{d}t < \infty.$$

(iii) *When* $\mu < 0$*, the mapping* $r \mapsto \dot{\pi}\big(a(r)\big)\dot{a}(r)$ *is non-decreasing.*

Remark 5. *Assumption 1-(ii) implies, in particular, that the company's profit function π is convex. This aligns with the interpretation of π as the compensation package of the company's management, which may include convex features such as stock options. Research on CEO compensation packages [20, 33] suggests that convex compensation schemes may mitigate excessive risk-avoiding behavior of firm's managers. Furthermore, the authors of [13] find that actual CEO contracts tend to be convex and develop a principal-agent model demonstrating that convex compensation schemes are optimal for medium and high performance outcomes.*

Setting $(R_0, X_0) = (r, x)$, we consider the following controlled dynamics:

$$\begin{cases} X_t^{\nu,\eta} = X_0 + \int_0^t \mu X_s^{\nu,\eta} \mathrm{d}s + \nu_t - \int_0^t \eta_s \mathrm{d}s, \\ R_t^{\eta} = R_0 + \int_0^t \eta_s \mathrm{d}s, \end{cases} \tag{19}$$

for $t \in [0, \infty)$. Then, the explicit dynamics of $X^{\nu,\eta}$ reads (cf. (6))

$$X_t^{\nu,\eta} = Y_t \left(x + \widehat{\nu}_t - \widehat{\Lambda}_t \right), \tag{20}$$

where

$$Y_t \triangleq \mathrm{e}^{\mu t}, \qquad \widehat{\nu}_t \triangleq \int_{[0,t]} Y_s^{-1} \mathrm{d}\nu_s, \qquad \widehat{\Lambda}_t \triangleq \int_0^t Y_s^{-1} \eta_s \mathrm{d}s, \tag{21}$$

for all $t \in [0, \infty)$. In this context the filtration $\mathbb{F}$ is trivial, i.e., all processes are deterministic, including all admissible controls for the firm and the investor.

Theorem 2. *Let the controlled dynamics be as in* (19) *with $\mu \le 0$ and let the profit function of the investor $\Pi(r, x)$ satisfy Assumption 1. Then, for any $\eta \in \mathcal{A}_F$ the investor's optimal investment policy is given by*

$$\nu_t^* \triangleq \int_{[0,t]} Y_s \mathrm{d}\widehat{\nu}_s^*, \qquad \textit{for } t \ge 0, \tag{22}$$

where

$$\widehat{\nu}_t^* \triangleq \left[Y_t^{-1} a(R_t^{\eta}) - x + \widehat{\Lambda}_t^{\eta} \right]^+ \quad \textit{with} \quad a(r) = G(r, \alpha\delta). \tag{23}$$

Moreover, the pair $(R_t^{\eta}, X_t^{\nu^*,\eta})_{t \ge 0}$ satisfies (16) *with $\mathcal{I} \triangleq \{(r, x) \in [0, \infty)^2 : x > a(r)\}$ and $\mathcal{M} \triangleq \{(r, x) : 0 < x \le a(r)\}$.*

Proof. For a fixed $\eta \in \mathcal{A}_F$ the dynamics of R^{η} is determined and so is the function $\widehat{\Lambda} = \widehat{\Lambda}^{\eta}$. For the ease of notation, using that η is fixed throughout, we simply denote $(R^{\eta}, \widehat{\Lambda}^{\eta}) = (R, \widehat{\Lambda})$. The investor's problem is to maximize over $\nu \in \mathcal{A}_I$ (equivalently over $\widehat{\nu}$) the quantity:

$$\mathcal{J}_{r,x}^I(\eta, \nu) = \int_0^{\infty} \mathrm{e}^{-\rho t} \Pi\big(R_t, Y_t[x + \widehat{\nu}_t - \widehat{\Lambda}_t]\big) \mathrm{d}t - \alpha \int_{[0,\infty)} \mathrm{e}^{-\rho t} Y_t \mathrm{d}\widehat{\nu}_t.$$

The integrability condition for $\nu \in \mathcal{A}_I$ implies $\lim_{t \to \infty} \mathrm{e}^{-\rho t} Y_t \widehat{\nu}_t = 0$, because it is easy to verify by integration by parts that

$$0 \le \int_0^{\infty} \mathrm{e}^{-\rho s} Y_s \widehat{\nu}_s \mathrm{d}s \le \rho^{-1} \widehat{\nu}_0 + \rho^{-1} \int_{[0,\infty)} \mathrm{e}^{-\rho s} Y_s \mathrm{d}\widehat{\nu}_s = \rho^{-1} \nu_0 + \rho^{-1} \int_{[0,\infty)} \mathrm{e}^{-\rho s} \mathrm{d}\nu_s < \infty.$$

Integration by parts then yields (recalling that $\delta = \rho - \mu$):

$$\begin{aligned} \mathcal{J}_{r,x}^I(\eta, \nu) &= \int_0^{\infty} \mathrm{e}^{-\rho t} \Big(\Pi\big(R_t, Y_t[x + \widehat{\nu}_t - \widehat{\Lambda}_t]\big) - \alpha\delta Y_t \widehat{\nu}_t \Big) \mathrm{d}t - \alpha \lim_{t \to \infty} \mathrm{e}^{-\rho t} Y_t \widehat{\nu}_t \\ &= \int_0^{\infty} \mathrm{e}^{-\rho t} \Big(\Pi\big(R_t, Y_t[x + \widehat{\nu}_t - \widehat{\Lambda}_t]\big) - \alpha\delta Y_t \widehat{\nu}_t \Big) \mathrm{d}t \triangleq \bar{\mathcal{J}}_{r,x}^I(\eta, \widehat{\nu}). \end{aligned} \tag{24}$$

For every $t \geq 0$, the functional

$$\widehat{\nu} \mapsto \Pi\big(R_t, Y_t[x + \widehat{\nu} - \widehat{\Lambda}_t]\big) - \alpha\delta Y_t \widehat{\nu},$$

is strictly concave and in view of Assumption 1-(i) it admits a unique (pointwise) maximizer $\widehat{\nu}^*$ on $[0, \infty)$. If such maximizer turns out to be an admissible control then $\sup_{\nu \in \mathcal{A}_I} \mathcal{J}^I_{r,x}(\eta, \nu) = \bar{\mathcal{J}}^I_{r,x}(\eta, \widehat{\nu}^*)$ and $\widehat{\nu}^*$ must be optimal for the original problem.

By first order conditions we look for $\widehat{\nu}$ such that

$$\Pi_x\big(R_t, Y_t[x + \widehat{\nu}_t - \widehat{\Lambda}_t]\big)Y_t - \alpha\delta Y_t = 0, \quad \text{for all } t \geq 0. \tag{25}$$

Since $\widehat{\nu}_t$ must be also positive, and recalling the inverse function $G(r, z) = (\Pi_x(r, \cdot))^{-1}(z)$, we have

$$\widehat{\nu}^*_t = \big[G(R_t, \alpha\delta)Y_t^{-1} - x + \widehat{\Lambda}_t\big]^+ = \big[a(R_t)Y_t^{-1} - x + \widehat{\Lambda}_t\big]^+.$$

Since $\mu \leq 0$, then $t \mapsto Y_t^{-1}$ is non-decreasing. So are also $t \mapsto R_t$ and $t \mapsto \widehat{\Lambda}_t$. Then $\widehat{\nu}^*$ is non-decreasing thanks to Assumption 1-(i) and it is immediate to see that it is also continuous except for a possible lump-sum investment at time zero of size

$$\widehat{\nu}^*_0 = \big[a(r) - x\big]^+. \tag{26}$$

The investment strategy associated to $\widehat{\nu}^*$ is obtained by setting

$$\nu^*_t \triangleq \int_0^t Y_s \mathrm{d}\widehat{\nu}^*_s, \qquad \text{for } t \geq 0. \tag{27}$$

To prove that ν^* is admissible, it is enough to verify that

$$\int_0^\infty \mathrm{e}^{-\rho t} Y_t \widehat{\nu}^*_t \mathrm{d}t < \infty.$$

Recall that $r \mapsto a(r)$ is non-decreasing. Then, for any $\eta \in \mathcal{A}_F$,

$$\begin{aligned} 0 &\leq \mathrm{e}^{-\rho t} Y_t \widehat{\nu}^*_t \leq \mathrm{e}^{-\rho t} a(r + \eta_{\max} t) + \mathrm{e}^{-\rho t} \eta_{\max} \int_0^t Y_t Y_s^{-1} \mathrm{d}s \\ &\leq \mathrm{e}^{-\rho t}\Big(a(r + \eta_{\max} t) + \eta_{\max} t\Big), \end{aligned}$$

where we used $Y_t/Y_s \leq 1$ because $\mu \leq 0$. The above expression is integrable in view of Assumption 1-(i), which means that ν^* is admissible and optimal for the investor.

It is easy to verify

$$X^{\nu^*,\eta}_t = Y_t\big(x + \widehat{\nu}^*_t - \widehat{\Lambda}_t\big) \geq a(R^\eta_t), \qquad \text{for all } t \geq 0. \tag{28}$$

If $\widehat{\nu}^*_0 > 0$ then $X^{\nu^*,\eta}_0 = a(r)$ (cf. (26)) and, finally,

$$\mathrm{d}\widehat{\nu}^*_t = 1_{\{X^{\nu^*,\eta}_t = a(R^\eta_t)\}} \mathrm{d}\widehat{\nu}^*_t = 1_{\{X^{\nu^*,\eta}_t = a(R^\eta_t)\}} \mathrm{d}\overline{\nu}^*_t, \tag{29}$$

with $\overline{\nu}^*_t = Y_t^{-1} a(R^\eta_t) - x + \widehat{\Lambda}^\eta_t$, for $t \geq 0$. In summary, the investor's optimal strategy is to keep the dynamics $(R^\eta_t, X^{\nu^*,\eta}_t)_{t \geq 0}$ above the threshold $r \mapsto a(r)$, $r \geq 0$. Then (16) holds with $\mathcal{I} = \{(r, x) : x > a(r)\}$ and $\mathcal{M} = \{(r, x) : 0 < x \leq a(r)\}$ as claimed. □

Remark 6. *If $X_0 > a(R_0)$, then $\nu^*_t = 0$ for $t \in [0, \tau(\eta)]$ where*

$$\tau(\eta) \triangleq \inf\{t \geq 0 : X^{0,\eta}_t \leq a(R^\eta_t)\}. \tag{30}$$

Moreover

$$\nu^*_t = a(R^\eta_t) + R^\eta_t - a(R^\eta_{\tau(\eta)}) - R^\eta_{\tau(\eta)}, \quad \text{for } t \geq \tau(\eta).$$

Having obtained an optimal strategy for the investor, we now turn our attention to determining the firm's optimal emission reduction policy. It is worth noticing that

$$X_t^{\nu^*,\eta} = Y_t\big(x - \widehat{\Lambda}_t^\eta + \big[Y_t^{-1}a(r+\Lambda_t^\eta) - x + \widehat{\Lambda}_t^\eta\big]^+\big) = \max\big\{Y_t(x-\widehat{\Lambda}_t^\eta), a(r+\Lambda_t^\eta)\big\}, \tag{31}$$

where, to streamline notation, we write

$$\Lambda_t \triangleq \int_0^t \eta_s \mathrm{d}s,$$

and, recalling also $\tau(\eta)$ from (30), we have $X_t^{\nu^*,\eta} = a(r+\Lambda_t^\eta)$, for $t \geq \tau(\eta)$. For any $\eta \in \mathcal{A}_F$ the firm's payoff reads

$$\begin{aligned}\mathcal{J}^F_{r,x}(\eta,\nu^*) &= \int_0^\infty \mathrm{e}^{-\bar{\rho}t}\pi(X_t^{\nu^*,\eta})\mathrm{d}t\\ &= \int_0^\infty \mathrm{e}^{-\bar{\rho}t}\pi\big(\max\big\{Y_t(x-\widehat{\Lambda}_t^\eta), a(r+\Lambda_t^\eta)\big\}\big)\mathrm{d}t\\ &= \int_0^{\tau(\eta)} \mathrm{e}^{-\bar{\rho}t}\pi\big(Y_t[x-\widehat{\Lambda}_t^\eta]\big)\mathrm{d}t + \int_{\tau(\eta)}^\infty \mathrm{e}^{-\bar{\rho}t}\pi\big(a(r+\Lambda_t^\eta)\big)\mathrm{d}t.\end{aligned} \tag{32}$$

When $R_0 = r$ and $X_0 = x = a(r)$, we have $\tau(\eta) = 0$ for any $\eta \in \mathcal{A}_F$. Then

$$\sup_{\eta\in\mathcal{A}_F} \mathcal{J}^F_{r,a(r)}(\eta,\nu^*) = \int_0^\infty \mathrm{e}^{-\bar{\rho}t}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}t, \tag{33}$$

and $\eta_t^* = \eta_{\max}$ for all $t \geq 0$ is optimal, because π is increasing (Assumption 1-(ii)). We then deduce the next simple result.

Proposition 1. *For $(R_0, X_0) = (r, a(r))$, $r \in (0,\infty)$, an equilibrium pair (η^*,ν^*) is given by*

$$\eta_t^* = \eta_{\max}, \quad \nu_t^* = a(r+\eta_{\max}t) + \eta_{\max}t - a(r), \quad \textit{for all } t \geq 0.$$

Proof. Optimality of $\eta^* = \eta_{\max}$ is guaranteed by (33). The optimal ν^* is given by (27) and it is easy to calculate it explicitly thanks to Remark 6, because $\tau(\eta^*) = 0$. □

Remark 7. *At this stage we can easily see that there may be well-posedness issues for the game if we let both players use singular controls, as anticipated in Remark 1.*

First of all we notice that the expression for $\widehat{\nu}_t^*$ in (23) continues to hold also if we generalize the dynamics of R^η to $R_t = R_0 + \xi_t$, with $t \mapsto \xi_t$ càdlàg, nondecreasing, $\mathbb{F}$-adapted, with $\xi_0 \geq 0$ and $\mathbb{E}\big[\int_{[0,\infty)} \mathrm{e}^{-\bar{\rho}t}\mathrm{d}\xi_t\big] < \infty$. Moreover, the expression of the boundary $a(r)$ remains the same. Then, the firm's problem in (33) turns into

$$\sup_\xi \mathcal{J}^F_{r,a(r)}(\xi,\nu^*) = \sup_\xi \int_0^\infty \mathrm{e}^{-\bar{\rho}t}\pi\big(a(r+\xi_t)\big)\mathrm{d}t.$$

It is clear that the expression on the right-hand side above can be made arbitrarily large taking an increasing sequence of admissible controls $\{(\xi_t^n)_{t\geq 0}, n \in \mathbb{N}\}$ with $\xi_0^n = n$. Since $\widehat{\nu}^*$ is optimal for the investor against any choice of the firm's abatement policy ξ, we deduce that there is no equilibrium in this game with finite payoff for the firm. A slight modification of this argument allows to reach the same conclusion also when $X_0 = x > a(r) = a(R_0)$, so the issue is not specific to one initial state of the dynamics.

Now we turn our attention to finding an optimal strategy for the firm when $x > a(r)$. For the ease of exposition, we split the analysis into the case when $\mu = 0$ and when $\mu < 0$.

4.1. **Equilibrium with $\mu = 0$.** Let us start from $\mu = 0$, in which case we have

$$w(r,x) = \sup_{\eta \in \mathcal{A}_F} \Big(\int_0^{\tau(\eta)} \mathrm{e}^{-\bar\rho t} \pi(x - \Lambda_t^\eta) \mathrm{d}t + \int_{\tau(\eta)}^\infty \mathrm{e}^{-\bar\rho t} \pi\big(a(r + \Lambda_t^\eta)\big) \mathrm{d}t \Big).$$

We need to introduce some quantities of interest in order to state the form of our equilibrium. The firm's payoff with no emission reduction reads

$$w_0(r,x) \triangleq \mathcal{J}^F_{r,x}(0,\nu^*) = \frac{1}{\bar\rho}\pi(x), \quad x > a(r),$$

because $\tau(0) = \infty$. Instead, when $\eta = \eta_{\max}$ we have $\tau_M \triangleq \tau(\eta_{\max})$ uniquely defined as the solution of

$$x - \eta_{\max}\tau_M = a(r + \eta_{\max}\tau_M). \tag{34}$$

Therefore, the payoff with maximum emission abatement reads

$$w_1(r,x) \triangleq \mathcal{J}^F_{r,x}(\eta_{\max},\nu^*) = \int_0^{\tau_M} \mathrm{e}^{-\bar\rho t}\pi(x - \eta_{\max} t)\mathrm{d}t + \int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\pi\big(a(r + \eta_{\max} t)\big)\mathrm{d}t,$$

which is finite by Assumption 1-(ii). The functions w_0 and w_1 determine a set

$$\mathcal{B} \triangleq \{(r,x) \in [0,\infty) \times \mathbb{R}_+ : w_0(r,x) < w_1(r,x)\},$$

which will be useful in finding an equilibrium in this case. For future reference, we notice that $\tau_M = \tau_M(r,x)$ is continuously differentiable in both variables with

$$\frac{\partial \tau_M}{\partial x} = \frac{1}{\eta_{\max}\big(1 + \dot a(r + \eta_{\max}\tau_M)\big)} \quad \text{and} \quad \frac{\partial \tau_M}{\partial r} = -\frac{\dot a(r + \eta_{\max}\tau_M)}{\eta_{\max}\big(1 + \dot a(r + \eta_{\max}\tau_M)\big)}.$$

Next, we characterise the set $\mathcal{B}$.

Lemma 1. *There is a unique function $r \mapsto b(r)$ such that*

$$\mathcal{B} = \{(r,x) \in [0,\infty) \times \mathbb{R}_+ : x < b(r)\}.$$

Moreover, $b \in C^1(\mathbb{R}_+)$ is increasing with $b(r) > a(r)$ for all $r \in [0,\infty)$.

Proof. Let us start by calculating $\partial_r w_1(r,x)$:

$$\begin{aligned}
\frac{\partial w_1(r,x)}{\partial r} &= \mathrm{e}^{-\bar\rho\tau_M}\pi(x - \eta_{\max}\tau_M)\frac{\partial\tau_M}{\partial r} - \mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r + \eta_{\max}\tau_M)\big)\frac{\partial\tau_M}{\partial r} \\
&\quad + \lim_{h\to 0}\int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\frac{1}{h}\left\{\pi\big(a(r + h + \eta_{\max}t) - \pi\big(a(r + \eta_{\max}t)\big)\right\}\mathrm{d}t \\
&= \lim_{h\to 0}\int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\frac{1}{h}\int_0^h \dot\pi\big(a(r + \xi + \eta_{\max}t)\big)\dot a(r + \xi + \eta_{\max}t)\mathrm{d}\xi\,\mathrm{d}t \\
&= \lim_{h\to 0}\frac{1}{h}\int_0^h \Big(\int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r + \xi + \eta_{\max}t)\big)\dot a(r + \xi + \eta_{\max}t)\mathrm{d}t\Big)\mathrm{d}\xi \\
&= \lim_{h\to 0}\frac{1}{h\eta_{\max}}\int_0^h\Big(\bar\rho\int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\pi\big(a(r+\xi+\eta_{\max}t)\big)\mathrm{d}t - e^{-\bar\rho\tau_M}\pi\big(a(r+\xi+\eta_{\max}\tau_M)\big)\Big)\mathrm{d}\xi \\
&= \frac{1}{\eta_{\max}}\Big(\bar\rho\int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\pi\big(a(r + \eta_{\max}t)\big)\mathrm{d}t - e^{-\bar\rho\tau_M}\pi\big(a(r + \eta_{\max}\tau_M)\big)\Big) \\
&= \int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r + \eta_{\max}t)\big)\dot a(r + \eta_{\max}t)\mathrm{d}t > 0,
\end{aligned} \tag{35}$$

where the second equality holds because of (34); to compute the limit we use monotonicity and continuity of π and a, and Assumption 1-(ii); the strict inequality holds by strict monotonicity of π and a. Then, $r \mapsto w_1(r,x)$ is strictly increasing, whereas $r \mapsto w_0(r,x)$ is constant. That shows that for each $x \in \mathbb{R}_+$ there is a unique $c(x) \geq 0$ such that permanent emission abatement is strictly more profitable than no action, i.e.,

$$\mathcal{B} = \{(r, x) \in [0, \infty) \times \mathbb{R}_+ : r > c(x)\}.$$

For the x-derivatives it is immediate that $\partial_x w_0(r, x) = \dot{\pi}(x)/\bar{\rho}$ and the same argument as above yields

$$\begin{aligned}\frac{\partial w_1}{\partial x}(r, x) &= \int_0^{\tau_M} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}(x - \eta_{\max}t)\mathrm{d}t + \mathrm{e}^{-\bar{\rho}\tau_M}\Big(\pi(x - \eta_{\max}\tau_M) - \pi\big(a(r + \eta_{\max}\tau_M)\big)\Big)\frac{\partial \tau_M}{\partial x} \\ &= \int_0^{\tau_M} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}(x - \eta_{\max}t)\mathrm{d}t > 0,\end{aligned} \tag{36}$$

where the second equality holds by (34).

Since w_0 and w_1 are both continuous, then $w_0(c(x), x) = w_1(c(x), x)$ and by the implicit function theorem

$$\dot{c}(x) = -\frac{\partial_x(w_0 - w_1)(c(x), x)}{\partial_r(w_0 - w_1)(c(x), x)} = \frac{\bar{\rho}^{-1}\dot{\pi}(x) - \int_0^{\tau_M} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}(x - \eta_{\max}t)\mathrm{d}t}{\int_{\tau_M}^{\infty} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}\big(a(r + \eta_{\max}t)\big)\dot{a}(r + \eta_{\max}t)\mathrm{d}t} > 0,$$

where the inequality holds because, by Assumption 1-(ii),

$$\int_0^{\tau_M} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}(x - \eta_{\max}t)\mathrm{d}t \leq \dot{\pi}(x)\bar{\rho}^{-1}(1 - \mathrm{e}^{-\bar{\rho}\tau_M}).$$

Then $c \in C^1(\mathbb{R}_+)$ and it is strictly increasing. We can also define the strictly increasing inverse boundary $b(r) = c^{-1}(r)$, so that $b \in C^1((0, \infty))$ and $\mathcal{B} = \{(r, x) \in [0, \infty) \times \mathbb{R}_+ : x < b(r)\}$, as claimed.

Finally, we can show that $b(r) > a(r)$ for all $r \in [0, \infty)$ arguing by contradiction. Indeed, assume there is $r_0 \in [0, \infty)$ such that $a(r_0) = b(r_0)$. Then, $\tau_M(r_0, b(r_0)) = 0$ and

$$w_1(r_0, b(r_0)) = \int_0^{\infty} \mathrm{e}^{-\bar{\rho}t}\pi\big(a(r_0 + \eta_{\max}t)\big)\mathrm{d}t > \frac{\pi\big(a(r_0)\big)}{\bar{\rho}} = w_0(r_0, a(r_0)).$$

The inequality implies $b(r_0) > a(r_0)$, hence a contradiction. □

Now we can state our result concerning the firm's optimal strategy.

Proposition 2. *Let $\mu = 0$. Then, the firm's equilibrium payoff reads as*

$$w(r, x) = \begin{cases} w_0(r, x), & x > b(r), \\ w_1(r, x), & a(r) \leq x \leq b(r). \end{cases} \tag{37}$$

The firm's optimal strategy is constant and equal to zero if $x > b(r)$ (i.e., $\eta^*_t \equiv 0$ for $t \geq 0$), whereas it is constant and equal to $\eta_{\max}$ if $a(r) \leq x \leq b(r)$ (i.e., $\eta^*_t = \eta_{\max}$ for $t \geq 0$).

Proof. Our strategy of proof is to check that the function w from (37) solves the HJB equation and then to apply a small adaptation of the verification theorem (Corollary 1). Recall that in the present case, $\mathcal{L} \equiv 0$ and the HJB equation takes the simple form $-\bar{\rho}w(r, x) + \mathcal{H}(r, x; w) = -\pi(x)$, for $x > a(r)$, with $\mathcal{H}(r, x; w)$ as in (12).

For $x > b(r)$ we have $(w_r - w_x)(r, x) = -\bar{\rho}^{-1}\dot{\pi}(x)$, thus $\mathcal{H}(r, x; w) = 0$. Then it is clear that the HJB holds for $x \geq b(r)$ because $-\bar{\rho}w_0(r, x) = -\pi(x)$. It is immediate to verify that $w_x(r, a(r)) = \partial_x w_1(r, a(r)) = 0$ by the explicit form of the derivative calculated in (36) because $\tau_M = 0$ in this case. Moreover, for $a(r) < x \leq b(r)$

$$(w_r - w_x)(r, x) = \int_{\tau_M}^{\infty} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}\big(a(r + \eta_{\max}t)\big)\dot{a}(r + \eta_{\max}t)\mathrm{d}t - \int_0^{\tau_M} \mathrm{e}^{-\bar{\rho}t}\dot{\pi}(x - \eta_{\max}t)\mathrm{d}t. \tag{38}$$

It is easy to see from the first expression above that $(w_r - w_x)(r, a(r)) > 0$. Moreover, for $a(r) < x < b(r)$ we have, by integration by parts, using (34) and Assumption 1-(ii),

$$\begin{aligned}
&\bar\rho w_1(r,x) - \pi(x)\\
&= \bar\rho \int_0^{\tau_M} \mathrm{e}^{-\bar\rho t}\pi\big(x - \eta_{\max} t\big)\mathrm{d}t + \bar\rho \int_{\tau_M}^{\infty} \mathrm{e}^{-\bar\rho t}\pi\big(a(r+\eta_{\max} t)\big)\mathrm{d}t - \pi(x)\\
&= -\eta_{\max} \int_0^{\tau_M} e^{-\bar\rho t}\dot\pi(x - \eta_{\max} t)\mathrm{d}t + \eta_{\max} \int_{\tau_M}^{\infty} e^{-\bar\rho t}\dot\pi(a(r+\eta_{\max} t))\dot a(r+\eta_{\max} t)\mathrm{d}t.
\end{aligned}$$

Comparing to (38) we see that

$$\eta_{\max}\big(\partial_r w_1(r,x) - \partial_x w_1(r,x)\big) = \bar\rho w_1(r,x) - \pi(x). \tag{39}$$

The right-hand side of the above expression is positive because $a(r) < x < b(r)$ (this follows from the definition of the set $\mathcal{B}$ and Lemma 1). Therefore, $\partial_r w_1(r,x) - \partial_x w_1(r,x) > 0$ and $\eta_{\max}\big(\partial_r w_1(r,x) - \partial_x w_1(r,x)\big) = \mathcal{H}(r,x;w_1)$. Thus, (39) shows that w_1 solves the HJB in $a(r) < x \le b(r)$, $r \ge 0$. Notice that indeed (39) also implies $\partial_r w_1(r,b(r)) - \partial_x w_1(r,b(r)) = 0$.

We have shown that w solves the HJB equation at all points (t,x) with $x \ge a(r)$ and $x \ne b(r)$. We cannot directly apply Corollary 1 with Remark 3, because w defined as in (37) is not continuously differentiable. However, the mapping $(r,x) \mapsto (\partial_r w - \partial_x w)(r,x)$ is well-defined as a function in $L^\infty_{\ell oc}$ on the set $\{(r,x) : x \ge a(r)\}$ (and $(r,x) \mapsto \mathcal{H}(r,x;w)$ is even continuous). Since there is no dynamics, unless $\eta_t \ne 0$, such regularity is sufficient to make sense of the change of variable formula at the beginning of the proof of Theorem 1. The rest follows by the same arguments as in that proof and we omit details to avoid repetitions. To check the transversality condition (17) for the function w, notice that

$$w(r,x) \le \frac{\pi(x)}{\bar\rho} + \int_0^\infty \mathrm{e}^{-\bar\rho t}\pi(a(r+\eta_{\max} t))\mathrm{d}t = \frac{\pi(x)}{\bar\rho} + \mathrm{e}^{r\bar\rho/\eta_{\max}} \int_{r/\eta_{\max}}^\infty \mathrm{e}^{-\bar\rho t}\pi(a(\eta_{\max} t))\mathrm{d}t.$$

Then, recalling that w is increasing in r,

$$\begin{aligned}
\mathrm{e}^{-\bar\rho t} w(R_t^{\eta^*}, X_t^{\nu^*,\eta^*}) &\le \mathrm{e}^{-\bar\rho t} w(r+\eta_{\max} t, X_t^{\nu^*,\eta^*})\\
&\le \frac{\mathrm{e}^{-\bar\rho t}\pi(X_t^{\nu^*,\eta^*})}{\bar\rho} + \mathrm{e}^{r\bar\rho/\eta_{\max}} \int_{r/\eta_{\max}+t}^\infty \mathrm{e}^{-\bar\rho t}\pi(a(\eta_{\max} t))\mathrm{d}t.
\end{aligned}$$

The second term converges to zero as $t \to \infty$ by Assumption 1-(ii). To estimate the first term, from (31) we deduce $X_t^{\nu^*,\eta^*} \le \max\{x, a(r+\eta_{\max} t)\}$. From Assumption 1-(ii), we deduce that

$$\lim_{t\to\infty} \mathrm{e}^{-\bar\rho t}\pi(a(r+\eta_{\max} t)) = 0,$$

which implies $\lim_{t\to\infty} \bar\rho^{-1}\mathrm{e}^{-\bar\rho t}\pi(X_t^{\nu^*,\eta^*}) = 0$ as well. □

Combining Theorem 2 and Proposition 2 we have obtained an equilibrium, summarized in the next corollary.

Corollary 2. *Let the controlled dynamics be as in* (19) *with* $\mu = 0$. *Then, an equilibrium pair is given by*

$$(\eta^*, \nu^*) = \begin{cases} (0,0), & \textit{for } x > b(r),\\ (\eta_{\max}, \nu^a), & \textit{for } a(r) \le x \le b(r), \end{cases} \tag{40}$$

with ν^a *as in* (22).

4.2. **Equilibrium with** $\mu < 0$ **.** Now we consider $\mu < 0$. Almost the entirety of our arguments hold for a generic $\pi(x)$ satisfying Assumption 1-(ii) and we only need Assumption 1-(iii) in the proof of Lemma 3. For any $\eta \in \mathcal{A}_F$ the firm's payoff reads

$$\mathcal{J}^F_{r,x}(\eta,\nu^*) = \int_0^{\tau(\eta)} \mathrm{e}^{-\bar\rho t}\pi\big(Y_t[x-\widehat\Lambda^\eta_t]\big)\mathrm{d}t + \int_{\tau(\eta)}^\infty \mathrm{e}^{-\bar\rho t}\pi\big(a(r+\Lambda^\eta_t)\big)\mathrm{d}t,$$

which is finite by Assumption 1-(ii). Motivated by the calculations for $\mu = 0$ we start by considering $\eta_t = \eta_{\max}$ and obtain

$$\widehat\Lambda_t = \eta_{\max}\int_0^t \mathrm{e}^{|\mu|s}\mathrm{d}s = \frac{\eta_{\max}}{|\mu|}\big(\mathrm{e}^{|\mu|t}-1\big) \triangleq f_M(t).$$

For later use we notice that

$$|\mu| f_M(t) = \frac{\eta_{\max}}{Y_t} - \eta_{\max} \text{ and } \dot f_M(t) = \eta_{\max}\mathrm{e}^{|\mu|t} = \frac{\eta_{\max}}{Y_t}. \tag{41}$$

The associated payoff is denoted by w_1 with a slight abuse of notation. It reads

$$\begin{aligned} w_1(r,x) &= \mathcal{J}^F_{r,x}(\eta_{\max},\nu^*) \\ &= \int_0^{\tau_M} \mathrm{e}^{-\bar\rho t}\pi\big(Y_t[x-f_M(t)]\big)\mathrm{d}t + \int_{\tau_M}^\infty \mathrm{e}^{-\bar\rho t}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}t, \end{aligned} \tag{42}$$

where

$$\tau_M = \tau_M(r,x) = \inf\{t \ge 0 : x - f_M(t) \le e^{|\mu|t}a(r+\eta_{\max}t)\}$$

Clearly, $\tau_M = 0$ for $x \le a(r)$ and it is easy to show that for $x > a(r)$, τ_M is the unique solution of

$$x - f_M(\tau_M) = \mathrm{e}^{|\mu|\tau_M}a\big(r+\eta_{\max}\tau_M\big). \tag{43}$$

For later use we observe that $(r,x)\mapsto \tau_M(r,x)$ is continuously differentiable on $\{x > a(r)\}$, with derivatives that we now proceed to calculating. Taking x-derivative of (43),

$$1 = \Big(\dot f_M(\tau_M) + |\mu|\mathrm{e}^{|\mu|\tau_M}a\big(r+\eta_{\max}\tau_M\big) + \mathrm{e}^{|\mu|\tau_M}\dot a\big(r+\eta_{\max}\tau_M\big)\eta_{\max}\Big)\frac{\partial\tau_M}{\partial x}.$$

Using the form of $\dot f_M(t)$ we get

$$1 = \Big[|\mu|\mathrm{e}^{|\mu|\tau_M}a\big(r+\eta_{\max}\tau_M\big) + \eta_{\max}\mathrm{e}^{|\mu|\tau_M}\Big(1+\dot a\big(r+\eta_{\max}\tau_M\big)\Big)\Big]\frac{\partial\tau_M}{\partial x}.$$

Using (43) and the explicit form of $|\mu|f_M(t)$ we get

$$1 = \Big[|\mu|x - \eta_{\max}\mathrm{e}^{|\mu|\tau_M} + \eta_{\max} + \eta_{\max}\mathrm{e}^{|\mu|\tau_M}\Big(1+\dot a\big(r+\eta_{\max}\tau_M\big)\Big)\Big]\frac{\partial\tau_M}{\partial x}.$$

Hence,

$$1 = \Big[|\mu|x + \eta_{\max} + \eta_{\max}\mathrm{e}^{|\mu|\tau_M}\dot a\big(r+\eta_{\max}\tau_M\big)\Big]\frac{\partial\tau_M}{\partial x}, \tag{44}$$

which also yields $\frac{\partial\tau_M}{\partial x} > 0$. Taking r-derivative of (43) and arguing in a similar way as above we have

$$-\dot a(r+\eta_{\max}\tau_M) = \big(\eta_{\max} + |\mu|a(r+\eta_{\max}\tau_M) + \eta_{\max}\dot a(r+\eta_{\max}\tau_M)\big)\frac{\partial\tau_M}{\partial r}. \tag{45}$$

Then, $\frac{\partial\tau_M}{\partial r} < 0$.

The function w_1 is our candidate for the firm's value at equilibrium when $x > a(r)$ is sufficiently close to $a(r)$. We will make this statement rigorous later. For now let us state an initial result concerning w_1. We recall that in this section

$$\mathcal{M} = \{(r,x)\in[0,\infty)\times\mathbb{R}_+ : 0 < x \le a(r)\} \text{ and } \mathcal{I} = \{(r,x)\in[0,\infty)\times\mathbb{R}_+ : x > a(r)\}, \tag{46}$$

because of Theorem 2

Proposition 3. *For w_1 as in (42) we have $w_1 \in C^1(\overline{\mathcal{I}})$ and*

$$\bar\rho w_1(r,x) - \pi(x) = \mu x \partial_x w_1(r,x) + \eta_{\max}\big[\partial_r w_1(r,x) - \partial_x w_1(r,x)\big], \quad \textit{for } (r,x) \in (0,\infty)^2. \tag{47}$$

Proof. We start by computing $\partial_r w_1$ and $\partial_x w_1$. Their explicit expressions will imply $w_1 \in C^1(\overline{\mathcal{I}})$ as claimed. Let us first consider the derivative with respect to r.

By following the same arguments as in (35), we obtain

$$\frac{\partial w_1}{\partial r}(r,x) = \int_{\tau_M}^{\infty} \mathrm{e}^{-\bar\rho t}\dot\pi(a(r+\eta_{\max}t))\dot a(r+\eta_{\max}t)\mathrm{d}t > 0. \tag{48}$$

This integral is finite because, by integration by parts,

$$\int_{\tau_M}^{\infty} \mathrm{e}^{-\bar\rho t}\dot\pi(a(r+\eta_{\max}t))\dot a(r+\eta_{\max}t)\mathrm{d}t = \frac{\bar\rho}{\eta_{\max}}\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}t - \frac{1}{\eta_{\max}}\mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r+\eta_{\max}\tau_M)\big),$$

and the final expression is finite by Assumption 1-(ii).

Turning now to the derivative with respect to x, using (43) and similar calculations to (36), we obtain:

$$\frac{\partial w_1}{\partial x}(r,x) = \lim_{h\to 0}\frac{1}{h}\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\int_0^h \dot\pi\big(Y_t[x+\xi-f_M(t)]\big)Y_t\mathrm{d}\xi\,\mathrm{d}t.$$

Using the monotonicity of $\dot\pi$ and the monotone convergence theorem, it is then easy to conclude that

$$\frac{\partial w_1}{\partial x}(r,x) = \int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x-f_M(t)]\big)Y_t\mathrm{d}t > 0. \tag{49}$$

The above integral is finite because it is computed over a bounded interval and the integrand is continuous. Combining the expressions for the two derivatives, we obtain

$$\begin{aligned}(\partial_r w_1 - \partial_x w_1)(r,x) = &\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r+\eta_{\max}t)\big)\dot a(r+\eta_{\max}t)\mathrm{d}t\\ &- \int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x-f_M(t)]\big)Y_t\mathrm{d}t.\end{aligned} \tag{50}$$

Moreover, $\mu x\partial_x w_1(r,x) < 0$ for $x > a(r)$ with

$$\mu x\partial_x w_1(r,x) = \mu x\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x-f_M(t)]\big)Y_t\mathrm{d}t,$$

by (49) and $\mu < 0$.

Now let us calculate $\bar\rho w_1(r,x) - \pi(x)$ and compare it to $\mu x\partial_x w_1(r,x) + \eta_{\max}(\partial_r w_1 - \partial_x w_1)(r,x)$. Rewriting $\bar\rho\mathrm{e}^{-\bar\rho t}\mathrm{d}t = -\mathrm{d}\mathrm{e}^{-\bar\rho t}$ and integrating by parts, we have

$$\begin{aligned}&\bar\rho w_1(r,x) - \pi(x)\\ &= -\pi(x) - \int_0^{\tau_M}\pi\big(Y_t[x-f_M(t)]\big)\mathrm{d}\mathrm{e}^{-\bar\rho t} - \int_{\tau_M}^{\infty}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}\mathrm{e}^{-\bar\rho t}\\ &= \int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x-f_M(t)]\big)Y_t\big(\mu x - \mu f_M(t) - \dot f_M(t)\big)\mathrm{d}t\\ &\quad + \eta_{\max}\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r+\eta_{\max}t)\big)\dot a(r+\eta_{\max}t)\mathrm{d}t,\end{aligned}$$

where we also used (43) to cancel a term resulting from integration by parts. Notice that $-\mu f_M(t) - \dot f_M(t) = -\eta_{\max}$ due to (41) and therefore, it is easy to verify that (47) holds. □

It is clear from the form of (50) that for sufficiently large values of x it must be $(\partial_r w_1 - \partial_x w_1)(r,x) < 0$ because $\lim_{x\to\infty} \tau_M(r,x) = \infty$ and $\dot{\pi}$ is non-decreasing (cf. Assumption 1). Then, w_1 cannot be a solution of the HJB equation in the whole space but at most on the set where $(\partial_r w_1 - \partial_x w_1)(r,x) \geq 0$. This motivates the next part of the analysis.

By definition of $\tau(0) = \tau_{r,x}(0)$ (cf. (30)) we have

$$x\mathrm{e}^{\mu\tau_{r,x}(0)} = a(r) \iff \tau_{r,x}(0) = \frac{1}{|\mu|}\ln\frac{x}{a(r)}.$$

We want to consider a class of firm's strategies of the form $\eta^\tau_t \triangleq \eta_{\max} 1_{\{t\geq\tau\}}$, for a generic $\tau \geq 0$. These are strategies according to which the firm is idle until time τ and then it starts exerting emission abatement at the maximum rate. The associated payoff reads

$$\mathcal{J}^F_{r,x}(\eta^\tau,\nu^*) = \int_0^{\tau\wedge\tau_{r,x}(0)} \mathrm{e}^{-\bar{\rho}t}\pi\big(xY_t\big)\mathrm{d}t + \mathrm{e}^{-\bar{\rho}(\tau\wedge\tau_{r,x}(0))} w_1(r, xY_{\tau\wedge\tau_{r,x}(0)}) \triangleq \mathcal{J}_{r,x}(\tau),$$

where, at time $\tau\wedge\tau(0)$ the firm receives the payoff associated to exerting maximum control. Then, with a slight abuse of notation, we set

$$w_0(r,x) \triangleq \sup_{\tau\geq 0} \mathcal{J}_{r,x}(\tau). \tag{51}$$

The problem in (51) is a deterministic optimal stopping problem. An explicit solution seems difficult to obtain but we can still rely on the general optimal stopping theory to find a characterization of the optimal stopping rule. Clearly $w_0(r,x) \geq \mathcal{J}_{r,x}(0) = w_1(r,x)$ and next we establish continuity of w_0.

Lemma 2. *For any compact $K \subset \overline{\mathcal{I}}$ there is $c_K > 0$ such that*

$$\big|w_0(r_1,x_1) - w_0(r_2,x_2)\big| \leq c_K\big(|x_2 - x_1| + |r_2 - r_1|\big), \quad \text{for } (r,x)\in K.$$

Proof. For $r_1, r_2 \in [0,\infty)$ and $x_1, x_2 \in [0,\infty)$, set $\tau_1 = \tau_{r_1,x_1}(0)$ and $\tau_2 = \tau_{r_2,x_2}(0)$ for notational simplicity. Then,

$$\begin{aligned}&\big|w_0(r_1,x_1) - w_0(r_2,x_2)\big|\\ &\leq \int_0^\infty \mathrm{e}^{-\bar{\rho}t}|\pi(x_2Y_t) - \pi(x_1Y_t)|\mathrm{d}t + \int_{\tau_1\wedge\tau_2}^{\tau_1\vee\tau_2} \mathrm{e}^{-\bar{\rho}t}\pi(x_1Y_t)\vee\pi(x_2Y_t)\mathrm{d}t\\ &\quad+\bar{\rho}|\tau_2-\tau_1|w_1(r_1,x_1) + \sup_\tau\big|w_1(r_1,x_1Y_{\tau\wedge\tau_1}) - w_1\big(r_1, x_1Y_{\tau\wedge\tau_2}\vee a(r_1)\big)\big|\\ &\quad+\sup_\tau\big|w_1\big(r_1,x_1Y_{\tau\wedge\tau_2}\vee a(r_1)\big) - w_1(r_2,x_2Y_{\tau\wedge\tau_2})\big|,\end{aligned}$$

where we used $w_1(r,xY_t) \leq w_1(r,x)$ by monotonicity of w_1 and $Y_t \leq 1$. Since $(r,x)\mapsto\tau_{r,x}(0)$ and $(r,x)\mapsto w_1(r,x)$ are locally Lipschitz continuous in $\overline{\mathcal{I}}$, and $r\mapsto a(r)$ is locally Lipschitz in $\mathbb{R}_+$, then it is not hard to see that $(r,x)\mapsto w_0(r,x)$ is also locally Lipschitz continuous in $\overline{\mathcal{I}}$, as claimed. □

Thanks to continuity of w_0, standard optimal stopping theory guarantees that it is optimal to stop at

$$\tau_* = \inf\{t\geq 0 : w_0(r,xY_t) = w_1(r,xY_t)\}\wedge\tau_{r,x}(0). \tag{52}$$

Moreover,

$$s \mapsto \int_0^{s\wedge\tau_*} \mathrm{e}^{-\bar{\rho}t}\pi(xY_t)\mathrm{d}t + \mathrm{e}^{-\bar{\rho}(s\wedge\tau_*)}w_0(r,xY_{s\wedge\tau_*}), \tag{53}$$

must be constant (a deterministic martingale). Now, it is natural to split the state space into the so-called *continuation* and *stopping* sets, defined respectively as

$$\mathcal{C} \triangleq \{(r,x)\in[0,\infty)\times\mathbb{R}_+ : w_0(r,x) > w_1(r,x)\} \quad \text{and} \quad \mathcal{S} = \mathcal{C}^c \triangleq \big([0,\infty)\times\mathbb{R}_+\big)\setminus\mathcal{C}.$$

By construction $(r, a(r)) \in \mathcal{S}$ for any $r \in [0, \infty)$. Moreover, writing

$$\mathrm{e}^{-\bar{\rho}(\tau\wedge\tau_{r,x}(0))} w_1(r, xY_{\tau\wedge\tau_{r,x}(0)}) = w_1(r,x) + \int_0^{\tau\wedge\tau_{r,x}(0)} \mathrm{e}^{-\bar{\rho}t}\Big(\mu x Y_t \partial_x w_1\big(r, xY_t\big) - \bar{\rho} w_1\big(r, xY_t\big)\Big)\mathrm{d}t,$$

we deduce

$$\varphi(r,x) \triangleq w_0(r,x) - w_1(r,x) = \sup_{\tau} \int_0^{\tau\wedge\tau_{r,x}(0)} \mathrm{e}^{-\bar{\rho}t} h(r, xY_t)\mathrm{d}t, \tag{54}$$

where

$$\begin{aligned} h(r,x) &\triangleq -|\mu| x \partial_x w_1(r,x) - \bar{\rho} w_1(r,x) + \pi(x) \\ &= -\eta_{\max}\big(\partial_r w_1 - \partial_x w_1\big)(r,x). \end{aligned} \tag{55}$$

Thanks to this formulation, it is possible to establish the form of the optimal stopping rule (52).

Lemma 3. *We have $h \in C^1(\overline{\mathcal{I}})$. For $x \geq a(r)$ and $r > 0$ we have $\partial_r h < 0$ and $\partial_x h > 0$. Moreover, $h(r, a(r)) < 0$ and $\lim_{x\to\infty} h(r,x) > 0$.*

Before proving the lemma, we obtain one important consequence thereof.

Proposition 4. *It is optimal to stop at*

$$\tau_* = \tau_b \triangleq \inf\{t \geq 0 : xY_t \leq b(r)\}, \tag{56}$$

where $b(r) > a(r)$ is the unique solution of $h(r, b(r)) = 0$ for $r \in [0, \infty)$. Moreover, $b \in C^1(\mathbb{R}_+)$ is increasing.

Proof. By Lemma 3, $t \mapsto h(r, xY_t)$ is decreasing. Therefore, the maximum on the right-hand side of (54) is uniquely attained at τ_b. Moreover, by the implicit function theorem we have

$$\dot{b}(r) = -\frac{\partial_r h(r, b(r))}{\partial_x h(r, b(r))} > 0, \quad r \in \mathbb{R}_+, \tag{57}$$

which concludes the proof. □

It remains to prove the lemma.

Proof of Lemma 3. We first observe that, due to $\partial_x w_1(r, a(r)) = 0$ (cf. (49)), we have $h(r, a(r)) = -\bar{\rho} w_1(r, a(r)) + \pi(a(r))$. However, Assumption 1-(ii) and (42) also imply

$$w_1(r, a(r)) = \int_0^\infty \mathrm{e}^{-\bar{\rho}t} \pi(a(r + \eta_{\max} t))\mathrm{d}t > \bar{\rho}^{-1}\pi\big(a(r)\big),$$

hence $h(r, a(r)) < 0$. By the definition of h and (50) we have by monotone convergence

$$\begin{aligned} \lim_{x\to\infty} h(r,x) = -\eta_{\max} \lim_{x\to\infty} \big[\partial_r w_1(r,x) - \partial_x w_1(r,x)\big] &= \eta_{\max} \lim_{x\to\infty} \int_0^\infty \mathrm{e}^{-\bar{\rho}t} \dot{\pi}\big(Y_t[x - f_M(t)]\big) Y_t \mathrm{d}t \\ = \eta_{\max} \int_0^\infty \mathrm{e}^{-\bar{\rho}t} \lim_{x\to\infty} \dot{\pi}\big(Y_t[x - f_M(t)]\big) Y_t \mathrm{d}t > 0, \end{aligned}$$

where the final inequality holds by (50) and Assumption 1-(ii). This concludes the proof of the final statement in the lemma.

Next we calculate the gradient of h. First we recall (50) and notice that we can rewrite, by integration by parts,

$$
\begin{aligned}
h(r,x) &= -\eta_{\max}\big(\partial_r w_1 - \partial_x w_1\big)(r,x) \\
&= -\int_{\tau_M}^{\infty} \mathrm{e}^{-\bar\rho t}\dot\pi(a(r+\eta_{\max}t))\dot a(r+\eta_{\max}t)\eta_{\max}\mathrm{d}t + \eta_{\max}\int_0^{\tau_M} \mathrm{e}^{-\bar\rho t}\dot\pi(Y_t[x-f_M(t)])Y_t\mathrm{d}t \\
&= \mathrm{e}^{-\bar\rho\tau_M}\pi(a(r+\eta_{\max}\tau_M)) - \bar\rho\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\pi(a(r+\eta_{\max}t))\mathrm{d}t \\
&\quad + \eta_{\max}\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi(Y_t[x-f_M(t)])Y_t\mathrm{d}t.
\end{aligned} \tag{58}
$$

Computing the derivative with respect to r, arguing as in (35) to justify interchanging the limit and the integral, we get

$$
\begin{aligned}
\partial_r h(r,x) &= -\bar\rho\mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r+\eta_{\max}\tau_M)\big)\frac{\partial\tau_M}{\partial r} + \mathrm{e}^{-\bar\rho\tau_M}\dot\pi\big(a(r+\eta_{\max}\tau_M)\big)\dot a(r+\eta_{\max}\tau_M)\Big(1+\eta_{\max}\frac{\partial\tau_M}{\partial r}\Big) \\
&\quad + \bar\rho\mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r+\eta_{\max}\tau_M)\big)\frac{\partial\tau_M}{\partial r} - \bar\rho\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r+\eta_{\max}t)\big)\dot a(r+\eta_{\max}t)\mathrm{d}t \\
&\quad + \eta_{\max}\mathrm{e}^{-\bar\rho\tau_M}\dot\pi\big(Y_{\tau_M}[x-f_M(\tau_M)]\big)Y_{\tau_M}\frac{\partial\tau_M}{\partial r} \\
&= \mathrm{e}^{-\bar\rho\tau_M}\dot\pi\big(a(r+\eta_{\max}\tau_M)\big)\Big[\dot a(r+\eta_{\max}\tau_M)\Big(1+\eta_{\max}\frac{\partial\tau_M}{\partial r}\Big) + \eta_{\max}Y_{\tau_M}\frac{\partial\tau_M}{\partial r}\Big] \\
&\quad - \eta_{\max}\bar\rho\int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r+\eta_{\max}t)\big)\dot a(r+\eta_{\max}t)\mathrm{d}t.
\end{aligned}
$$

The finiteness of the integral can be justified by another integration by parts and Assumption 1-(ii). This shows the differentiability with respect to r. To show that the derivative is negative, recall that $\frac{\partial\tau_M}{\partial r} < 0$ and consider the representation in the second line of (58). The second term depends on r only through τ_M, its derivative with respect to r is therefore strictly negative. To evaluate the first term, consider $r_1 < r_2$. Then, using Assumption 1-(iii)

$$
\begin{aligned}
&\int_{\tau_M(r_1)}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r_1+\eta_{\max}t)\big)\dot a(r_1+\eta_{\max}t)\mathrm{d}t - \int_{\tau_M(r_2)}^{\infty}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r_2+\eta_{\max}t)\big)\dot a(r_2+\eta_{\max}t)\mathrm{d}t \\
&= \int_{\tau_M(r_1)}^{\infty}\mathrm{e}^{-\bar\rho t}\Big(\dot\pi\big(a(r_1+\eta_{\max}t)\big)\dot a(r_1+\eta_{\max}t) - \dot\pi\big(a(r_2+\eta_{\max}t)\big)\dot a(r_2+\eta_{\max}t)\Big)\mathrm{d}t \\
&- \int_{\tau_M(r_2)}^{\tau_M(r_1)}\mathrm{e}^{-\bar\rho t}\dot\pi\big(a(r_2+\eta_{\max}t)\big)\dot a(r_2+\eta_{\max}t)\mathrm{d}t \le 0.
\end{aligned}
$$

Thus, $\frac{\partial h}{\partial r} < 0$. Further, to compute the derivative with respect to x, we use integration by parts to rewrite h as follows:

$$
\begin{aligned}
h(r,x) &= -\frac{\eta_{\max}}{|\mu|x+\eta_{\max}}\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\frac{\mathrm{d}}{\mathrm{d}t}\pi\big(Y_t[x-f_M(t)]\big)\mathrm{d}t - \int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\frac{\mathrm{d}}{\mathrm{d}t}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}t \\
&= \frac{\eta_{\max}}{|\mu|x+\eta_{\max}}\pi(x) - \frac{\bar\rho\eta_{\max}}{|\mu|x+\eta_{\max}}\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\pi\big(Y_t[x-f_M(t)]\big)\mathrm{d}t \\
&\quad - \frac{\eta_{\max}}{|\mu|x+\eta_{\max}}\mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r+\eta_{\max}\tau_M)\big) - \int_{\tau_M}^{\infty}\mathrm{e}^{-\bar\rho t}\frac{\mathrm{d}}{\mathrm{d}t}\pi\big(a(r+\eta_{\max}t)\big)\mathrm{d}t.
\end{aligned}
$$

Once again, we can apply monotone convergence theorem to show that h is differentiable with respect to x, with

$$\begin{aligned}
&\partial_x h(r,x)\\
&= \frac{\eta_{\max}}{|\mu|x+\eta_{\max}}\dot\pi(x) - \frac{|\mu|\eta_{\max}}{\big(|\mu|x+\eta_{\max}\big)^2}\Big(\pi(x) - \bar\rho\int_0^{\tau_M} \mathrm{e}^{-\bar\rho t}\pi\big(Y_t[x-f_M(t)]\big)\mathrm{d}t\Big)\\
&\quad - \frac{\bar\rho\eta_{\max}}{|\mu|x+\eta_{\max}}\int_0^{\tau_M}\mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x-f_M(t)]\big)Y_t\mathrm{d}t\\
&\quad + \frac{\eta_{\max}|\mu|}{\big(|\mu|x+\eta_{\max}\big)^2}\mathrm{e}^{-\bar\rho\tau_M}\pi\big(a(r+\eta_{\max}\tau_M)\big)\\
&\quad + \mathrm{e}^{-\bar\rho\tau_M}\dot\pi\big(a(r+\eta_{\max}\tau_M)\big)\dot a(r+\eta_{\max}\tau_M)\frac{\partial\tau_M}{\partial x}\frac{|\mu|x}{|\mu|x+\eta_{\max}}.
\end{aligned}$$

To show that the derivative is positive, recall that $\frac{\partial\tau_M}{\partial x} > 0$ and consider the representation in the second line of (58). The first term depends on x only through τ_M, its derivative with respect to x is therefore strictly positive. To evaluate the second term, consider $x_1 < x_2$. Then,

$$\begin{aligned}
&\int_0^{\tau_M(x_2)} \mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x_2-f_M(t)]\big)Y_t\mathrm{d}t - \int_0^{\tau_M(x_1)} \mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x_1-f_M(t)]\big)Y_t\mathrm{d}t\\
&= \int_{\tau_M(x_1)}^{\tau_M(x_2)} \mathrm{e}^{-\bar\rho t}\dot\pi\big(Y_t[x_2-f_M(t)]\big)Y_t\mathrm{d}t + \int_0^{\tau_M(x_1)} \mathrm{e}^{-\bar\rho t}\Big(\dot\pi\big(Y_t[x_2-f_M(t)]\big) - \dot\pi\big(Y_t[x_1-f_M(t)]\big)\Big)Y_t\mathrm{d}t,
\end{aligned}$$

where both terms are nonnegative by our assumptions. Thus $\frac{\partial h}{\partial x} > 0$.

Continuity of ∇h is easily deduced by the explicit expressions for the derivatives. □

Proposition 5. *Let $\mu < 0$ and $(R_0, X_0) = (r, x)$. Take w_0 and w_1 as in (51) and (42), respectively. Then, the firm's equilibrium payoff reads as*

$$w(r,x) = \begin{cases} w_0(r,x), & x > b(r),\\ w_1(r,x), & a(r) \le x \le b(r). \end{cases} \tag{59}$$

The firm's optimal strategy reads $\eta^*_t = \eta_{\max}1_{\{t\ge\tau_b\}}$, with $\tau_b = \tau_*$ as in (56).

Proof. The proof consists of showing that w solves the HJB equation and then applying the verification theorem (in the form of Corollary 1 with Remark 3).

First, we want to show that $w \in C^1(\overline{\mathcal{I}})$. The result is clear for $a(r) \le x \le b(r)$ by (48) and (49) and because $w = w_1$ in that set. Indeed, w_1 is everywhere continuously differentiable. For $x > b(r)$ it is convenient to recall that (cf. (56))

$$\tau_*^{r,x} = \frac{1}{|\mu|}\ln\frac{x}{b(r)}.$$

Therefore

$$\frac{\partial\tau_*^{r,x}}{\partial x} = \frac{1}{|\mu|x} \quad\text{and}\quad \frac{\partial\tau_*^{r,x}}{\partial r} = -\frac{1}{|\mu|}\frac{\dot b(r)}{b(r)},$$

are both continuous (cf. (57)). Recall that for $x > b(r)$, $w(r,x) = w_0(r.x) = w_1(r,x) + \varphi(r,x)$ with (cf. (54))

$$\varphi(r,x) = \int_0^{\tau_*^{r,x}} \mathrm{e}^{-\bar\rho t}h\big(r, xY_t\big)\mathrm{d}t, \tag{60}$$

where we also used that $\tau_*^{r,x} \le \tau_{r,x}(0)$. Then, ∇w_0 is continuous in the set $\{(r,x) : x \ge b(r)\}$ if and only if $\nabla\varphi$ is such. Differentiating in x, using the continuity of the derivatives of h shown above to justify exchanging the limit and the integral, and recalling $h(r,b(r))=0$ yields

$$\begin{aligned}\partial_x\varphi(r,x)&=\mathrm{e}^{-\bar\rho\tau_*^{r,x}}h(r,b(r))\frac{\partial\tau_*^{r,x}}{\partial x}+\int_0^{\tau_*^{r,x}}\mathrm{e}^{-\bar\rho t}\partial_x h\big(r,xY_t\big)Y_t\mathrm{d}t\\&=\int_0^{\tau_*^{r,x}}\mathrm{e}^{-\bar\rho t}\partial_x h\big(r,xY_t\big)Y_t\mathrm{d}t.\end{aligned}$$

Differentiating in r yields

$$\begin{aligned}\partial_r\varphi(r,x)&=\mathrm{e}^{-\bar\rho\tau_*^{r,x}}h(r,b(r))\partial_r\tau_*^{r,x}+\int_0^{\tau_*^{r,x}}\mathrm{e}^{-\bar\rho t}\partial_r h(r,xY_t)\mathrm{d}t\\&=\int_0^{\tau_*^{r,x}}\mathrm{e}^{-\bar\rho t}\partial_r h(r,xY_t)\mathrm{d}t.\end{aligned}$$

We immediately deduce $\nabla\varphi$ continuous in the set $\{(r,x):x\geq b(r)\}$. Since $w_0(r,b(r))=w_1(r,b(r))$, it remains to verify that $\nabla w_0(r,b(r))=\nabla w_1(r,b(r))$. However, $\tau_*^{r,b(r)}=0$ implies $\nabla\varphi(r,b(r))=0$ and the claim follows.

Now we show that w solves the HJB equation. Recall that $h(r,x)<0$ for $a(r)\leq x<b(r)$ by Lemma 3. Comparing to (47) yields $(\partial_r w_1-\partial_x w_1)(r,x)=-\eta_{\max}^{-1}h(r,x)>0$ for $a(r)\leq x<b(r)$ and therefore $\eta_{\max}(\partial_r w_1-\partial_x w_1)(r,x)=\mathcal{H}(r,x;w)$. Moreover, it is clear that $w_x(r,a(r))=\partial_x w_1(r,a(r))=0$ for all $r>0$ by (49). Thus, w is a solution of the HJB equation in the set $\{(r,x):a(r)\leq x<b(r)\}$, i.e., by Proposition 3

$$\begin{aligned}\mu x\partial_x w(r,x)-\bar\rho w(r,x)+\mathcal{H}(r,x;w)+\pi(x)=0,&\quad (r,x):a(r)<x<b(r),\\ w_x(r,a(r))=0,&\quad r\in(0,\infty).\end{aligned}$$

Next, we prove that $(\partial_r w_0-\partial_x w_0)(r,x)\leq 0$ for $x\geq b(r)$. Since $x\mapsto h(r,xY_t)$ and $x\mapsto\tau_{r,x}(0)$ are increasing, then for $x_1<x_2$

$$\begin{aligned}\varphi(r,x_1)=\sup_\tau\int_0^{\tau\wedge\tau_{r,x_1}(0)}\mathrm{e}^{-\bar\rho t}h(r,x_1Y_t)\mathrm{d}t&\leq\sup_\tau\int_0^{\tau\wedge\tau_{r,x_2}(0)}\mathrm{e}^{-\bar\rho t}h(r,x_1Y_t)\mathrm{d}t\\&\leq\sup_\tau\int_0^{\tau\wedge\tau_{r,x_2}(0)}\mathrm{e}^{-\bar\rho t}h(r,x_2Y_t)\mathrm{d}t=\varphi(r,x_2),\end{aligned}$$

where the first inequality holds because by replacing $\tau_{r,x_1}(0)$ with $\tau_{r,x_2}(0)$ we enlarge the set of admissible stopping rules, and the second inequality holds by monotonicity of h. Since $r\mapsto h(r,xY_t)$ and $r\mapsto\tau_{r,x}(0)$ are decreasing, by analogous arguments we conclude that for $r_1<r_2$

$$\begin{aligned}\varphi(r_1,x)=\sup_\tau\int_0^{\tau\wedge\tau_{r_1,x}(0)}\mathrm{e}^{-\bar\rho t}h(r_1,xY_t)\mathrm{d}t&\geq\sup_\tau\int_0^{\tau\wedge\tau_{r_2,x}(0)}\mathrm{e}^{-\bar\rho t}h(r_2,xY_t)\mathrm{d}t\\&\geq\sup_\tau\int_0^{\tau\wedge\tau_{r_2,x}(0)}\mathrm{e}^{-\bar\rho t}h(r_2,xY_t)\mathrm{d}t=\varphi(r_2,x).\end{aligned}$$

Thus, for (r,x) such that $x\geq b(r)$ we have

$$\big(\partial_r w_0-\partial_x w_0\big)(r,x)=\big(\partial_r w_1-\partial_x w_1\big)(r,x)+\big(\partial_r\varphi-\partial_x\varphi\big)(r,x)\leq 0,$$

as needed. Since τ_* from (56) is optimal and using the fact that the mapping in (53) is constant, we deduce that[1]

$$\mu x\partial_x w_0(r,x)-\bar\rho w_0(r,x)+\pi(x)=0,\quad\text{for } x\geq b(r).$$

[1]In this case also direct differentiation of the function w_0 would easily lead to the same result.

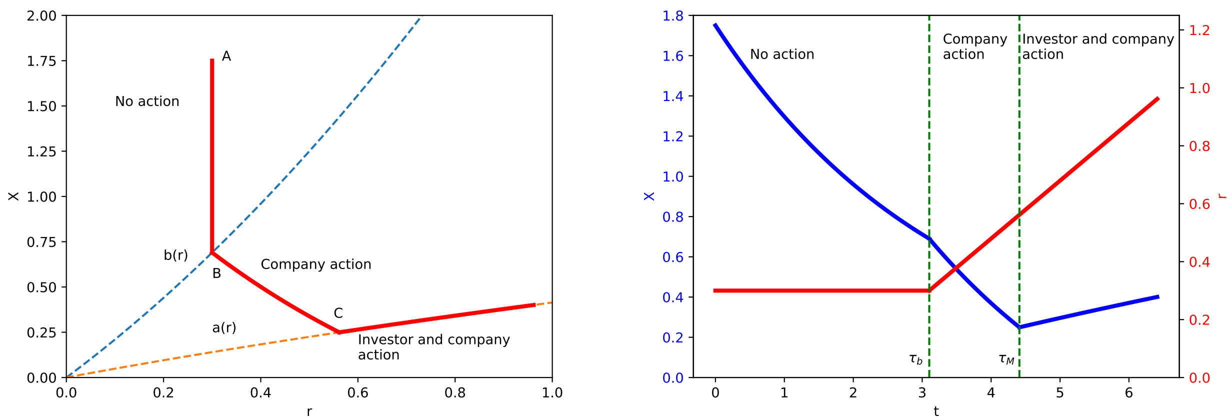


FIGURE 1. Illustration of the optimal strategies in the deterministic case with $\mu < 0$. Left: X as function of R. Right: X (in blue) and R (in red) as function of time. See text for a detailed discussion.

In conclusion we have shown that $w \in C^1(\overline{\mathcal{I}})$ solves

$$\mu x \partial_x w(r,x) - \bar{\rho} w(r,x) + \mathcal{H}(r,x;w) + \pi(x) = 0, \quad (r,x) : x > a(r),$$
$$w_x(r, a(r)) = 0, \quad r \in (0,\infty).$$

To check the transversality condition (17) for the function w, using (51) and (42) we get

$$|w(r,x)| \leq \frac{\pi(x)}{\bar{\rho}} + w_1(r,x) \leq \frac{2\pi(x)}{\bar{\rho}} + \mathrm{e}^{r\bar{\rho}/\eta_{\max}} \int_{r/\eta_{\max}}^{\infty} \mathrm{e}^{-\bar{\rho}t} \pi(a(\eta_{\max} t)) \mathrm{d}t$$

and then proceed as in the proof of Proposition 2.

By construction, the optimal investment policy ν^* satisfies (16) (cf. Theorem 2) and therefore we can apply our verification theorem (Corollary 1 and Remark 3) to deduce optimality of $\eta^*_t \triangleq \eta_{\max} 1_{\{t \geq \tau_*\}}$, in the sense that $w(r,x) = \sup_\eta \mathcal{J}^F_{r,x}(\eta, \nu^*) = \mathcal{J}^F_{r,x}(\eta^*, \nu^*)$, for $(r,x) \in \overline{\mathcal{I}}$. □

Thanks to Theorem 2 and Proposition 5, the form of the Nash equilibrium follows by a direct application of Corollary 1 and Remark 3.

Corollary 3. *Let the controlled dynamics be as in* (19) *with* $\mu < 0$*. Then, an equilibrium pair is given by*

(61) $$(\eta^*, \nu^*) = (\eta_{\max} 1_{\{t \geq \tau_b\}}, \nu^a),$$

with ν^a *as in* (22) *and* τ_b *as in* (56).

The equilibrium strategies of Corollary 3 are illustrated in Figure 1 in the situations when the initial values (X_0, R_0) are in the no action region for both the company and the investor, that is $X_0 > b(R_0)$. The starting point is the point A in the left graph, corresponding to $t = 0$. In the first period, the company and the investor do nothing, and the production capacity of the firm decreases exponentially until it hits the value $b(R_0)$ (point B in the left graph). At this time (τ_b), the firm starts to invest in pollution abatement, but the investor takes no action. As a result, during the second period, the production capacity declines faster than in the first period, and the abatement investment grows at the maximum rate. The second period continues until the production capacity X_t hits the moving boundary $a(R_t)$ (point C in the left graph). At this time (τ_M) the investor starts to invest to keep the production capacity at the moving boundary.

4.3. **Strong solution of the HJB system.** A posteriori we can show that the investor's equilibrium payoff v solves the variational problem (14). Then, the pair (v, w) solves the HJB system (13)–(14) because we have shown that w solves (13) in the proof of Proposition 5. However, since the function v may not be smooth, we need a precise notion of solution of the HJB system (13)–(14) (we call it *strong solution*). In particular, we will adapt the definition of the set $\mathcal{M}$ in order to account for the fact that v_x is only defined almost everywhere.

Definition 2 (**Strong solution**). *For $\sigma = 0$, recall that $\mathcal{L} = \mu x \partial_x$. The pair (v, w) is a* strong solution *of the HJB system if:*

(i) *v is locally Lipschitz on $(0, \infty)^2$ with $v_x \le \alpha$ a.e.*
(ii) *Letting $\mathcal{M} \triangleq \{(r, x) : v_x(r, x)$ exists and $v_x(r, x) = \alpha\}$, $\mathcal{I} \triangleq \mathcal{M}^c$ and $\partial\mathcal{M} = \overline{\mathcal{M}} \cap \overline{\mathcal{I}}$, it holds $w \in C^1(\overline{\mathcal{I}})$ and*
$$\begin{cases} (\mathcal{L}w - \bar{\rho}w)(r, x) + \mathcal{H}(r, x; w) + \pi(x) = 0, & (r, x) \in \mathcal{I}, \\ w_x(r, x) = 0, & (r, x) \in \partial\mathcal{M}. \end{cases}$$
(iii) *Letting $\eta^*(r, x) \triangleq \eta_{\max} 1_{\{w_r > w_x\}}(r, x)$,*
$$\max\left\{(\mathcal{L}v - \rho v)(r, x) + (v_r(r, x) - v_x(r, x))\,\eta^*(r, x) + \Pi(r, x), v_x(r, x) - \alpha\right\} = 0,$$
for a.e. $(r, x) \in [0, \infty) \times \mathbb{R}_+$.

Next we show that v satisfies (i) and (iii) of the above definition. For the statement and proof of the next proposition it is convenient to introduce sets

$$\mathcal{O}_a \triangleq \{(r, x) : 0 < x < a(r)\}, \quad \mathcal{O}_{a,b} \triangleq \{(r, x) : a(r) < x < b(r)\} \quad \text{and} \quad \mathcal{O}_b \triangleq \{(r, x) : b(r) < x\}.$$

Moreover, we are going to use the following notation: given a set S that can be partitioned as $S = A \cup B$, we say that a function $\varphi : S \to \mathbb{R}$ belongs to $C(\overline{A}) \cap C(\overline{B})$ if φ is continuous separately on A and B with continuous extensions to the closure of both sets; this allows the function φ to be discontinuous across the boundary $\overline{A} \cap \overline{B}$, thus $C(S) \subsetneq C(\overline{A}) \cap C(\overline{B})$.

Proposition 6. *The investor's equilibrium payoff v satisfies (i) and (iii) in Definition 2. The set $\mathcal{M}$ is explicitly given by $\mathcal{M} = \{(r, x) : 0 < x \le a(r)\}$ and its boundary reads $\partial\mathcal{M} = \{(r, x) : x = a(r)\}$. Finally, $v_x \in C(\overline{\mathcal{O}}_a) \cap C(\overline{\mathcal{O}}_{a,b}) \cap C(\overline{\mathcal{O}}_b)$ (possibly discontinuous across $r \mapsto b(r)$) and $v_r \in C(\overline{\mathcal{O}}_{a,b})$.*

Proof. First of all we obtain an analytical expression for v using that $v(r, x) = \mathcal{J}^I_{r,x}(\eta^*, \nu^*)$. From the explicit formulae for η^* and ν^* we obtain the following expressions: for $(r, x) \in \overline{\mathcal{O}}_b$ we have

$$\begin{aligned} v(r, x) = &\int_0^{\tau_b^{r,x}} \mathrm{e}^{-\rho t}\Pi\big(r, xY_t\big)\mathrm{d}t + \mathrm{e}^{-\rho\tau_b^{r,x}} \int_0^{\tau_M^{r,b(r)}} \mathrm{e}^{-\rho t}\Pi\big(r + \eta_{\max}t, Y_t[b(r) - f_M(t)]\big)\mathrm{d}t \\ &+ \mathrm{e}^{-\rho\tau_b^{r,x}} \int_{\tau_M^{r,b(r)}}^{\infty} \mathrm{e}^{-\rho t}\Pi\big(r + \eta_{\max}t, a(r + \eta_{\max}t)\big)\mathrm{d}t \\ &- \alpha\mathrm{e}^{-\rho\tau_b^{r,x}} \int_{\tau_M^{r,b(r)}}^{\infty} \mathrm{e}^{-\rho t}\Big(\eta_{\max}\dot{a}(r + \eta_{\max}t) + \eta_{\max} + |\mu|a(r + \eta_{\max}t)\Big)\mathrm{d}t; \end{aligned} \tag{62}$$

for $(r, x) \in \overline{\mathcal{O}}_{a,b}$ we have

$$\begin{aligned} v(r, x) = &\int_0^{\tau_M^{r,x}} \mathrm{e}^{-\rho t}\Pi\big(r + \eta_{\max}t, Y_t[x - f_M(t)]\big)\mathrm{d}t \\ &+ \int_{\tau_M^{r,x}}^{\infty} \mathrm{e}^{-\rho t}\Pi\big(r + \eta_{\max}t, a(r + \eta_{\max}t)\big)\mathrm{d}t \\ &- \alpha \int_{\tau_M^{r,x}}^{\infty} \mathrm{e}^{-\rho t}\Big(\eta_{\max}\dot{a}(r + \eta_{\max}t) + \eta_{\max} + |\mu|a(r + \eta_{\max}t)\Big)\mathrm{d}t; \end{aligned} \tag{63}$$

for $(r,x)\in\mathcal{O}_a$ we have

$$
\begin{aligned}
v(r,x)=&-\alpha\big(a(r)-x\big)+\int_0^\infty \mathrm{e}^{-\rho t}\Pi\big(r+\eta_{\max}t,a(r+\eta_{\max}t)\big)\mathrm{d}t\\
&-\alpha\int_0^\infty \mathrm{e}^{-\rho t}\Big(\eta_{\max}\dot a(r+\eta_{\max}t)+\eta_{\max}+|\mu|a(r+\eta_{\max}t)\Big)\mathrm{d}t.
\end{aligned}\tag{64}
$$

Notice that for any $\tau\ge 0$

$$
\begin{aligned}
\int_\tau^\infty \mathrm{e}^{-\rho t}\dot a(r+\eta_{\max}t)\mathrm{d}t&=\frac{1}{\eta_{\max}}\int_\tau^\infty \mathrm{e}^{-\rho t}\mathrm{d}a(r+\eta_{\max}t)\\
&=\frac{1}{\eta_{\max}}\Big(-\mathrm{e}^{-r\tau}a(r+\eta_{\max}\tau)+\rho\int_\tau^\infty \mathrm{e}^{-\rho t}a(r+\eta_{\max}t)\mathrm{d}t\Big).
\end{aligned}
$$

It is now easy to check that $(r,x)\mapsto v(r,x)$ is locally Lipschitz on $[0,\infty)\times\mathbb{R}_+$ using also the explicit formulae for $\nabla\tau_b^{r,x}$ and $\nabla\tau_M^{r,x}$.

Since

$$
v(r,x)=\sup_{\nu\in\mathcal{A}_I}\mathcal{J}^I_{r,x}(\eta^*,\nu),
$$

then by dynamic programming arguments for singular control, justified by continuity of v (cf. [10]), we know that for any $\nu\in\mathcal{A}_I$

$$
t\mapsto \mathrm{e}^{-\rho t}v\big(R_t^{\eta^*},X_t^{\nu,\eta^*}\big)+\int_0^t \mathrm{e}^{-\rho s}\Pi(R_s^{\eta^*},X_s^{\nu,\eta^*})\mathrm{d}s-\alpha\int_{[0,t]}\mathrm{e}^{-\rho s}\mathrm{d}\nu_s \tag{65}
$$

is a nonincreasing function (a deterministic supermartingale) and

$$
t\mapsto \mathrm{e}^{-\rho t}v\big(R_t^{\eta^*},X_t^{\nu^*,\eta^*}\big)+\int_0^t \mathrm{e}^{-\rho s}\Pi(R_s^{\eta^*},X_s^{\nu^*,\eta^*})\mathrm{d}s-\alpha\int_{[0,t]}\mathrm{e}^{-\rho s}\mathrm{d}\nu^*_s \tag{66}
$$

is constant (a deterministic martingale). In particular, from the supermartingale property we deduce, by choosing $\nu_0=\delta>0$ and $t=0$, that $v(r,x)\ge v(r,x+\delta)-\alpha\delta$. Hence,

$$
\partial_x v(r,x)\le\alpha,\quad \text{for a.e. } (r,x)\in[0,\infty)\times\mathbb{R}_+. \tag{67}
$$

Then (i) in Definition 2 holds.

Thanks to the Lipschitz regularity of v we can apply a change of variable formula to (65) with $\nu\equiv 0$ because the dynamics $(R^{\eta^*},X^{0,\eta^*})$ is absolutely continuous in time. That yields

$$
\begin{aligned}
v(r,x)&\ge \mathrm{e}^{-\rho t}v\big(R_t^{\eta^*},X_t^{0,\eta^*}\big)+\int_0^t \mathrm{e}^{-\rho s}\Pi(R_s^{\eta^*},X_s^{0,\eta^*})\mathrm{d}s\\
&=v(r,x)+\int_0^t \mathrm{e}^{-\rho s}\big(\mathcal{L}v+\big(v_r-v_x\big)\eta^*-\rho v+\Pi\big)(R_s^{\eta^*},X_s^{0,\eta^*})\mathrm{d}s.
\end{aligned}
$$

Dividing by t and letting $t\downarrow 0$ we deduce

$$
\mathcal{L}v(r,x)+\big(v_r(r,x)-v_x(r,x)\big)\eta^*(r,x)-\rho v(r,x)+\Pi(r,x)\le 0,\quad a.e.\ (r,x)\in[0,\infty)\times\mathbb{R}_+.
$$

Since $t\mapsto\nu_t^*$ is also absolutely continuous when $(R_0,X_0)=(r,x)$ is such that $x\ge a(r)$ (cf. Remark 6), then an analogous change of variable argument, combined with (66) yields

$$
\mathcal{L}v(r,x)+\big(v_r(r,x)-v_x(r,x)\big)\eta^*(r,x)-\rho v(r,x)+\Pi(r,x)=0, \tag{68}
$$

for a.e. $(r,x)\in[0,\infty)\times\mathbb{R}_+$ such that $x>a(r)$. Combining the results above we obtain that v satisfies (iii) in Definition 2. Actually, upon closer inspection of the formulae in (62) and (63) we notice that v and v_x are continuous separately in the set $\mathcal{O}_{a,b}$ and $\mathcal{O}_b$, with continuous extensions to the boundary of the two domains (we are not claiming continuity of derivatives across the boundaries). Since $\eta^*(r,x)$ is also constant in those two sets, we deduce that v_r is continuous in $\overline{\mathcal{O}}_{a,b}$ and indeed (68) holds in the classical sense at all points (r,x) with $x>a(r)$ and $x\ne b(r)$.

It remains to show that $\mathcal{M} = \{(r,x) : 0 < x \le a(r)\}$. From the explicit formulae for v it is immediate to deduce $v_x(r,x) = \alpha$ for all (r,x) such that $0 < x \le a(r)$. Then we must show that $v_x(r,x) < \alpha$ at all points in $x > a(r)$ where the derivative exists.

Let us fix $(r,x) \in \mathcal{O}_{a,b}$ and let $\widehat{\nu}^* = \widehat{\nu}^{*;r,x}$ be optimal for $v(r,x)$ and given by (23). Then, from (24) we get

$$v(r,x) = \int_0^\infty \mathrm{e}^{-\rho t}\Big(\Pi\big(r + \eta_{\max} t, Y_t[x + \widehat{\nu}^*_t - f_M(t)]\big) - \alpha\delta Y_t\Big)\mathrm{d}t.$$

Notice that in this case $\eta^*_t = \eta_{\max}$ for all $t \ge 0$ because the dynamics $(R^{\eta^*}, X^{\nu^*,\eta^*})$ is bound to evolve in $\overline{\mathcal{O}}_{a,b}$. Moreover, the firm's optimal control remains the same also when $(R_0, X_0) = (r, x-\varepsilon)$ for any small $\varepsilon > 0$, because $X_0 \le b(R_0)$ and $\widehat{\nu}^{*;r,x}$ is admissible but suboptimal for the payoff $\bar{\mathcal{J}}^I_{r,x-\varepsilon}(\nu,\eta^*)$ in (24). We then have

$$v(r,x-\varepsilon) \ge \int_0^\infty \mathrm{e}^{-\rho t}\Big(\Pi\big(r + \eta_{\max} t, Y_t[x - \varepsilon + \widehat{\nu}^*_t - f_M(t)]\big) - \alpha\delta Y_t\Big)\mathrm{d}t.$$

Subtracting the two expressions, yields

$$\begin{aligned}&v(r,x) - v(r,x-\varepsilon)\\ &\le \int_0^\infty \mathrm{e}^{-\rho t}\Big(\Pi\big(r + \eta_{\max} t, Y_t[x + \widehat{\nu}^*_t - f_M(t)]\big) - \Pi\big(r + \eta_{\max} t, Y_t[x - \varepsilon + \widehat{\nu}^*_t - f_M(t)]\big)\Big)\mathrm{d}t.\end{aligned}$$

If (r,x) is a point where $v_x(r,x)$ exists, we can divide by ε and let $\varepsilon \downarrow 0$ to obtain

$$\begin{aligned}v_x(r,x) &\le \int_0^\infty \mathrm{e}^{-\rho t} Y_t \Pi_x\big(r + \eta_{\max} t, Y_t[x + \widehat{\nu}^*_t - f_M(t)]\big)\mathrm{d}t\\ &= \int_0^{\tau^{r,x}_M} \mathrm{e}^{-\rho t} Y_t \Pi_x\big(r + \eta_{\max} t, Y_t[x - f_M(t)]\big)\mathrm{d}t\\ &\quad + \int_{\tau^{r,x}_M}^\infty \mathrm{e}^{-\rho t} Y_t \Pi_x\big(r + \eta_{\max} t, a(r + \eta_{\max} t)\big)\mathrm{d}t.\end{aligned}$$

By definition of $a(r)$ we have, for $t \ge \tau^{r,x}_M$

$$Y_t \Pi_x\big(r + \eta_{\max} t, a(r + \eta_{\max} t)\big) = \alpha\delta Y_t$$

and therefore, using also $\mathrm{e}^{-\rho t} Y_t = \mathrm{e}^{-\delta t}$,

$$\int_{\tau^{r,x}_M}^\infty \mathrm{e}^{-\rho t} Y_t \Pi_x\big(r + \eta_{\max} t, a(r + \eta_{\max} t)\big)\mathrm{d}t = \alpha \int_{\tau^{r,x}_M}^\infty \delta \mathrm{e}^{-\delta t}\mathrm{d}t = \alpha Y_{\tau^{r,x}_M} \mathrm{e}^{-\rho \tau^{r,x}_M}.$$

Then

$$v_x(r,x) \le \alpha Y_{\tau^{r,x}_M} \mathrm{e}^{-\rho \tau^{r,x}_M} + \int_0^{\tau^{r,x}_M} \mathrm{e}^{-\rho t} Y_t \Pi_x\big(r + \eta_{\max} t, Y_t[x - f_M(t)]\big)\mathrm{d}t \triangleq \Gamma(r,x).$$

Since $a(r) < x < b(r)$, for sufficiently small $\varepsilon > 0$ we can repeat analogous arguments to estimate

$$\begin{aligned}&v(r,x+\varepsilon) - v(r,x)\\ &\ge \int_0^\infty \mathrm{e}^{-\rho t}\Big(\Pi\big(r + \eta_{\max} t, Y_t[x + \varepsilon + \widehat{\nu}^*_t - f_M(t)]\big) - \Pi\big(r + \eta_{\max} t, Y_t[x + \widehat{\nu}^*_t - f_M(t)]\big)\Big)\mathrm{d}t,\end{aligned}$$

where, again, $\widehat{\nu}^*_t = \widehat{\nu}^{*;r,x}_t$ is independent of ε. Thus, dividing by ε and passing to the limit we conclude $v_x(r,x) \ge \Gamma(r,x)$. Combining with the previous bound we get $v_x(r,x) = \Gamma(r,x)$ for $a(r) < x < b(r)$. From this representation we immediately deduce continuity of v_x in the set $\mathcal{O}_{a,b}$. Moreover, v_x is extended continuously to $\overline{\mathcal{O}}_{a,b}$ thus lifting the regularity of v_x from $L^\infty(\mathcal{O}_{a,b})$ to $C(\overline{\mathcal{O}}_{a,b})$. It is also clear that $\Gamma(r,a(r)) = \alpha$ because $\tau^{r,a(r)}_M = 0$ and therefore v_x is continuous across the boundary $r \mapsto a(r)$. Next we are going to show that $\Gamma(r,x) < \alpha$ in $\mathcal{O}_{a,b}$.

Taking a derivative in x of Γ we obtain

$$\begin{aligned}\Gamma_x(r,x) &= \Big(\Pi_x\big(r+\eta_{\max}\tau_M^{r,x}, a(r+\eta_{\max}\tau_M^{r,x})\big) - \delta\alpha\Big)\mathrm{e}^{-\rho\tau_M^{r,x}} Y_{\tau_M^{r,x}} \frac{\partial \tau_M^{r,x}}{\partial x} \\ &\quad + \int_0^{\tau_M^{r,x}} \mathrm{e}^{-\rho t}(Y_t)^2 \Pi_{xx}\big(r+\eta_{\max}t, Y_t[x-f_M(t)]\big)\mathrm{d}t \\ &= \int_0^{\tau_M^{r,x}} \mathrm{e}^{-\rho t}(Y_t)^2 \Pi_{xx}\big(r+\eta_{\max}t, Y_t[x-f_M(t)]\big)\mathrm{d}t < 0,\end{aligned}$$

where the second equality holds by definition of the boundary $a(r)$ and the strict inequality is by strict concavity of $\Pi(r,\cdot)$. Then, $\Gamma(r,x) < \alpha$ for all $x > a(r)$, which implies $v_x(r,x) < \alpha$ for $a(r) < x \le b(r)$.

Now we look at $x > b(r)$. It is clear from the form of v in (62) that for each $r \in [0,\infty)$, $v(r,\cdot)$ is twice continuously differentiable for $x > b(r)$. Then, for fixed $r \in [0,\infty)$ the HJB equation reads

$$\mu x v_x(r,x) - \rho v(r,x) + \Pi(r,x) = 0, \quad \text{for all } x > b(r).$$

However, using that $\Pi(r,\cdot)$ is twice continuously differentiable (cf. Assumption 1-(i)) we deduce from the equation above that actually $v(r,\cdot)$ is three times continuously differentiable.

Setting $u \triangleq v_x - \alpha$, we differentiate the equation above with respect to x and obtain

$$\mu x u_x(r,x) - \delta u(r,x) + \big(\Pi_x(r,x) - \alpha\delta\big) = 0, \quad \text{for all } x > b(r).$$

Let us start by noticing that $\Pi_x(r,x) - \alpha\delta < 0$ for $x \ge b(r) > a(r)$ by strict concavity of $\Pi(r,\cdot)$ and the fact that $\Pi_x(r,a(r)) - \alpha\delta = 0$. We also know from (67) that $u(r,x) \le 0$ for $x > b(r)$. Then, by the maximum principle we deduce $u(r,x) < 0$ for $x > b(r)$. This is directly seen by the representation

$$u(r,x) = \mathrm{e}^{-\delta(t\wedge\tau_b^{r,x})} u(r, xY_{t\wedge\tau_b^{r,x}}) + \int_0^{t\wedge\tau_b^{r,x}} \mathrm{e}^{-\delta s}\big(\Pi_x(r,xY_s) - \alpha\delta\big)\mathrm{d}s,$$

where we recall $\tau_b^{r,x} = \inf\{s \ge 0 : xY_s \le b(r)\}$.

Although $v_x(r,\cdot)$ can be extended continuously to $b(r)$ from above and from below, we are unable to establish the relationship between $v_x(r,b(r)-)$ and $v_x(r,b(r)+)$. In particular, it may occur that $v_x(r,\cdot)$ does not exists at $b(r)$. However, if $v_x(r,b(r))$ exists, then it must be strictly smaller than α because $v_x(r,b(r)-) < \alpha$. Hence, $(r,b(r)) \notin \mathcal{M}$. Otherwise $v_x(r,b(r))$ does not exists and $(r,b(r)) \notin \mathcal{M}$. So in all cases $(r,b(r)) \notin \mathcal{M}$. Then we have proven that $\mathcal{M} = \{(r,x) : 0 < x \le a(r)\}$ as claimed, which also implies $\partial\mathcal{M} = \{(r,x) : x = a(r)\}$.

Finally, the set $\{(r,x) : x = b(r)\} \subset \mathcal{I}$ is of zero measure and it can be neglected in the variational inequality for v. □

Combining the above proposition with the fact that w satisfies (ii) in Definition 2 we deduce the following corollary (cf. the proof of Proposition 5 and notice that the definition of $\mathcal{M}$ in that proof is given by (46), which turns out to agree with the result in Proposition 6).

Corollary 4. *The pair of equilibrium payoffs (v,w) is a strong solution of the HJB system.*

5. An algorithm for the construction of an equilibrium in the general case

In view of Theorem 1, finding an equilibrium in our model boils down to finding a solution of (13)–(14). In full generality we are not able to obtain an analytical solution to the problem. Therefore, we proceed by developing a numerical method that combines finite differences for both (13) and (14) with a penalization method that reduces the nonlinear problem in (14) to an easier semilinear one.

The penalization method follows a well-trodden path in PDE theory (cf., e.g., [31, Chapter 9]) which approximates (14) by relaxing the hard constraint $v_x \le \alpha$ into a soft constraint. More precisely, given (small) $\epsilon > 0$, we want to find v^ϵ that satisfies

$$\mathcal{G}^\rho[v^\epsilon, \eta^*](r,x) = -\Pi(r,x) - \chi^\epsilon[v^\epsilon](r,x), \tag{69}$$

with $\mathcal{G}^\rho[\cdot,\cdot](r,x)$ defined in (15), and

$$\chi^\epsilon[v^\epsilon](r,x) \triangleq \frac{1}{\epsilon}(v^\epsilon_x(r,x) - \alpha)^+. \tag{70}$$

Under suitable assumptions, it is often possible to show that as $\epsilon \to 0$ the solution v^ϵ of the penalized problem converges to a solution of the original problem (14). In our case, the proof appears very complicated due to the (expected) low regularity of the function η^* and, more in general, due to the coupling between (69) and (13). However, we observe such convergence numerically.

Since the simultaneous solution of (13) and (14) (or (69)) requires knowledge of the function η^* and of the sets $\mathcal{M}$ and $\mathcal{I}$, we need to argue in a sort of iterative way (with the number of iterations denoted by ℓ). We initialize our algorithm by taking $\eta^* \equiv 0$ in (14), and $\ell = 0$. It is shown below that the resulting variational inequality admits an explicit solution, which we denote by $\hat{v}(r,x)$. The sets $\hat{\mathcal{M}} \triangleq \{\hat{v}_x = \alpha\}$ and $\hat{\mathcal{I}} \triangleq (\hat{\mathcal{M}})^c$ can be calculated explicitly (cf. (75)) with $\hat{\mathcal{I}} = \{(r,x) : x > \hat{a}(r)\}$ and the function $r \mapsto \hat{a}(r)$ is found in (77). In this iteration, the boundary of $\hat{\mathcal{M}}$ is given by $\partial\hat{\mathcal{M}} = \{(r,\hat{a}(r)), r \ge 0\}$, and it can be used to solve the zero-order iteration of the problem for the firm.

The next step is to calculate the solution $w^{(\ell)} = w^{(0)}$ of (13) with $\hat{\mathcal{M}}$ and $\hat{\mathcal{I}}$ instead of $\mathcal{M}$ and $\mathcal{I}$. This is done by finite-difference scheme as detailed in (79) of Section 5.2. Once we have obtained the function $w^{(0)}$ we can define the function $\eta^{*(\ell)} = \eta^{*(0)}$ as a *proxy* for the firm's optimal control:

$$\eta^{*(0)}(r,x) = \eta_{\max} 1_{\{\partial_r w^{(0)} > \partial_x w^{(0)}\}}(r,x). \tag{71}$$

That concludes the initialization of the algorithm.

Next, for the first iteration of our scheme, we set $\ell = 1$ and we approximate (14) by (69). Then, we want to find $v^{\epsilon(\ell)} = v^{\epsilon(1)}$ that satisfies (69) in the form

$$\mathcal{G}^\rho[v^{\epsilon(\ell)}, \eta^{*(\ell-1)}](r,x) = -\Pi(r,x) - \chi^\epsilon[v^{\epsilon(\ell)}](r,x), \tag{72}$$

for $(r,x) \in [0,\infty) \times (0,\infty)$ with boundary conditions $v^{\epsilon(\ell)}(0,x) = 0$ and $v^{\epsilon(\ell)}(r,0) = 0$. The boundary conditions are motivated by the form of the investor's equilibrium payoff in Section 4. The solution of (72) is obtained again by finite differences as described in Section 5.2. Once we have obtained a solution $v^{\epsilon(\ell)}$ of (72) we can determine numerically the sets $\mathcal{M}^{(\ell)}_\epsilon \triangleq \{v^{\epsilon(\ell)}_x = \alpha\}$ and $\mathcal{I}^{(\ell)}_\epsilon \triangleq ([0,\infty) \times \mathbb{R}_+) \setminus \mathcal{M}^{(\ell)}_\epsilon$. It turns out that

$$\mathcal{I}^{(\ell)}_\epsilon = \big\{(r,x) : x > a^{\epsilon(\ell)}(r)\big\},$$

where $r \mapsto a^{\epsilon(\ell)}(r)$ is a continuous function on $[0,\infty)$. In order to conclude the first iteration we calculate a solution $w^{(\ell)} = w^{(1)}$ of (13) with $\mathcal{M}^{(\ell)}_\epsilon$, $\mathcal{I}^{(\ell)}_\epsilon$ instead of $\mathcal{M}$, $\mathcal{I}$. That also yields a new proxy for the firm's optimal control:

$$\eta^{*(\ell)}(r,x) = \eta_{\max} 1_{\{\partial_r w^{(\ell)} > \partial_x w^{(\ell)}\}}(r,x). \tag{73}$$

We must notice that also $w^{(\ell)}$ and $\eta^{*(\ell)}$ depend on ϵ via the sets $\mathcal{M}^{(\ell)}_\epsilon$ and $\mathcal{I}^{(\ell)}_\epsilon$. However, we suppress such dependence in our notation for ease of exposition.

The procedure continues as follows: Given $w^{(\ell)}$, $\eta^{*(\ell)}$, $v^{\epsilon(\ell)}$, $\mathcal{M}^{(\ell)}_\epsilon$, $\mathcal{I}^{(\ell)}_\epsilon$ we find $v^{\epsilon(\ell+1)}$ by solving (72) and then we determine the sets $\mathcal{M}^{(\ell+1)}_\epsilon$, $\mathcal{I}^{(\ell+1)}_\epsilon$ with boundary $r \mapsto a^{\epsilon(\ell+1)}(r)$; subsequently we find $w^{(\ell+1)}$ by solving (13) with $\mathcal{M}^{(\ell+1)}_\epsilon$, $\mathcal{I}^{(\ell+1)}_\epsilon$ instead of $\mathcal{M}$, $\mathcal{I}$ and we obtain $\eta^{*(\ell+1)}$ as in (73). This iteration continues until a stopping criteria prescribed in Algorithm 1 (step 9, Section 5.2) is reached.

Remark 8. *The regularization parameter ϵ plays a crucial role in finding an approximation for the solution of* (14)*, enabling us to make the problem more amenable to numerical techniques. As $\ell \to \infty$ we observe numerically that $w^{(\ell)}$, $\eta^{*(\ell)}$, $v^{\epsilon(\ell)}$, $a^{\epsilon(\ell)}$ converge to limits that we denote w^ϵ, $\eta^{*\epsilon}$, v^ϵ, a^ϵ. Then, letting ϵ go to zero, we also observe numerically that the functions w^ϵ, $\eta^{*\epsilon}$, v^ϵ, a^ϵ have a well-defined limit, which we denote w, η^*, v, a. In practice, in our numerical implementation, we fix a small ϵ and take the resulting solutions of the iterative procedure described above as our proxy for the true solution of the system* (13)–(14).

5.1. **The investor's problem in isolation.** From now on we work under the assumption:

Assumption 2. *The profit functions Π and π are given by* (18).

In order to initialize our algorithm we need to start by considering an investor who acts in isolation, i.e., with no emission reduction ever performed by the firm. The investor's expected payoff then reads

$$\mathcal{J}^I_{r,x}(\nu) \triangleq \mathbb{E}_{r,x}\left[\int_0^\infty \mathrm{e}^{-\rho t}\Pi(r, X_t^{\nu,0})\mathrm{d}t - \alpha \int_0^\infty \mathrm{e}^{-\rho t}\mathrm{d}\nu_t\right],$$

and the corresponding value function reads

$$\hat{v}(r,x) = \sup_{\nu \in \mathcal{A}^I} \mathcal{J}^I_{r,x}(\nu).$$

Setting $\hat{\mathcal{M}} = \{\hat{v}_x = \alpha\}$, the analytical expression of $\hat{v}$ can be determined by the direct solution of

$$\begin{cases} (\mathcal{L}\hat{v} - \rho\hat{v})(r,x) = -\Pi(r,x), & (r,x) \in \hat{\mathcal{I}} = (\hat{\mathcal{M}})^c, \\ \hat{v}_x(r,x) = \alpha, & (r,x) \in \partial\hat{\mathcal{M}}, \\ \hat{v}_{xx}(r,x) = 0, & (r,x) \in \partial\hat{\mathcal{M}}, \end{cases} \tag{74}$$

with $|\hat{v}(r,x)| \leq c(r)(1+x)$ for some $c(r) > 0$ and using the ansatz $\partial\hat{\mathcal{M}} = \{(r, \hat{a}(r)), r \geq 0\}$, for some $\hat{a}(r)$ to be determined by imposing the third condition in the system above.

Under Assumption 2, lengthy by straightforward calculations yield:

$$\hat{v}(r,x) = B(r)x^{-m} + \lambda x^\beta r^\gamma, \tag{75}$$

with constants

$$\begin{aligned} \lambda &= \frac{1}{\sigma^2/2(m+\beta)(n-\beta)}, \\ m &= \frac{\mu - \sigma^2/2 + \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2\rho}}{\sigma^2}, \\ n &= \frac{-(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\sigma^2\rho}}{\sigma^2}, \end{aligned} \tag{76}$$

and where

$$B(r) = \frac{\kappa\lambda(1-\beta)\beta}{m(m+1)} r^{\frac{\gamma(m+1)}{1-\beta}}$$

with

$$\kappa \triangleq \left(\frac{\lambda\beta}{\alpha}\left(\frac{m-\beta+2}{m+1}\right)\right)^{\frac{\beta+m}{1-\beta}}.$$

Additionally,

$$\hat{a}(r) = \kappa^{\frac{1}{\beta+m}} r^{\frac{\gamma}{1-\beta}}. \tag{77}$$

The optimal investment in this setting is given by

$$\hat{\nu}_t = \int_0^t X_s^0 \mathrm{d}\hat{\lambda}_s, \tag{78}$$

where $\hat\lambda_t = \sup_{0\le s\le t}\Big(\hat a(r)/X^0_s - x\Big)^+$, $t \ge 0$.

Remark 9. *It is worth noticing that if $r = 0$, then the investor never invests in this setup. That provides us with a boundary condition for the firm's value function. Indeed, for $\nu \equiv 0$ we have $w(0, x) = x/(\bar\rho - \mu)$, whenever $\bar\rho - \mu > 0$.*

Remark 10. *When $x \uparrow \infty$ the firm is not going to mitigate its emissions because $\lim_{x\to\infty} \mathcal{J}^F_{r,x}(\eta, \nu) = \infty$ for any pair $(\eta, \nu) \in \mathcal{A}_F \times \mathcal{A}_I$ and all $r \in [0, \infty)$. For $\eta \equiv 0$ the investor is again faced with a problem with value $\hat v$ and optimal boundary $\hat a$. Based on this heuristics we postulate that for large values of x the firm's payoff should be given by $\mathcal{J}^F_{r,x}(0, \hat\nu)$, where $\hat\nu$ is given in (78), whereas the investor's payoff is again $\hat v$. Analogous calculations to the ones above yield*

$$\mathcal{J}^F_{r,x}(0, \hat\nu) = C(r)x^{-m} + \lambda x$$

with m and λ as in (76) with $\beta = 1$, and

$$C(r) \triangleq \frac{\lambda}{m}\hat a(r)^{m+1},$$

where $\hat a$ is given in (77).

5.2. **A numerical scheme .** Our approach to solve the problem described in Sections 2–5 will rely on the algorithm explained below. We will employ a finite-difference scheme to solve both (13) and (69). More precisely, we adopt the first-order backward difference for first-order derivatives with respect to r, followed by fourth-order central discretizations for the first and second-order derivatives with respect to x (cf., e.g., [25, Chapter 2]).

Given a sufficiently smooth function φ, let $\varphi_{i,j} = \varphi(r_i, x_j)$ at points on uniform grid partitions $\{r_0, \dots, r_M\}$ and $\{x_0, \dots, x_N\}$ of $[0, \infty)$ with $x_0 = r_0 = 0$ and large but fixed r_N and x_N. We approximate first and second order derivatives as

$$\begin{aligned}
&\varphi_r(r_i, x_j) \approx \frac{\varphi_{i,j} - \varphi_{i-1,j}}{\Delta_r},\\
&\varphi_x(r_i, x_j) \approx \frac{\varphi_{i,j-2} - 8\varphi_{i,j-1} + 8\varphi_{i,j+1} - \varphi_{i,j+2}}{12\Delta_x},\\
&\varphi_{xx}(r_i, x_j) \approx \frac{-\varphi_{i,j-2} + 16\varphi_{i,j-1} - 30\varphi_{i,j} + 16\varphi_{i,j+1} - \varphi_{i,j+2}}{12\Delta_x^2},
\end{aligned} \tag{79}$$

for $i \in \{1, \dots, M\}$ and $j \in \{2, \dots, N-2\}$, with $\Delta_r = r_i - r_{i-1}$ and $\Delta_x = x_j - x_{j-1}$, for any pair (i, j). We assume the following initial conditions:

$$\begin{cases} v^\epsilon(r_0, x_j) = 0,\\ v^\epsilon(r_i, x_0) = 0,\\ w(r_0, x_j) = (\bar\rho - \mu)^{-1}x_j,\\ w(r_i, x_0) = 0, \end{cases} \tag{80}$$

for every $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$. The conditions at $r_0 = 0$ are in keeping with Remark 9. For the condition at $x_0 = 0$ we should notice that the geometric Brownian motion cannot start from zero and we intuitively assign zero value to a firm with zero profitability. However, this condition is somewhat superfluous because, already starting from the first iteration of our algorithm, the controlled dynamics for $X^{\nu,\eta}$ is not allowed to visit $x = 0$.

Although in principle our problem is set on $[0, \infty)^2$, in practice we must select (large) maximum elements r_N and x_N of our state space in order to compute the solution. However, this requires us to specify at least one more boundary condition for the PDEs at either points (r_N, x_j) or (r_i, x_N) for (i, j). We choose to specify the values of $w(r_i, x_N)$ and $v^\epsilon(r_i, x_N)$, for which we have natural candidates, thanks to Remark 10. Indeed, we assume

$$\begin{cases} v^{\epsilon}(r_i, x_N) = \hat{v}(r_i, x_N), \\ w(r_i, x_N) = C(r_i) x_N^{-m} + \lambda x_N, \end{cases} \tag{81}$$

for every $i \in \{1, \ldots, M\}$.

Next, Algorithm 1 describes our strategy to derive optimal numerical solutions for $w(r,x)$, $v^{\epsilon}(r,x)$, $\eta^*(r,x)$, and $a^{\epsilon}(r)$. All PDEs in the algorithm are solved using finite-difference scheme with the approximation of derivatives as described above.

Algorithm 1. *Given $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}_+$, $\rho, \bar{\rho}, \alpha, \epsilon > 0$, $\beta, \gamma \in (0,1)$, a threshold $\eta_{\max} > 0$ and desired precision levels ϖ and ϖ', consider the following steps:*

(1) *Compute $\hat{v}(r,x)$ and $\hat{a}(r)$ via (75) and (77), respectively. Store $a^{\epsilon(0)}(r) \leftarrow \hat{a}(r)$, $v^{\epsilon(0)}(r,x) \leftarrow \hat{v}(r,x)$ and compute $\chi^{\epsilon}[v^{\epsilon(0)}](r,x)$ as in (70), for all (r,x).*
(2) *Solve numerically (13) with $\hat{\mathcal{M}}$ and $\hat{\mathcal{I}}$ instead of $\mathcal{M}$ and $\mathcal{I}$ and with $\eta^* \equiv 0$. Denote the solution $w^{(0)}(r,x)$ and then compute $\eta^{*(0)}(r,x)$ via (71), for all (r,x).*
(3) *Increase $\ell \leftarrow \ell + 1$.*
(4) *From $\eta^{*(\ell-1)}$ solve (69) and find $v^{\epsilon(\ell)}(r,x)$, for all (r,x) as follows: for $k \geq 1$, solve*

$$\mathcal{G}^{\rho}[v_k^{\epsilon(\ell)}, \eta^{*(\ell-1)}](r,x) = -\Pi(r,x) - \chi^{\epsilon}[v_{k-1}^{\epsilon(\ell)}](r,x),$$

with $v_0^{\epsilon(\ell)} = v^{\epsilon(\ell-1)}$. Iterate until $\|v_k^{\epsilon(\ell)} - v_{k-1}^{\epsilon(\ell)}\| \leq \varpi$.
(5) *Store $v^{\epsilon(\ell)}(r,x) \leftarrow v_k^{\epsilon(\ell)}(r,x)$.*
(6) *Compute $a^{\epsilon(\ell)}(r)$ as*

$$a^{\epsilon(\ell)}(r) = \partial \left\{ (r,x) : v_x^{\epsilon(\ell)}(r,x) = \alpha \right\}. \tag{82}$$

(7) *From $a^{\epsilon(\ell)}(r)$, solve (13) and find $w^{(\ell)}(r,x)$, for all (r,x), by considering the following new iteration: for fixed ℓ and each $k \geq 1$, solve*

$$\begin{cases} (\mathcal{L} w_k^{(\ell)} - \bar{\rho} w_k^{(\ell)})(r,x) + \mathcal{P}(w_k^{(\ell)}, \eta_{k-1}^{(\ell)})(r,x) = -\pi(x), & (r,x) : x > a^{\epsilon(\ell)}(r), \\ \partial_x w_k^{(\ell)}(r,x) = 0, & (r,x) : x = a^{\epsilon(\ell)}(r), \\ |w_k^{(\ell)}(r,x)| \leq c(1+x), & (r,x) \in [0,\infty) \times \mathbb{R}_+, \end{cases}$$

where $\eta_0^{(\ell)} = 0$, $\mathcal{P}(\varphi, \eta) \triangleq \big(\varphi_r - \varphi_x\big)\eta$, and $\eta_k^{(\ell)}(r,x) = \eta_{\max} 1_{\{\partial_r w_k^{(\ell)} > \partial_x w_k^{(\ell)}\}}(r,x)$, until $\|w_k^{(\ell)}(r,x) - w_{k-1}^{(\ell)}(r,x)\| \leq \varpi$.
(8) *Store $w^{(\ell)}(r,x) \leftarrow w_k^{(\ell)}(r,x)$ and compute $\eta_k^{*(\ell)}(r,x)$ via (73).*
(9) *Check: If*

$$\max(\|w^{(\ell)}(r,x) - w^{(\ell-1)}(r,x)\|, \|v^{\epsilon(\ell)}(r,x) - v^{\epsilon(\ell-1)}(r,x)\|) > \varpi',$$

go back to step 3.
Otherwise, proceed to the next step.
(10) *Return $w(r,x) \equiv w^{(\ell)}(r,x)$, $v^{\epsilon}(r,x) \equiv v^{\epsilon(\ell)}(r,x)$, $\eta^*(r,x) \equiv \eta^{*(\ell)}(r,x)$ and $a(r) \equiv a^{\epsilon(\ell)}(r)$, for all (r,x).*

In summary, Algorithm 1 has been designed to approximate optimized solutions for both equilibrium payoffs $w(r,x)$ and $v^{\epsilon}(r,x)$, along with the firm's optimal strategy η^* and the boundary function $a^{\epsilon}(r)$ that triggers investor's actions. The algorithm initiates with the assumption that the firm takes no initial action to reduce pollution ($\eta^* = 0$). It then proceeds by calculating the investor's response given by $\hat{a}(r)$ and $\hat{v}(r,x)$ (step 1), together with $w^{(0)}(r,x)$ and $\eta^{*(0)}$ via step 2. Then, for each $\ell \geq 1$, step 4 computes $v^{\epsilon(\ell)}(r,x)$ via a sub loop with iterations $k \geq 1$; step 6 constructs $a^{\epsilon(\ell)}(r)$ and step 7 obtains $w^{(\ell)}(r,x)$ by implementing a sub loop with iterations $k \geq 0$. It's important to note that the k-dependent sub-loops contributing to the construction of both $w(r,x)$ and $v^{\epsilon}(r,x)$ are run independently. The iterative ℓ-dependent loop continues until the predefined stopping criterion at step 9 is achieved, refining our variables of interest, and ultimately converging towards the optimal equilibrium solutions of our problem.

Remark 11. *Algorithm 1 draws inspiration from Howard's algorithm (or policy iteration), widely used in dynamic programming and optimization. Seminal works on this methodology are attributed to Bellman and can be found in [4, 5]. Howard extended Bellman's approach to stationary infinite-horizon Markovian dynamic programming problems in [21]. Howard's algorithm is celebrated for its effectiveness in solving sequential decision-making problems and has been widely applied in diverse fields such as economics, engineering, and finance. Our approach incorporates the core principles of Howard's algorithm while tailoring them to the specific requirements and features of our problem.*

6. Numerical Results

In this section, we perform a detailed numerical analysis of the equilibria discussed in the previous sections **under Assumption 2**. We first look at the form of the optimal strategies and of the equilibrium payoffs in the deterministic setting from Section 4, i.e., $\sigma = 0$, with decreasing revenues $\mu \leq 0$. Then we will implement the algorithm described in Section 5 in order to derive equilibrium payoffs and optimal strategies in the full stochastic problem. Numerical results are obtained with MATLAB (R2022b). Section 6.1 addresses the deterministic problem and Section 6.2 the stochastic one. In all the numerical examples, the cost of investment in (8) is set to $\alpha = 1$ and the firm's maximum investment rate is set to $\eta_{\max} = 1$. For the solution of (72) we set the regularization parameter to $\epsilon = 10^{-4}$.

Unless otherwise specified, in the fully stochastic case the values of μ and σ are borrowed from [32] and they are equal to 0.0741 and 0.3703, respectively. In [32] the authors study the profit dynamics of an Australian company in the Metals and Mining sector. Finally, we set the precision levels required for the numerical algorithm from Section 5 to $\varpi = \varpi' = 10^{-3}$.

6.1. Deterministic setting.

In Section 4.2, we have presented the solution for the deterministic setting with $\mu < 0$. This solution includes an explicit formula for the boundary $a(r)$ given by (23), and $b(r)$ specified in Proposition 4. Note that the construction of $b(r)$ depends on the solution of two coupled non-linear equations: $h(r, b(r)) = 0$ and (43), where $h(\cdot,\cdot)$ is described by (55).

In order to construct numerical solutions of $b(r)$, we have implemented the implicit Euler method with Newton-Raphson method as follows. Consider a finite partition $\{r_0, \dots, r_M\}$ of $r \in [0, \infty)$. From (57) we can see that

$$b_{i+1} = b_i + \Delta_r \cdot g(r_{i+1}, b_{i+1}), \tag{83}$$

for $i \in \{0, 1, \dots, M-1\}$, where $\Delta_r \triangleq r_{i+1} - r_i$ and the function $g(\cdot)$ is the right-hand side of (57) with explicit expression obtained using the formulae for $\partial_r h$ and $\partial_x h$ from the proof of Lemma 3. Notice that derivatives of τ_M appearing in $\partial_r h$ and $\partial_x h$ are explicit thanks to (44) and (45), whereas τ_M is calculated from (34).

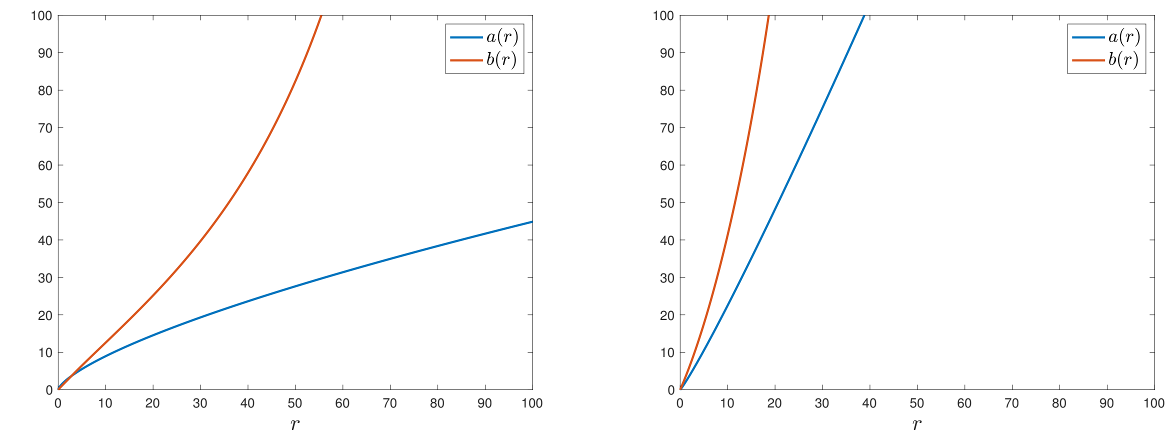
By fixing an $i \in \{0, 1, \dots, M-1\}$, let $\tilde{b} = b_{i+1}$ in (83). We want to find $\tilde{b}$ that solves $\tilde{b} - b_i - \Delta_r \cdot g(r_{i+1}, \tilde{b}) = 0$, which is equivalent to finding the zero of a function

$$s(\tilde{b}) \triangleq \tilde{b} - b_i - \Delta_r \cdot g(r_{i+1}, \tilde{b}). \tag{84}$$

To solve (84), it turns out that the Newton's iteration is given by (see [16, Chapter 8])

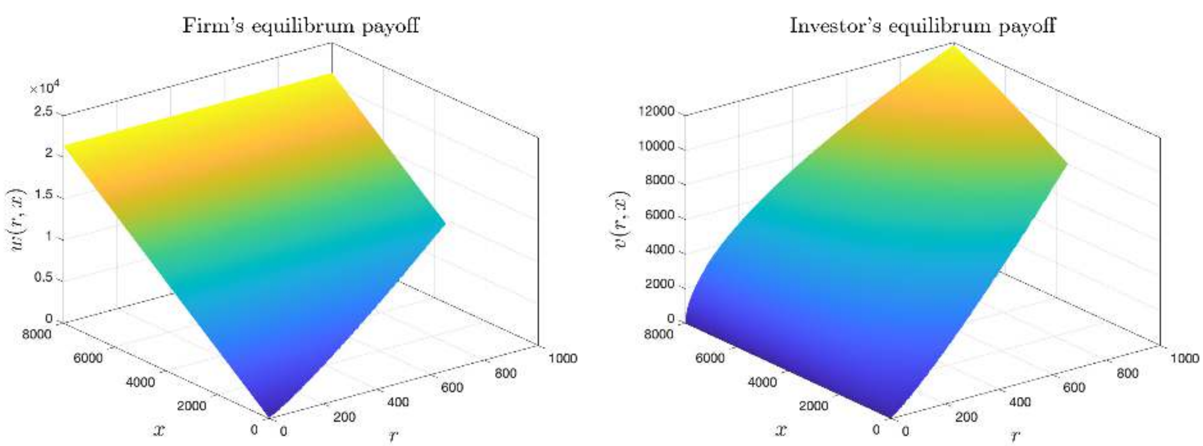
$$\tilde{b}_{k+1} = \tilde{b}_k - \frac{s(\tilde{b}_k)}{\dot{s}(\tilde{b}_k)}, \quad \text{with} \quad \dot{s}(\tilde{b}_k) = 1 - \Delta_r \cdot \frac{\partial}{\partial b} g(r_{i+1}, \tilde{b}_k). \tag{85}$$

We iterate (85) until $\|\tilde{b}_{k+1} - \tilde{b}_k\| < \tilde{\varpi}$, where $\tilde{\varpi} = 10^{-3}$ is a prescribed precision level. Hence, b_{i+1} is set to be equal to the resulting $\tilde{b}$. This procedure is repeated for all $i \in \{0, 1, \dots, M-1\}$, obtaining an approximation of $b(r)$ on $\{r_0, \dots, r_M\}$. In each k-subloop described in (85), the values of $\tau_M(r_{i+1}, \tilde{b}_k)$ for every fixed $i \in \{0, 1, \dots, M-1\}$ are obtained by solving (43) with the MATLAB function FZERO.



(a) (b)

FIGURE 2. Comparison between functions $a(r)$ and $b(r)$ evaluated for $\rho = \bar{\rho} = 0.3$, $\mu = -0.0741$, $\beta = 0.5$, $\gamma = 0.35$ (Figure 2(a)) and $\gamma = 0.55$ (Figure 2(b)). The function $a(r)$ is given by (23), while the boundary $b(r)$ is described in Proposition 4.



(a) (b)

FIGURE 3. Firm and Investor equilibrium expected payoff functions for the deterministic setting discussed in Section 4.2, with $\mu = -0.0741$, $\rho = \bar{\rho} = 0.3$, $\beta = 0.5$ and $\gamma = 0.55$. Figure 3(a) shows $w(r, x)$ and Figure 3(b) shows $v(r, x)$, for $x > a(r)$.

Figure 2 presents numerical simulations for the functions $a(r)$ and $b(r)$. The equilibrium payoffs for both firm and investor are illustrated in Figure 3. The investor's utility function's parameters are $\beta = 0.5$ and $\gamma = 0.55$. Notably, solving the HJB equations is not necessary in this deterministic setting because we have derived explicit solutions to the proposed optimal control problem. Consequently, after obtaining the functions $a(r)$ and $b(r)$ using the parameters mentioned above, we can directly compute the value functions for both players, which are given by: $w(r,x)$ as in (59) and $v(r,x)$ as in (62), (63) and (64), with equilibrium pair (η^*,ν^*) as in (61).

6.2. **Stochastic setting.** An important step to obtain numerically w and v^ϵ is the construction of the boundary a^ϵ. It turns out that the shape of a^ϵ is qualitatively similar to the one of the initial condition $a^{\epsilon(0)} \equiv \hat{a}$ (i.e., the solution to the investor's problem in isolation discussed in Section 5.1). It is clear by its explicit expression (77) that $\hat{a}$ is convex if $\gamma > 1-\beta$ and concave if $\gamma < 1-\beta$. Plots of $\hat{a}$ are provided in Figure 4 for parameter values: $\rho = \bar{\rho} = 0.3$, $\mu = 0.0741$, $\sigma = 0.3703$, $\beta = 0.55$ and $\gamma = \{0.1, 0.25, 0.3, 0.45, 0.5, 0.6, 0.75, 0.9\}$.

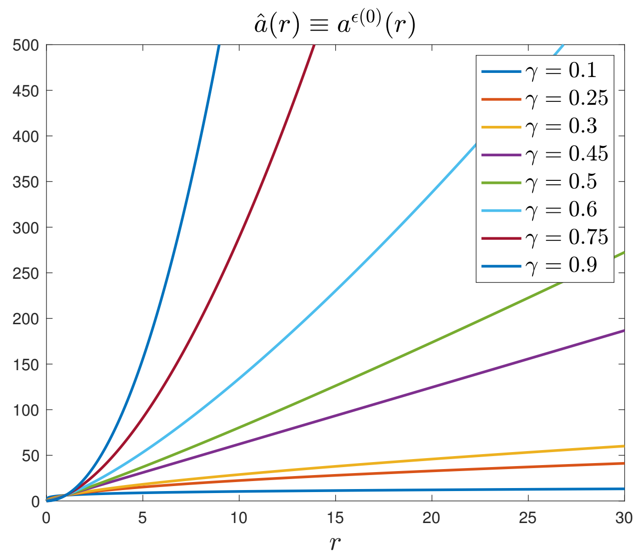
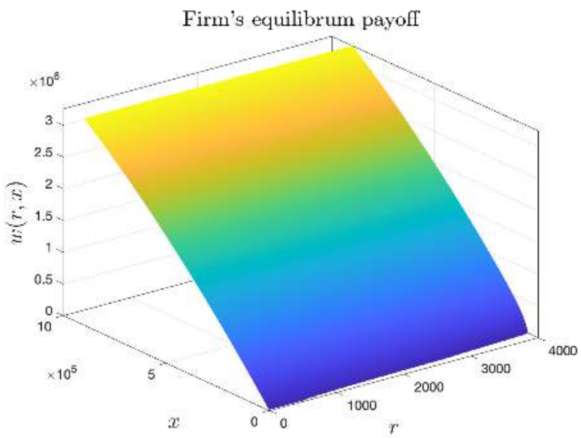


FIGURE 4. Function $\hat{a}(r)$, as in (77), evaluated for $\rho = \bar{\rho} = 0.3$, $\mu = 0.0741$, $\sigma = 0.3703$, $\beta = 0.55$, and $\gamma = \{0.1, 0.25, 0.3, 0.45, 0.6, 0.75, 0.9\}$.

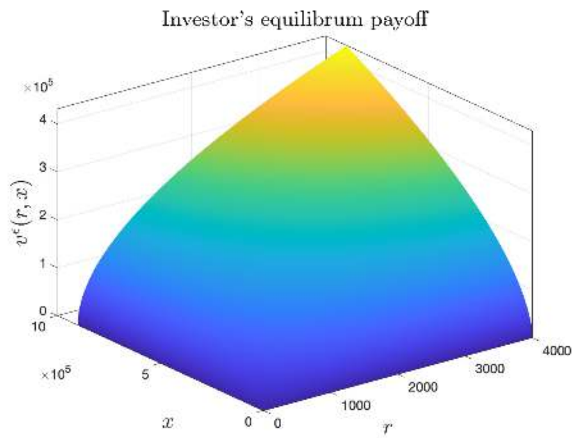
Figure 5 displays the equilibrium payoffs w (Figure 5(a)) and v^ϵ (Figure 5(b)) obtained by solving (13) and (69). The performance of Algorithm 1 is illustrated by Figures 5(c) and 5(d). We observe that the overall algorithm converges after just 4 iterations. In the top panel of Figure 5(c), we show the error at the end of each sub-loop in the construction of $w(r,x)$ and $v(r,x)$. As expected we are always within the precision bound ϖ. In the bottom panel of Figure 5(c) we show the number of iterations required to construct the equilibrium payoff for both the firm and the investor (within the desired precision level ϖ). Finally, Figure 5(d) shows the algorithm's overall convergence, with the final error achieved at $\max(\|w^{(\ell)}(r,x) - w^{(\ell-1)}(r,x)\|, \|v^{\epsilon(\ell)}(r,x) - v^{\epsilon(\ell-1)}(r,x)\|) = 6.2523 \times 10^{-4}$.

In Figure 6 we illustrate the geometry of the regions in the state space where the firm and the investor act. More precisely, the white region corresponds to $\eta^* = 0$, while the green region represents $\eta^* = \eta_{\max}$ (cf. (73)). A boundary $r \mapsto b^\epsilon(r)$ of the set $\{(r,x) : \partial_r w(r,x) > \partial_x w(r,x)\}$, separates the firm's action region from the inaction one. The figure also displays the investor's optimal boundary $r \mapsto a^\epsilon(r)$ derived from (82). Taken together, functions a^ϵ and b^ϵ summarize the optimal strategies for both the firm and the investor. The firm mitigates emissions when $X_t^{\nu^*,\eta^*} \le b^\epsilon(R_t^{\eta^*})$, while the investor provides capital when $X_t^{\nu^*,\eta^*} \le a^\epsilon(R_t^{\eta^*})$. It is worth noticing that the shapes of both $r \mapsto b^\epsilon(r)$ and $r \mapsto a^\epsilon(r)$ are qualitatively similar to the shapes of the optimal boundaries $r \mapsto b(r)$ and $r \mapsto a(r)$ that we obtained in the deterministic setup of Section 4. Therefore, the form of the equilibrium we constructed theoretically in the deterministic framework coveys the same economic message as the one obtained numerically in the fully stochastic framework.

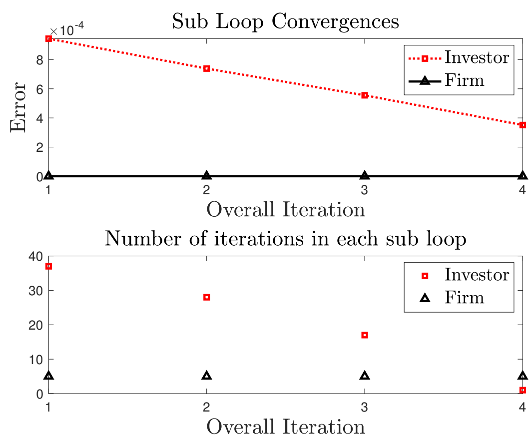
Figures 6(a)–6(c) also illustrate the sensitivity of $a^\epsilon(r)$ and $b^\epsilon(r)$ when we fix $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$, $\gamma = 0.5$, and perturb the values of μ and σ as shown in Table 1 below. We observe



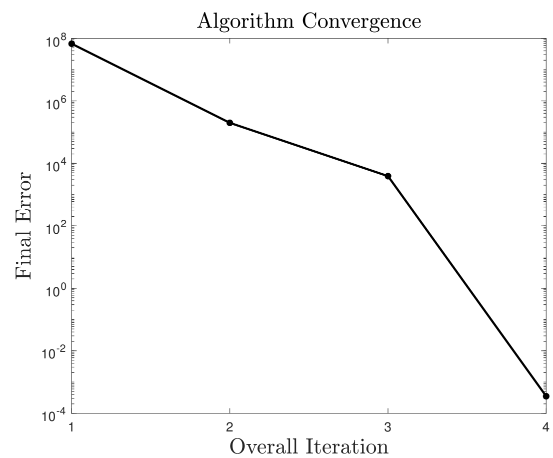
(a) Solution of (13) with $\pi(x) = x$.



(b) Solution of (69) with $\Pi(r,x) = x^{0.55}r^{0.5}$.



(c)



(d)

FIGURE 5. Firm and Investor equilibrium expected payoff functions obtained via Algorithm 1, where $\mu = 0.0741$, $\sigma = 0.3703$, $\rho = \bar{\rho} = 0.3$ and $\eta_{\max} = 1$. In Figure 5(a), $w(r,x)$ has been determined by solving (13) with $\pi(x) = x$. Figure 5(b) displays the final $v^\epsilon(r,x)$ by solving (69) with $\Pi(r,x) = x^{0.55}r^{0.5}$, $\alpha = 1$ and $\epsilon = 10^{-4}$. Figure 5(c) shows the performance of Algorithm 1 in each global iteration. The plots show the convergences and the number of iterations in each sub-loop for both the firm and the investor. The overall convergence of the algorithm can be seen in Figure 5(d).

that the investment boundary is higher for higher values of μ and for lower values of σ: since the investor is profit-seeking and risk-averse, they are more inclined to invest into a more profitable and less risky company. The company is risk-neutral, so when the volatility σ is lower but the drift μ is the same (Figure 6(c)), the company's mitigation actions are mostly determined by those of the investors: since the investment boundary is higher, the company is able to attract the same level of investment with less mitigation. When the volatility is the same but the drift is lower (Figure 6(b)), there are two competing effects: on the one hand, lower investment boundary motivates the company to mitigate more, on the other hand, lower profitability motivates it to mitigate less because the returns from mitigation are lower. Overall, we see that the mitigation boundary is lower but not by so much as in Figure 6(c). Additionally, Figure 6(d) shows the values of $\eta^*(r,x)$

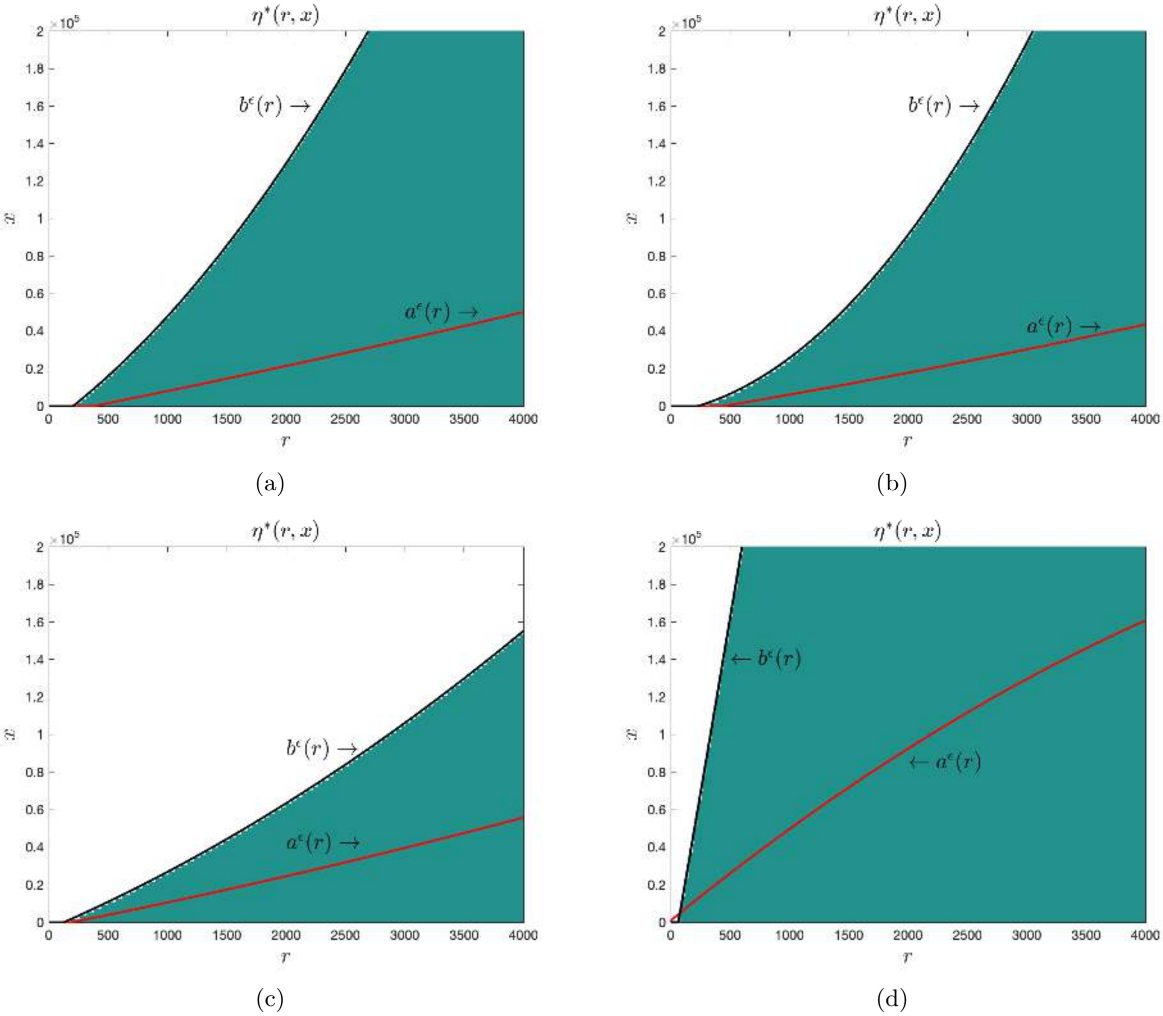


FIGURE 6. The curve $r \mapsto b^\epsilon(r)$ is the boundary of the firm's action set. The function $r \mapsto a^\epsilon(r)$ is the investor's optimal investment threshold. The white region is the inaction region for both players. In the green region the firm implements emission abatement at the maximum rate. Optimal investment keeps the firm's production capacity above a^ϵ. **Parameter values**. In Figure 6(a): $\mu = 0.0741$, $\sigma = 0.3703$, $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$ and $\gamma = 0.5$. In Figure 6(b): $\mu = 0.0445$, $\sigma = 0.3703$, $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$ and $\gamma = 0.5$. In Figure 6(c): $\mu = 0.0741$, $\sigma = 0.2222$, $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$ and $\gamma = 0.5$. In Figure 6(d): $\mu = 0.16$, $\sigma = 0.4$, $\rho = \bar{\rho} = 0.3$, $\beta = 0.75$ and $\gamma = 0.2$.

for concave functions $a^\epsilon(r)$ and $b^\epsilon(r)$, where $\mu = 0.16$, $\sigma = 0.4$, $\rho = \bar{\rho} = 0.3$, $\beta = 0.75$ and $\gamma = 0.2$. Similarly, numerical simulations show that the two boundaries become closer when ρ increases.

To conclude this section, Figure 7 presents numerical simulations derived from Algorithm 1 for a negative value of μ ($\mu = -0.0445$), alongside the parameters $\sigma = 0.3703$, $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$ and $\gamma = 0.5$. Figure 7(a) illustrates again the action and inaction regions for both players, together with the boundaries $a^\epsilon(r)$ and $b^\epsilon(r)$. Figure 7(b) shows optimal control actions through the dynamics of $(R_t^{\eta^*}, X_t^{\nu^*,\eta^*})$, $a^\epsilon(R_t^{\eta^*})$ and $b^\epsilon(R_t^{\eta^*})$, for initial conditions given by $R_0 = 709.11$ and $X_0 = 1.4 \times 10^3$. Here, the dynamic of the profit stream (5) was obtained through the classical

	μ	σ
Figure 6(a)	0.0741	0.3703
Figure 6(b)	0.0445	0.3703
Figure 6(c)	0.0741	0.2222

TABLE 1. Comparison between the parameters that show the sensitivity of $\eta^*(r, x)$.

Euler–Maruyama method. This figure presents one sample path of the processes $X_t^{\nu^*,\eta^*}$, $a(R_t^{\eta^*})$ and $b(R_t^{\eta^*})$. Firm's profits $X_t^{\nu^*,\eta^*}$ are maintained by the investor above the moving investment boundary $a(R_t^{\eta^*})$ (via Skorokhod reflection). We also observe that when the firm's profits are sufficiently high (i.e., $X_t^{\nu^*,\eta^*} > b(R_t^{\eta^*})$) there is no emission abatement and $R_t^{\eta^*}$ remains constant.

In order to analyze average quantities we conducted Monte Carlo simulations with 10^5 sample paths. Figures 7(c) and 7(d) illustrates the resulting average investment strategies for both the firm and the investor. These figures illustrate three key dynamics: the evolution of average cumulative spending in emission abatement, denoted as $t \mapsto \mathbb{E}[R_t^{\eta^*}]$ (Figure 7(c)); the evolution of the ratio between total abatement spending and current profits, represented by $t \mapsto \mathbb{E}[R_t^{\eta^*}/X_t^{\nu^*,\eta^*}]$; and the evolution of the ratio between the investor's cumulative financial investment and the firm's current profits, given by $t \mapsto \mathbb{E}[\nu_t^*/X_t^{\nu^*,\eta^*}]$ (Figure 7(d)). We see that in the long-run the average investment-to-profit ratio increases faster than the average abatement-to-profit ratio, whereas in the short-run the abatement increases faster than the investment, relatively to the profit levels. This seems to indicate that emission reduction brings a long-term financial benefit to the firm by allowing it to attract larger investment levels (relatively to the production capacity) compared to a situation where abatement actions are not taken.

7. CONCLUSIONS

In this paper we designed a game-theoretic model of stochastic investment and pollution abatement between two players. The model is set as a 2-player nonzero-sum stochastic game of singular control vs. classical control. We are interested in Nash equilibria.

One player is a representative privately-owned firm and the other one is a pool of investors owning the company, who are described in our model as a single representative investor. The representative firm maximizes future discounted cash-flows and can invest into costly emission abatement policies. The representative investor has both financial and environmental preferences and, based on the environmental performance of the company, sets an optimal investment policy.

We find theoretically an equilibrium and optimal strategies for both players when there is no Brownian component in the model (i.e., the dynamics are deterministic). In that equilibrium the representative firm invests at maximum rate in emission abatement when its financial performance is below an emission-dependent threshold. This spurs investment from the representative investor, who keeps the production capacity level of the representative firm above an optimal threshold depending on the total emission abatement over time.

For the general, stochastic, framework we formulate a verification theorem and we obtain equilibrium payoffs and optimal strategies via numerical resolution of a system of variational inequalities for the firm and the investor. The numerical results show that the structure of the equilibrium is qualitatively the same as the one found in the deterministic setup. Our results are illustrated via numerical simulation of the equilibrium trajectories and the of the equilibrium payoffs for both players.

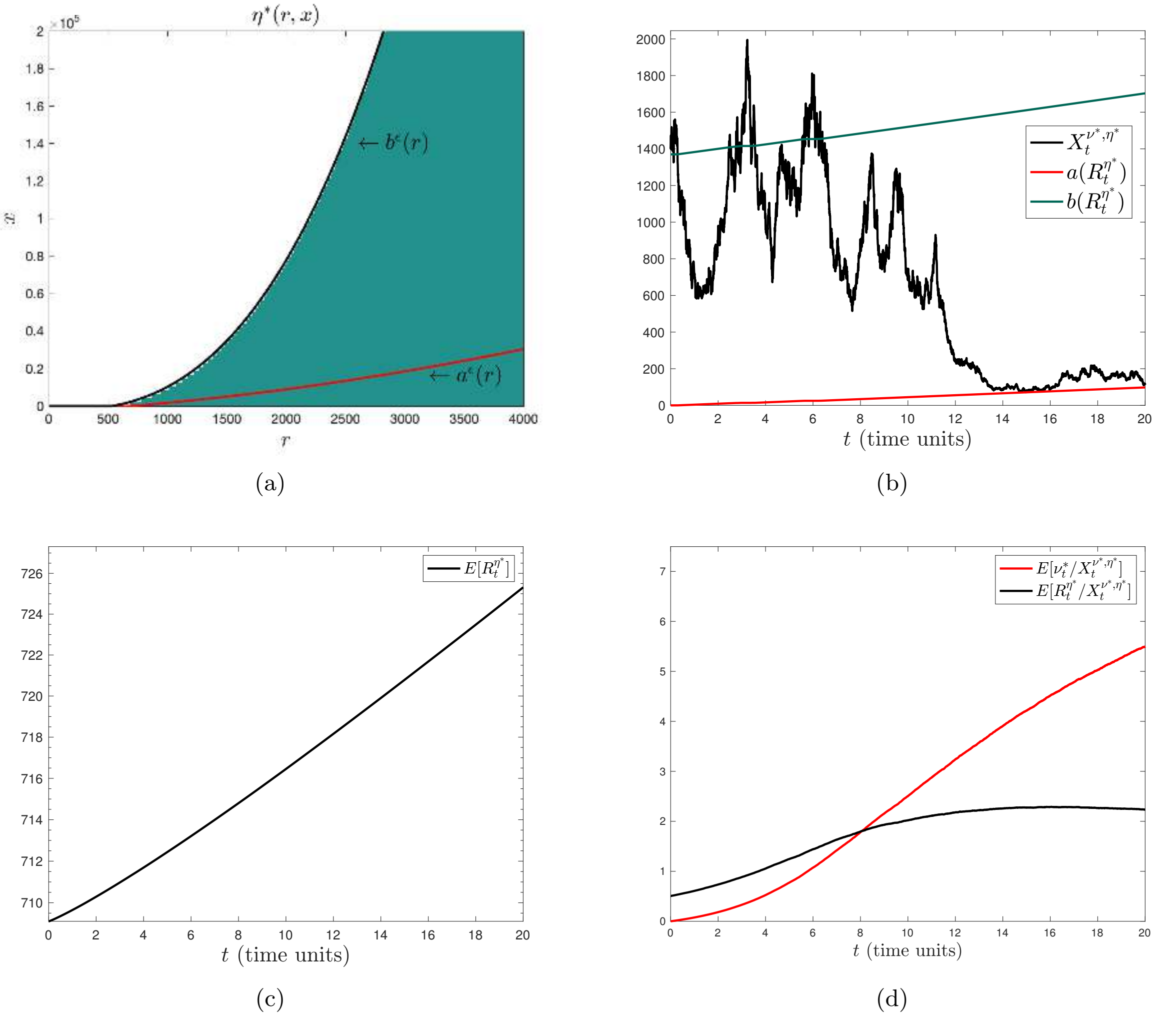


FIGURE 7. Figure 7(a) illustrates optimal boundaries $a^\epsilon(r)$ and $b^\epsilon(r)$ with $\mu = -0.0445$, $\sigma = 0.3703$, $\rho = \bar{\rho} = 0.2850$, $\beta = 0.55$ and $\gamma = 0.5$. Figure 7(b) shows one sample path of the firm's optimal production capacity $(X_t^{\nu^*,\eta^*})_{t\geq 0}$, along with the moving investment boundary $a(R_t^{\eta^*})$ and $b(R_t^{\eta^*})$, with initial conditions $R_0 = 709.11$ and $X_0 = 1.4 \times 10^3$. Figures 7(c) and 7(d) illustrate: (i) the dynamics of the average amount allocated by the firm towards emission reduction over time, $\mathbb{E}[R_t^{\eta^*}]$, (ii) the expected ratio of the firm's abatement spending to total profits, $\mathbb{E}[R_t^{\eta^*}/X_t^{\nu^*,\eta^*}]$, (iii) expected ratio of the investor's cumulative financial contribution to total profits, $\mathbb{E}[\nu_t^*/X_t^{\nu^*,\eta^*}]$. These quantities were approximated via Monte Carlo simulations with 10^5 sample paths.

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