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Abstract

The paper studies optimal fees and the conditions for existence of an equilibrium without forks in a market for cryptocurrencies, of the Bitcoin type. Once calibrated to the BTC.com high-frequency data, the model explains the realized volatility of the observed fees, the volatility amplification from the prices without to the ones with fees, as well as the relative stability of the implied optimal policies. The rate of return that investors would require from an asset with the same drift and diffusion of the BTC, but without costs, is a modest 3.5%, while the expected return from the crypto is 14.7%

Keywords: equilibrium in crypto markets, cryptocurrencies, Bitcoin, blockchain.

JEL-Codes: D41, D35, D58, G11, G12.

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The market for cryptocurrencies - such as the Bitcoin (BTC) - is active and attracts a lot of attention, from investors as well as regulators. Its size steadily increases, while cum-fee prices are highly volatile. Fees often have both a fixed and a variable component.

Given the scarce literature on crypto microeconomics, it is unclear if all the observed cum-fee prices are equilibrium ones, or whether there is no rational way to explain the observed prices, gross of fees, and their volatility, as well as the contemporaneous more tranquil behavior of orders.

A large consensus exists on the fact that market participants on the sell side, or miners, are rational agents or machines programmed by them. The main question is whether irrationality on the demand side is the only way to explain the observed outcomes or whether the joint rational behavior of demand and supply is able to explain trade, in quantities and prices. The model below provides a rational explanation of the fees, as compounded on a standard diffusive price, consistent with the rational behavior of both sides and with BTC market data.

To explain a crypto price - before the fees - a part of the literature looks at digital currencies in their role of means of payment or stores of value, similar to money. This is the approach taken by Biais, Bisière, Bouvard, Casamatta and Menkvield (2022), who support the idea that – at least theoretically – the intrinsic value of cryptocurrencies comes from the difference between their costs and benefits as means of payment, with noise generated by sunspots. Another part of the literature emphasizes the role of cryptocurrencies as risky financial assets (see for instance Kose, O'Hara and Saleh (2021)). Without denying the role of cryptos as means of payment, we focus on their value as risky assets. We take the price behavior before the fees as a given stochastic process, and we ask ourselves how we can rationalize the observed fees and the corresponding orders, and whether we can explain their observed variance. We focus on BTC.

The paper shows that, given a streamlined market structure, comprehensive of the main features of a cryptomarket, there is a triple made by a fee, an optimal sell/buy behavior of users, and an implementation of their orders by miners, which constitutes an equilibrium without forks. As such, it maximizes the welfare of both counterparts, and there is no incentive for them to deviate from it. The equilibrium is such that - as it happens in actual BTC-like crypto markets - trade, although very frequent, is not continuous, and the time between two successive trades is not constant. It is a stopping time, which can be characterized consistently with observations. The equilibrium can be implemented starting from the submission of a couple quantity-fee on the part of the users, as it often happens in reality. In such an equilibrium, miners, who have a stock of crypto assets, do not simply choose the highest fee around, but trade-off the advantages and disadvantages of executing the transaction because they exchange crypto assets for dollars - or any other non crypto currency - and get the fees. As a consequence, both the level and the volatility of the fees we model are consistent with actual data. In particular, they are consistent with the volatility amplification we observe on actual markets. We resort to the notion of equivalent safe rate to quantify the effect of such an high volatility on traders' utility.

The paper proceeds as follows: in Section 1 we describe a typical cryptomarket, as it emerges from the literature; in Section 2 we build a corresponding theoretical model and analyze the decisions of market participants; in Section 3 we solve for equilibrium trade and fees. In Section 4 we parametrize the model using actual Bitcoin data from BTC.com and show how the model can rationally explain the observed behavior of trades and fees, in level and volatility. Section 5 analyses the equivalent safe rate concept and Section 6 concludes.

1 Basic features of crypto, BTC-alike markets

The microeconomics of cryptocurrencies has been little explored yet. Notable exceptions are Kose et al. (2021), Haberman, Leshno and Moallemi (2021), and Halaburda, Haeringer, Gans and Gandal (2021), which give an exhaustive description of the Bitcoin ecosystem, the market we focus on, examining both the supply and demand side.

The market is characterized by a ledger, namely a chain of blocks (blockchain) in which the transactions are registered. The supply side is represented by miners, who register the transactions in the blockchain. To do that, they need to solve the encoding or hashing puzzle, by putting a costly IT effort. The demand side is represented by users.

A first feature of the markets is that blockchain is neither a storage for physical cryptos, nor a file which contains all crypto positions of miners, but simply a chain of the transactions which have been made. Miners are simply committed to executing the transactions they receive a bid for as soon as possible and to attaching transactions to the longest existing chain.

As soon as possible msu be consistent with the fact that miners need to solve the hashing puzzle, which requires time. The execution cannot be performed before that.

The reason for attaching transactions to the longest existing chain is to avoid so-called forks, or bifurcations of the blockchain, which would mine its validity (see Nakamoto (2008)). To explain that, recall that crypto transactions need the consensus of the other miners. Consensus makes the market different from other bilateral trading mechanisms. The longest chain rule is a way to guarantee the consensus, and is granted if the miner has chosen the longest blockchain. Suppose indeed that two miners operate a different transaction at the same time. Without the rule, it might well happen that some miners consent on one of the two transactions, because they receive it first, some on the other. The blockchain would incur into a so-called fork, which is undesirable. According to Nakamoto's suggestion, consensus is restored by maintaining the longest blockchain, and disregarding the other. So, it is generally understood that any miner must not only make an effort to encrypt his own transactions quickly, but also to reach the longest chain.[1]

[1] Forks can also be created on purpose, by dishonest miners. In this paper we will not exhamine dishonest miners, and we suggest that the reader consults Halaburda et al. (2021)

Miners collect trades in blocks. A second feature of the market is the presence of block rewards and fees. For each successful addition to the blockchain, the miner receives a so-called block reward, which is an amount of newly-issued BTCs, as well as a variable fee, in standard currency, say dollars. The block reward, which is fixed, is interpreted as being the bulk of the miner reward, which compensates for the mining costs. There is evidence, reported for instance in Huberman et al. (2021), that indeed the block reward covers the energy and hardware cost of mining. Similar evidence is provided by Websites. And miners tries to influence the power of the miners' network, by consolidating their purchasing power.

The way in which this fixed component is interpreted is usually the following, as proposed for instance by Budish (2018). Suppose that there are N identical miners. The probability of winning the hashing puzzle is considered to be $1/N$. If, in a reference currency (say, the dollar), each miner has a cost c to solve the puzzle - the hashing cost - and receives a block reward θ, whose dollar value is $e\theta$, if e is the exchange rate dollar to BTC. The participation constraint of the representative miner is $e\theta/N \geq c$. The equilibrium number of miners N^* is the one which makes this inequality constraint an equality: $N^* = e\theta/c$. [2]

The second part of the miner revenue is the fee, which is proportional either to the quantity transacted or to the value of the transaction in a reference currency. According to Easley, O'Hara and Basu (2019), "the endogenous development of transactions fees reflects an important step in the evolution of the Bitcoin blockchain from being a mining-based set of rules towards being a market-based system capable of adapting to changing economic conditions." We share that opinion and focus our analysis below on fees, to explain market equilibria.

A related feature of the standard, blockchain crypto market is that users can place an order - ask for a quantity of crypto or tokens - with a fee attached or without the fee attached.

If no fee is attached to the quantity of crypto demanded (sold or bought) by users, the miner can choose the fee at which he is willing to execute the transaction. The usual relation between demand and supply, in which the demand chooses the quantity and the supply decides at which price - in this case at which fee - to satisfy the given order, is at play.

The case in which the quantity demanded is accompanied by a fee is less trivial. The standard reasoning in the literature is that miners choose the highest fee around. This however does not reflect the fact that miners can net the orders received from different users, but supply the net balance. As a consequence,

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and the references therein for that.

[2] These markets is that they are permissionless, or free-entry markets. The purpose of the Bitcoin market however is to gradually reduce the reward, so as to make it vanish - and have no further issued coins - in the future. Reducing the reward means to have a different number of active miners. We will maintain the description of the block reward as the one which covers the mining costs, below. However, at least if we accept Budish (2018) interpretation, in which the block reward covers expected costs, as we do below, there is no reason why the number of miners should vary, as long as the value of the block reward does not vary in a way decorrelated with the hashing cost.

each bulk of trades increases the amount of traditional currency of the miner - because of the fee - but also his net amount of crypto asset, because of the block reward and net trade. Both matter to him. He does not simply choose the highest fee around, but evaluates the trade off between its net accrual of depletion of crypto and accrual of traditional currency. As a consequence, in our model we substitute the choice of the highest fee around with the evaluation of the overall impact of the trade. By so doing, we end up with miners choosing the optimizing fee, which is not necessarily the highest one.

Easley et al. (2019) build a model in which miners choose the highest fee around and users increase the offered fee till their order is executed. Each user trades-off the cost of waiting for the transaction to be executed and the level of the fee he is willing to pay. Miners execute first transactions with higher fees. The user-miner game results in two equilibria: one with fees and one without fees, from which it is not profitable to deviate. The equilibria we observe at present - as we show in the empirical part - are of the first type.

2 A BTC market model

We examine a market model whose features stylize the above description. The market is populated by N^* identical miners, where N^* is such that the expected block reward is equal to the hashing cost c, and a continuum of users, also named investors. Investors differ in risk aversion. Let i denote a specific investor. This Section specifies the objective of users as well as miners, and solves their optimization problems.

The intuition is as follows. Rational miners know that, because trading is costly for users, they will never submit orders continuously in time, but - as known from the previous literature on costly trade, starting from the seminal proof of Constantinides (1979) - they should change the ratio of crypto to non-crypto in their portfolios only if it becomes to high or too low, beyond the tolerance bounds. We argue that miners execute the users order as soon as the hashing puzzle is solved, if the demanded quantity happens to restore the ratio of risky to riskless asset of the users within some appropriate tolerance bounds. If this is not the case, they solve the puzzle and wait till the optimal sell or buy ratio for the user is reached. We also show below that, if miners optimally execute orders when either the lower or the upper ratio is reached, they can maximize both their own and the counterpart's welfare, guaranteeing an equilibrium.

Users and miners trade the non-crypto asset - to be understood as a safe asset or cash, dollars to take an example - versus the cryptocurrency. The interest rate r on cash is exogenous, and can be either positive or zero, $r \geq 0$. The price of the non-crypto asset is $S^0(t)$ at time t. Let its initial level be $S^0(0) = 1$. Then

$$S^0(t) = \exp(rt)$$

is the non-crypto price at t, at which it is traded against the crypto.

We denote with $S(t)$ the price of the crypto asset - say, BTC - at time t, in the reference currency. If the latter is the dollar, S is the exchange rate dollar to BTC, or the price in dollars of a BTC. We take $S(t)$ to be a Geometric Brownian motion with parameters $\alpha + r > 0$ and $\sigma > 0$:

$$dS(t)/S(t) = (\alpha + r)\, dt + \sigma dZ(t)$$

where dZ is a Brownian motion. Let the filtration generated by it be the model's underlying one.

We motivate such choice because it is the standard, basic model for risky assets.[3] Our idea is that of verifying, in the model implementation, if the crypto price does evolve as a lognormal. Within this idea, σ is the *intrinsic* volatility, which we will compare to the overall, *realized* volatility of the cum-fee price, to see whether the market structure generates volatility amplification.

Investors cannot trade the crypto among themselves, but have to go through miners. Miners are the one who encript the investors' order in the blockchain, and get both a block reward and a fee for that.

The block reward per transaction, which we denote with $\Theta(t)$, is a quantity of crypto asset. Based on the Bitcoin rules described above and modelled by Halaburda et al. (2021), it may for instance be the case that $\Theta(t) = c(t)N^*/S(t)$, where $c(t)$ is the hashing cost - in dollars - that the miners pay per transaction at t. In what follows, we will search for equilibria in which the representative miner trades with probability 1, so that $\Theta(t) = c(t)/S(t)$.

For the sake of simplicity, we identify each block with the transaction with a single investor i. One user trades at a time. The fee for each block, which we denote with $\epsilon_i \in [0, 1]$, makes the price at which investor i sells the crypto to the miner (the bid of the latter) smaller than the underlying or ask price, that miners offer to sell the currency. They are respectively $(1 - \epsilon_i)S$ and S.[4] Following the description summarized above of the actual functioning of the BTC market, the fee may formally chosen by the users. Any miner can then choose the best fee around, that we are going to characterize below. It will not necessarily be the highest one around.

Let us examine what drives purchases, sales and the crypto equilibrium quantity and bid price.

Denote with $C^0(0)$ and $C(0) = 0$ the initial aggregate quantity of cash and crypto tokens, namely the number with which the market opens up at time 0. The number of crypto tokens at time t does not remain fixed but, as described above, new enscriptions are done and orders are appended to the blockchain,

[3] In the mans of payment view of Biais et al. (2022), $\alpha + r$ would be the net transactional benefits of the cryptocurrency, per unit of time, i.e. the benefits that a fiat money cannot provide, such as the possibility to trade even when other currencies are in disarray, net of the costs of limited convertibility. The Brownian term σdZ would take the place of the sunspot, crash risk in Biais et al. It would lead not only to crashes, but also to exogenous positive increases in price.

[4] One could model separately the bid and ask fee. For the simplicity of the solution, we take the equivalent approach of modelling simply the fee as the percentage difference between the bid and ask price.

depending on the request from investors. This means that the quantity in the hands of all investors is

$$C(t) = L(t) - U(t)$$

where $L(t)$ and $U(t)$ are respectively the cumulative quantities of crypto for non-crypto cash purchased from and sold to miners by investors between 0 and t. We denote with $L_i(t), U_i(t)$ the individual processes, so that L(t)= $\int_i L_i(t) di and similarly for$U(t), C(t), C$^0(t)$.

The dollar value of non-crypto and crypto tokens in the users' hands at t is respectively $S^0(t)C^0(t)$ and $S(t)C(t)$.[5]

Investor - and consequently all investors together, since only one trades - have to satisfy a budget constraint, namely to finance the value of their net trade of cryptos, $-S(t)dL(t) + (1 - \epsilon_i)S(t)dU(t)$, with that of the net trade of non-crypto, namely $S^0(t)dC^0(t)$. The aggregate budget constraint is

$$S^0(t)dC^0(t) = -S(t)dL(t) + (1 - \epsilon_i)S(t)dU(t) \tag{1}$$

So, $dC^0(t)$, the change in the quantity of dollars in the hands of users, is a function of aggregate crypto sales $((1-\epsilon_i)S(t)dU(t))$ and purchases $(-S(t)dL(t))$.

It follows from the assumed dynamics of prices and quantities that the dollar value of aggregate cash and crypto in the hands of users follows the following SDEs:

$$\begin{aligned}
& d\left[S^0(t)C^0(t)\right] \\
&= r\left[S^0(t)C^0(t)\right]dt + S^0(t)dC^0(t) \\
&= r\left[S^0(t)C^0(t)\right] - S(t)dL(t) + (1 - \epsilon_i)S(t)dU(t) \\
& d\left[S(t)C(t)\right] = (\alpha + r)\left[S(t)C(t)\right]dt \\
& +\sigma\left[S(t)C(t)\right]dZ + S(t)dL(t) - S(t)dU(t)
\end{aligned}$$

The number of crypto tokens in the hands of miners is different, since they also receive the block reward Θ. It is

$$-C(t) + \Theta \times \mathcal{N}(t)$$

where $\mathcal{N}(t)$ is the total number of transactions executed up to t.

We assume that investors are perfectly rational and maximize the long-run rate of growth of expected utility of their terminal wealth, which is the liquidation value of cash and cryptos:

$$W_i(T) = Ci^0(T)\exp(rT) - (1 - \epsilon_i)S(T)U_i(T) + S(T)L_i(T)$$

Investors' wealth evidently depends on their cumulative positions in the crypto and cash, $C_i(T)$ and $C_i^0(T)$, and most specifically on purchases separately from sales, as well as on the fees applied by miners.

[5]Note that in the dollar-to-BTC interpretation, since $S(t)$ is the exchange rate between dollars and BTCs, or the price in dollars of a BTC, so that all the values are in dollars.

Investors are assumed to have an infinite-horizon power utility, $W^{1-\gamma_i}/(1-\gamma_i)$. We also assume that they are risk averse and non-myopic: $\gamma_i > 0, \gamma_i \neq 1$ and, to model the long-run nature of their goal, that their horizon tends to infinity. Their objective is

$$\max \lim_{T\to\infty} \inf \frac{1}{T} \ln \mathbb{E}\left[W_i(T)^{1-\gamma_i}\right]^{1/(1-\gamma_i)} \tag{2}$$

This means that they maximize the rate of growth - the ln - of the expected utility of wealth $\mathbb{E}\left[W_i(T)^{1-\gamma_i}\right]$ under risk aversion, per unit of time $(1/T)$.[6] Another way to read the objective function is to say that investors maximize the equivalent safe rate (ESR) of growth of their utility, or the rate that would produce the same expected utility they get under transaction fees.

The instruments investors have for optimizing are the timing and amount of purchases and sales, or the processes L_i and U_i.

Miners issue the crypto for cash, getting the rewards Θ and charging the fee ϵ_i as soon as they serve investor i. We model perfectly rational miners, who stand ready to encrypt all orders from investors in the blockchain, and maximize the rate of growth of expected utility of their final wealth, which is again power, non-myopic, when the horizon becomes infinite. We assume that miners are risk averse, non-myopic, $0 < \gamma \neq 1$:

$$\max \lim_{T\to\infty} \inf \frac{1}{T} \ln \mathbb{E}\left[V(T)^{1-\gamma}\right]^{1/(1-\gamma)} \tag{3}$$

Because their positions in cash and BTC are opposite to the users' ones, but they also have the cumulated block rewards, which we assume to deserve no interest rate, net of the costs to encrypt the transactions, their wealth is

$$\begin{aligned} V(T) = & -C^0(T)\exp(rT) + (1-\epsilon_i)S(T)U(T) - S(T)L(T) \\ & + \sum \mathbf{1}_{dL(\tau_i)>0 U(\tau_i)>0, \tau_i \leq T} \left[\Theta(\tau_i)S(\tau_i) - c(\tau_i)\right]. \end{aligned}$$

where $\mathbf{1}_E$ is the indicator function of the event E, τ_i are the stopping times where trade is carried out, and the sum is extended to all $\tau_i \leq T$.

Since the block reward Θ is meant to cover the hashing costs, whose dollar value at τ_i is $c(\tau_i)$, the sum is zero and we get $V(T) = -W(T)$, where W is the aggregate wealth over investors i.

The instrument miners have for optimizing is the accepted fee level ϵ_i .

2.1 Optimization for user i

In Sections 2.1 and 2.2 we discuss the solution for users and miners assuming that users do not submit a fee, while in Section 2.3 we examine the case in which they also submit the fee.

For any specific trade, if the fee is not chosen by the user, his problem is similar to the portfolio one formalized by Gerhold, Guasoni, Muhle-Karbe and

[6]The exponent $1/(1-\gamma_i)$ does not affect the trading policy.

Schachermayer (2011). Given a power utility and a lognormal underlying, the user decision problem with a long horizon entails two processes L_i and U_i that increase only when $\theta(t) \doteq S(t)C_i(t)/S^0(t)C_i^0(t)$, which here represents the ratio of BTC to dollars (crypto to non-crypto asset) in portfolio for user i, overpasses respectively a constant lower and a constant upper barrier or tolerance bound, denoted as l_i and $u_i, l_i \leq u_i$. The trading policy consists in setting the ratio back to the closest bound.

It follows that

$$dW_i = S^0(t)C_i^0(t)rdt + S^0(t)dC_i^0 + dS(L_i(t) - U_i(t)) \\ +S(t)dL_i(t) - S(t)dU_i(t)$$

where the changes in the crypto quantities dU_i and dL_i are different from zero only when $\theta = u_i$ and l_i respectively. At that point in time, also the quantity of cash dC_i^0 changes, and because of the self-financing constraint (1), it equates in absolute value the terms in dL_i and $dU\%_i$, excluding the fees.

$$dW_i = S^0(t)C_i^0(t)rdt + dS(L_i(t) - U_i(t)) - \epsilon_i S(t)dU_i(t)$$

In what follows, for the sake of simplicity, we restrict our formulas to parameter combinations which make both barriers positive, i.e. $0 < l_i \leq u_i$. To this end, we restrict the parameters so that the optimal crypto holdings would be positive in the absence of fees. We posit:

$$0 < \theta_i^* = \frac{\alpha}{\gamma_i \sigma^2 - \alpha} < 1 \tag{4}$$

We indeed know since Merton that, in the absence of a fee, investors would keep at all times a position in crypto to non crypto equal to

$$l_i = u_i = \frac{\alpha}{\gamma_i \sigma^2 - \alpha} \tag{5}$$

and that, when the fees are positive, this position must stay between the tolerance bounds l_i and u_i.

The function K_i is the value function of the problem,

$$\lim_{T\to\infty} K_i(S^0(t)C_i^0(t), S(t)C_i(t), t; T) \doteq \max \lim_{T\to\infty} \inf \frac{1}{T} \ln \mathbb{E}\left[W_i(T)^{1-\gamma_i}\right]^{1/(1-\gamma_i)}$$

if there exists a constant β_i - an artificial discount rate - which makes K_i itself, once discounted, finite and stationary. The discount rate β is defined by the requirement that, once K is discounted, the resulting value function

$$J(S^0(t)C_i^0(t), S(t)C_i(t), t; T) = \exp^{-(1-\gamma_i)(r-\beta_i)(T-t)} K(S^0(t)C_i^0(t), S(t)C_i(t), t; T)$$

is stationary:

$$\lim_{T\to\infty} J(S^0(t)C_i^0(t), S(t)C_i(t), t; T) = J(S^0(t)C_i^0(t), S(t)C_i(t))$$

Given that the utility function is power, we can safely assume that

$$J(S^0(t)C_i^0(t), S(t)C_i(t)) = \left[S^0(t)C_i^0(t)\right]^{1-\gamma_i} I_i(\theta)$$

and solve for I. According to both Dumas-Luciano (1991) and Gerhold et al. (2011), the problem (2) under (1) reduces to solving for the function I the ODE

$$-(1-\gamma_i)\beta_i I_i(\theta) + \alpha I_i'(\theta)\theta + \sigma^2 I_i''(\theta)\frac{\theta^2}{2} = 0 \tag{6}$$

- with $\beta > 0$, to be specified in the solution - under the so-called value-matching and smooth-pasting BCs, which require continuity of the first and second derivatives of the value function at the tolerance bounds, respectively

$$\begin{cases} (1+l_i)I_i'(l_i) - (1-\gamma_i)I_i(l_i) = 0 \\ (\frac{1}{1-\epsilon_i} + u_i)I_i'(u_i) - (1-\gamma_i)I_i(u_i) = 0 \end{cases} \tag{7}$$

and

$$\begin{cases} (1+l_i)I_i''(l_i) + \gamma_i I_i'(l_i) = 0 \\ (\frac{1}{1-\epsilon_i} + u_i)I_i''(u_i) + \gamma_i I_i'(u_i) = 0 \end{cases} \tag{8}$$

If we substitute the BCs at l_i (respectively, u_i) in equation (6) we get a second degree equation in the barriers l_i, u_i which is solved respectively by

$$l_i(\epsilon_i) = \frac{\alpha - \lambda(\epsilon_i)}{\gamma_i\sigma^2 - \alpha + \lambda(\epsilon_i)} \tag{9}$$

$$u_i(\epsilon_i) = \frac{1}{1-\epsilon_i}\frac{\alpha + \lambda(\epsilon_i)}{\gamma_i\sigma^2 - \alpha - \lambda(\epsilon_i)} \tag{10}$$

where λ parametrizes the departure of l_i and u_i from the value that the ratio of crypto to non-cripto assets would have if there were no fees (the Merton's model), $\alpha/(\gamma_i\sigma^2 - \alpha)$. The departure of the allocation from what it would be without fees is a function of the fees themselves, and therefore we write $\lambda(\epsilon_i)$. Since the function I_i depends on θ, as well as on the policy, parametrized by $\lambda(\epsilon_i)$, we also write $I_i(\theta, \epsilon_i)$. Let us transform the unknown function I_i into w as follows:

$$I_i(\theta, \epsilon_i) = \exp^{(1-\gamma_i)\int_0^{\ln\theta/l_i(\epsilon_i)} w(x)dx} \tag{11}$$

where $x = \ln(\theta/l_i)$. The I function that solves (6) is the solution of the Riccati equation in w, with the value matching BCs - which correspond to (7) above - written in terms of w itself, and valid at $x = 0$ and $x = ln(u_i/l_i)$:[7]

$$w'(x) + (1-\gamma_i)w(x)^2 + \left(\frac{2\alpha}{\sigma^2} - 1\right)w(x) - \gamma_i\frac{\alpha^2 - \lambda^2}{\gamma_i^2\sigma^4} = 0$$

$$w(0) = \frac{\alpha - \lambda}{\gamma_i\sigma^2}$$

$$w(\ln(u_i/l_i)) = \frac{\alpha + \lambda}{\gamma_i\sigma^2}$$

[7] Guasoni et al. give both the heuristic and formal proof of that and the analytical expression of I for small costs.

The last system is solved as follows: the Riccati equation gives a solution for the unknown function w, the BC at the lower boundary $x = 0$ is its initial condition, while the BC at the upper boundary $x = \ln(u_i/l_i)$ serves to determine λ.[8] The value of β_i which makes the value function stationary is then $\beta = (\mu^2 - \lambda^2)/(2\gamma_i\sigma^2)$. It obtains from the differential equation after substituting the BCs (both value matching and smooth pasting at u_i) in it.

It is shown in Gerhold et al. (2011) that, if $\frac{\alpha}{\gamma_i\sigma^2} \neq 1$, a condition which is always true when the Merton's θ_i is positive, as assumed here, λ is an analytic function of ϵ_i. It is of the order of magnitude $\epsilon_i^{1/3}$ and, up to $O(\epsilon_i^{4/3})$, it can be approximated by

$$\lambda = \gamma_i\sigma^2 \left(\frac{3}{4\gamma_i} K^{*2}(1-K^*)^2\right)^{1/3} \epsilon_i^{1/3} + \sigma^2 \left(\frac{5-2\gamma_i}{10} K^*(1-K^*) - \frac{3}{20}\right) \epsilon_i$$

where $K^* = \frac{\alpha}{\gamma_i\sigma^2}$.

Note that the solution makes sense for $\lambda < \alpha$.

2.2 Optimization for the miner

The miner - representative of the N^* ones - has to solve his own problem. He can do that by fixing the fee for the next trade, whenever the order comes without the latter, as we assume in this Section. We will see in Section 2.3 how he can implement an equilibrium when the order of the first user comes with its own fee.

For any specific trade with user i the dynamics of the miner's wealth V is the opposite of the one for his counterpart. W_i , with the exception of the addition

[8] For small fees, the Riccati equation has an explicit solution

$$w(x) = (\gamma_i - 1)^{-1} \begin{cases} \left[a \times \tan h(\tan h^{-1}(b/a - ax)) + \left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)\right] \\ \text{if } \gamma_i \in (0,1),\, \frac{\alpha}{\gamma_i\sigma^2} < 1 \text{ or } \gamma_i > 1,\, \frac{\alpha}{\gamma_i\sigma^2} > 1 \\ \left[a \times \tan h(\tan h^{-1}(b/a + ax)) + \left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)\right] \\ \text{if } \gamma_i > 1,\, \frac{\alpha}{\gamma_i\sigma^2} \in \frac{1}{2}\left(1 - \sqrt{1-1/\gamma_i}, 1 + \sqrt{1-1/\gamma_i}\right) \\ \left[a \times cotan(cotan^{-1}(b/a - ax)) + \left(\frac{\alpha}{\sigma^2} - \frac{1}{2}\right)\right] \\ \text{otherwise} \end{cases} \quad (12)$$

where

$$\begin{aligned} a &= a(\epsilon_i) = \sqrt{\left|(\gamma_i - 1)\left(\frac{\alpha^2 - \lambda^2}{\gamma_i\sigma^4}\right) - \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2\right|} \\ b &= b(\epsilon_i) = \left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right) - (\gamma_i - 1)\left(\frac{\alpha - \lambda}{\gamma_i\sigma^2}\right) \end{aligned}$$

and λ is the solution to the equation

$$w(\ln(u_i/l_i))) = \frac{\alpha + \lambda}{\gamma_i\sigma^2}.$$

The different functional forms depend on γ_i and $\frac{\alpha}{\gamma_i\sigma^2}$, because they entail a different discriminant for the Riccati equation. They hold under the restriction $\lambda < \alpha$.

of the crypto block rewards. We assume that a single user at a time comes, so that $dL(t) = dL_i(t)$, and similarly for dU. It follows that

$$\begin{aligned} & dV(t) \\ = & -S^0(t)C^0(t)rdt - S^0(t)dC^0(t) + dS(U(t) - L(t)) - S(t)dL(t) + S(t)dU(t) \\ & + S(t)\Theta(t)\mathbf{1}_{\{dU(t)>0\}U\{dL(t)>0\}}. \end{aligned}$$

Let K be the miner's value function, i.e.

$$\lim_{T\to\infty} K(S^0(t)C^0(t), S(t)C(t), t; T) \doteq \max \lim_{T\to\infty} \inf \frac{1}{T} \ln \mathbb{E}\left[V(T)^{1-\gamma}\right]^{1/(1-\gamma)}$$

It is easy to show, as in the investor's case, that, if we aim at a stationary value function, we must discount K at a rate $\beta > 0$. We can guess the discounted value function

$$\begin{aligned} & J(S^0(t)C^0(t), S(t)C(t), t; T) = \\ \exp^{-(1-\gamma)(r-\beta)(T-t)} & K(S^0(t)C^0(t), S(t)C(t), t; T) \end{aligned}$$

and assume that it has a stationary limit:

$$\begin{aligned} \lim_{T\to\infty} & J(S^0(t)C^0(t), S(t)C(t), t; T) = \\ & J(S^0(t)C^0(t), S(t)C(t)) \\ & = \left(-S^0(t)C^0(t)\right)^{\gamma} I(\theta). \end{aligned}$$

In the tolerance region for the user I must satisfy the following differential equation, because of the equation which moves the asset price S:

$$-(1-\gamma)\beta_s I(\theta) + \alpha I'(\theta)\theta + \sigma^2 I''(\theta)\frac{\theta^2}{2} = 0 \quad (13)$$

with continuity of the function at the boundaries. The latter is established by letting the value matching conditions be satisfied when trading, i.e. at the boundaries chosen by the counterpart:

$$\begin{cases} (1+l_i)I'(l_i) - (1-\gamma)I(l_i) = 0 \\ (\frac{1}{1-\epsilon_i} + u_i)I'(u_i) - (1-\gamma)I(u_i) = 0 \end{cases} \quad (14)$$

The value function of the miner must be maximized with respect to the choice of ϵ_i, considering that the boundary itself - chosen by his counterpart - depends on ϵ_i through λ. At that boundary the change in the miner's value function can be approximated as

$$(-S_0C_0)^{\gamma-1}\left[(\gamma I(u_i) - lI'(u_i))(-(1-\epsilon_i)SdU) + I'(u_i)(SdU)\right]$$

If we write down the usual FOC for maximization with respect to ϵ_i we get

$$\frac{\partial u_i}{\partial \epsilon_i}\left\{(-\gamma)I'(u_i) - I''(u_i)\left[u_i(1-\epsilon_i)+1\right]\right\} - I'(u_i)(1-\epsilon_i) = 0 \quad (15)$$

$$\frac{\partial u_i}{\partial \epsilon_i} = \frac{\partial u_i}{\partial \lambda}\frac{\partial \lambda}{\partial \epsilon_i}. \tag{16}$$

For small ϵ_i,

$$\frac{\partial u_i}{\partial \epsilon_i} = -\frac{\gamma_i \sigma^2}{(1-\epsilon_i)\left(\gamma_i\sigma^2 - \alpha - \lambda(\epsilon_i)\right)^2}\frac{\partial \lambda}{\partial \epsilon_i}$$

and, up to $O(\epsilon_i^{1/3})$

$$\frac{\partial \lambda}{\partial \epsilon_i} = \frac{1}{3}\gamma_i\sigma^2\left(\frac{3}{4\gamma_i}K^{*2}(1-K^*)^2\right)^{1/3}\epsilon_i^{-2/3} + \sigma^2\left(\frac{5-2\gamma_i}{10}K^*(1-K^*) - \frac{3}{20}\right)$$

So, the miner's problem consists in solving (13) subject to (14) and (15), with respect to the function I and the constants ϵ_i and β respectively, for given u_i. The solution technique can be envisaged following the user's one: solve the ODE (13) using the value matching conditions at 0 and $\ln(u_i/l_i)$ as initial condition for the Riccati equation and equation in ϵ_i. If we operate the substitution of the unknown function from I to w, analogous to (11), the ODE for I becomes a Riccati in the function w, and its value matching BCs become analogous to the user one, so that also the solution is analogous to the user one.
Given ϵ_i and u_i, the technique for solving for β consists in substituting (14) and (15) in (13). By so doing, we obtain a unique positive solution for β, provided that the condition in the following theorem holds.

Theorem 1 *A unique positive solution for the miner's problem exists if*

$$\epsilon_i > -\left[\frac{\left(\frac{5-2\gamma_i}{10}K^*(1-K^*) - \frac{3}{20}\right)}{\frac{1}{3}\gamma_i\left(\frac{3}{4\gamma_i}K^{*2}(1-K^*)^2\right)^{1/3}}\right]^{-3/2}$$

Proof. We prove first that there is a unique solution for β, then that, under the stated hypotheses, it is positive. Substituting both (14) and (15) in the ODE (13) at u_i, we get the unique solution

$$\beta = \alpha u_i \frac{1-\epsilon_i}{1+u_i(1-\epsilon_i)} + \frac{u_i^2\sigma^2(1-\epsilon_i)}{2}\frac{-\frac{\partial u_i}{\partial \epsilon_i}\gamma_{is} - (1-\epsilon_i)}{(1+\frac{\partial u_i}{\partial \epsilon_i})((u_i(1-\epsilon_i)+1)}$$

(17)

The solution is positive iff

$$\alpha > u_i\frac{-\frac{\partial u_i}{\partial \epsilon_i}\gamma_{is} - (1-\epsilon_i)}{(1+\frac{\partial u_i}{\partial \epsilon_i})^2}$$

A sufficient condition for that is that $\partial u_i/\partial \epsilon_i > 0$, since in that case the right hand side of the equation is negative. For small fees, the derivative has the required sign iff

$$\epsilon_i > -\left[\frac{\left(\frac{5-2\gamma_i}{10}K^*(1-K^*)-\frac{3}{20}\right)}{\frac{1}{3}\gamma_i\left(\frac{3}{4\gamma_i}K^{*2}(1-K^*)^2\right)^{1/3}}\right]^{-3/2}$$

■

Note that equation (14) for ϵ_i does not admit a null solution $\epsilon_i = 0$, since the derivative of λ - and therefore of u_i - tends to infinity as ϵ_i tends to zero. The only way for (15) to be satisfied in that case would be if both the first and second derivative of the value function at u_i were zero, which is inconsistent with the BCs.

2.3 Fee choice

Let us discuss now the fee level choice, ε. We have seen up to now how the optimal choice of the miner may include the fee, if the user's order comes without it. In that case he solves the hashing puzzle and executes the order straight away.

If the order of the user comes with a fee ε^*, the miner is willing to execute the trade only if it entails the exchange described by the boundaries $l_i(\lambda(\varepsilon^*))$ or $u_i(\lambda(\varepsilon^*))$, as determined by (9) and (10), depending on whether the order is a buy or sell one. If this is not the case, he waits until the first time the ratio θ of the user reaches either $l_i(\lambda(\varepsilon^*))$ or $u_i(\lambda(\varepsilon^*))$. He will wait until one of them is reached, even if he finishes the hashing before, to reach an equilibrium.

The SDESs and then ODEs for the two parties are not affected by this waiting time. Since $\ln(\theta_i)(t)$ is distributed as a Brownian motion with drift $-\alpha - \frac{\sigma^2}{2}$ and diffusion σ, the expected time for the miner to get an order at the upper barrier u_i is

$$\min_i \int_0^\infty \sqrt{\frac{(u_i-\theta_i(t))^2}{2\pi\sigma^2 x^3}} \exp\left[-\frac{(-\alpha-\frac{\sigma^2}{2}-u_i+\theta_i(t))^2}{2\sigma^2 x}\right] dx$$

The expected time to get an order at the lower barrier is

$$\min_i \int_0^\infty \sqrt{\frac{(l_i-\theta_i(t))^2}{2\pi\sigma^2 x^3}} \exp\left[-\frac{(-\alpha-\frac{\sigma^2}{2}-l_i+\theta_i(t))^2}{2\sigma^2 x}\right] dx$$

The expected time to implement the trade with fee ε^* is the maximum between the hashing time and the two minima above.

3 Equilibrium

An *equilibrium* in the previous market is a triple (ϵ_i, l_i, u_i), with $\epsilon_i \in [0,1]$, such that, assuming for simplicity that $\theta_i(0) = l_i$ or $\theta_i(0) = u_i$ for all users[9]

- the incoming investor's optimization problem is solved
- the miner's one is solved too
- and the market for the cryptocurrency against dollars clears

Based on the above discussion, we can state that

Theorem 2 *Under the hypothesis in Theorem 1, the crypto market has a single equilibrium, corresponding to the $\epsilon_i \in (0,1)$ such that users solve the ODE (6) with value matching and smooth pasting BCs (7) and (8), miners solve their ODE (13) under the value matching BCs (14) and FOC (15). The order may come with or without a fee. In the former case the order is executed as soon as the hashing puzzle is solved and the incoming user's ratio of BTC to dollars overcomes the tolerance bounds*

$$l_i = \frac{\alpha - \lambda(\epsilon_i)}{\gamma_i\sigma^2 - \alpha + \lambda(\epsilon_i)}$$

$$u_i = \frac{1}{1-\epsilon_i}\frac{\alpha + \lambda(\epsilon_i)}{\gamma_i\sigma^2 - \alpha - \lambda(\epsilon_i)}$$

where, up to $O(\epsilon_i^{4/3}$

$$\lambda(\epsilon_i) = \gamma_i\sigma^2\left(\frac{3}{4\gamma_i}K^{*2}(1-K^*)^2\right)^{1/3}\epsilon_i^{1//3} + \sigma^2\left(\frac{5-2\gamma_i}{10}K^*(1-K^*) - \frac{3}{20}\right)\epsilon_i)$$

$$K^* = \alpha/\gamma_i\sigma^2.$$

In the latter case ϵ_i is chosen so as to satisfy (15).

Proof. The user optimization problem has a unique solution if ODE (6) with value matching and smooth pasting BCs (7) and (8) holds. We proved that the same is true for the miner if the conditions of Theorem 1 hold. The market for both assets clears since we posited that the position of the one party in both asset is the opposite of the other party's one. ■

The equilibrium does not admit forks because miners are identical (a representative one exists).

Nor an equilibrium with zero neither one with 100% fees exist:

Corollary 3 *Under the conditions of the previous theorem, an equilibrium with zero or maximum fees (*100%*) does not exist.*

Proof. Lack of existence of an equilibrium with $\epsilon_i = 0\%$ has been proved in the previous Section. With $\epsilon_i = 100\%$ the value matching condition for the user cannot be satisfied. ■

[9] We could also start from a position for the investor at time 0 different from the intervention one. In that case, as observed in Dumas and Luciano (1991), the conditions should be modified to take the initial adjustment to the barrier into consideration.

4 Empirical analysis

The empirical analysis of the BTC market consists in calibrating the model to its actual fees and prices, in order to show for which variables there is volatility amplification.

4.1 Data

The inputs for the empirical analysis are as follows. The transaction fee and volume data are public, and come from the BTC.com website. We take all the daily "blocks of transactions", which correspond to our transactions[10]. They come at high frequency, approximately a block of transactions each 5 minutes from June 1 to July 7 2022. This makes a total of 37 days, since the market is open 24/7, and approximately 4500 transaction blocks. For each block of transactions we have the timing (up to the second), fee in BTC and total transaction volume in BTC. The volume is the number of BTC transacted, dL or dU in our terminology. We compute the percentage fee ϵ_i as a percentage of the BTC fee to the transaction volume.
We take the monthly exchange rate $/BTC from Yahoo Finance over the period October 2014 - June 2023. Over the same time span, we take as riskless rate the T-bill one, from the FED, at the beginning of each month.

4.2 Preliminary statistical analysis

First of all, we verify that log returns on the BTC price in dollars are normal, and compute the corresponding drift and diffusion. Figure 1 presents the histogram of returns, the theoretical Gaussian distribution which best fits it (upper left), the comparison between the corresponding distributions (lower left), the qq-plot of the monthly returns, which shows no departure from Gaussianity (upper right), and the probabilities of the Gaussian best fit distribution compared to the empirical ones (lower right), over the period under exam.

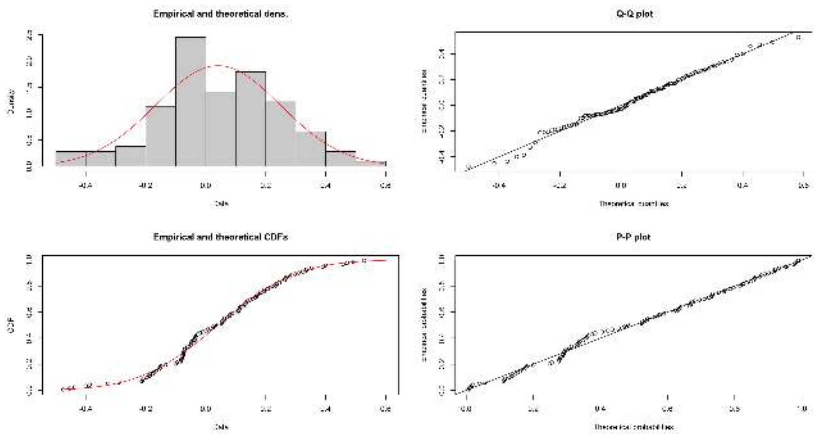
The Jarque-Bera test provides a p-value of 0.9211, which, together with the plot, is a clear evidence of Gaussianity.

The corresponding α parameter, computed by taking the average log return over the same time span, in excess of the monthly, variable riskless rate over the same period, and annualizing it, is 0.11742. The riskless rate in turn is computed by taking the beginning-of-the month rate on the T-Bills, as described above, over the same time span used for the $/BTC exchange rate, and annualizing it. It varies over the period under exam and its average is $r = 0.031028$. As a result, the return on BTC is $\alpha + r = 14.8\%$.

The standard deviation of the exchange rate obtains from the same data. The daily or intrinsic volatility is $\sigma/\sqrt{365} = 0.72904/\sqrt{365} = 0,03816$, which is already very high.

[10]In principe we should distinguish buy and sell orders. Block data come without that specification. This is not a problem for us, since we simplified the model so as to have the difference between the sell and buy price.

Figure 1: Distribution fit for monthly returns



The Figure presents the histogram of monthly returns on BTC and the theoretical Gaussian distribution which best fits it in the upper left plot, with the values of returns on the horizontal axis and frequencies or their density on the vertical; the comparison between the corresponding distributions, with the values of returns and the cumulative frequencies or density in the lower left plot; the qq-plot of returns, with empirical and theoretical quantiles on the axes,in the upper right plot; the pp-plot of returns with empirical and theoretical best-fit probabilities in the lower right plot.

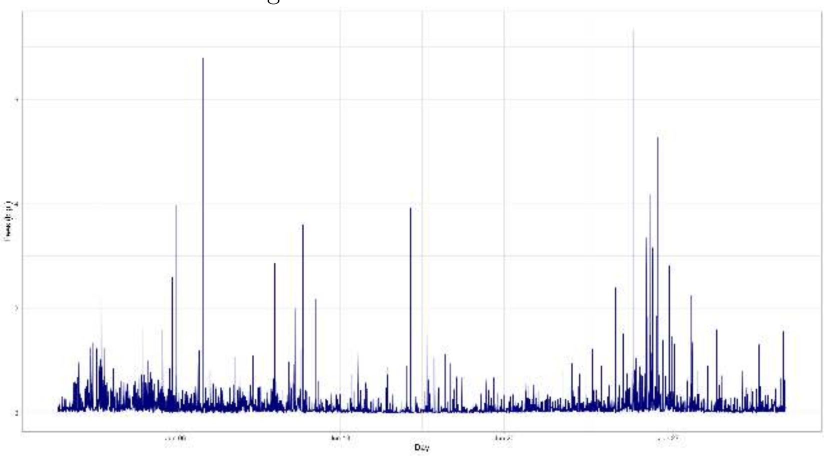
The drift and standard deviation of the crypto satisfy the hypothesis of Theorem 1, so that we expect a unique equilibrium.
Let us examine now the fees. Actual fees are different, because ach time the miner can trade with a different user i.
We take the time series of the high frequency fees from the BTC.com website. Once we get rid of the negligible transactions, we get a time series for ϵ_i with 4418 data points, which confirms that the fee is never 0 or 100%, consistently with Corollary 3. The fee behavior is highly erratic, even at first sight, as one can see from Figure 2. In Figure 2, fees are in basis points:

Figure 3 is an histogram of the same fees, in basis points, which reveals high skewness and kurtosis.

To give the fee statistics, we give the range over over the 37-days period of the daily statistics:[11]

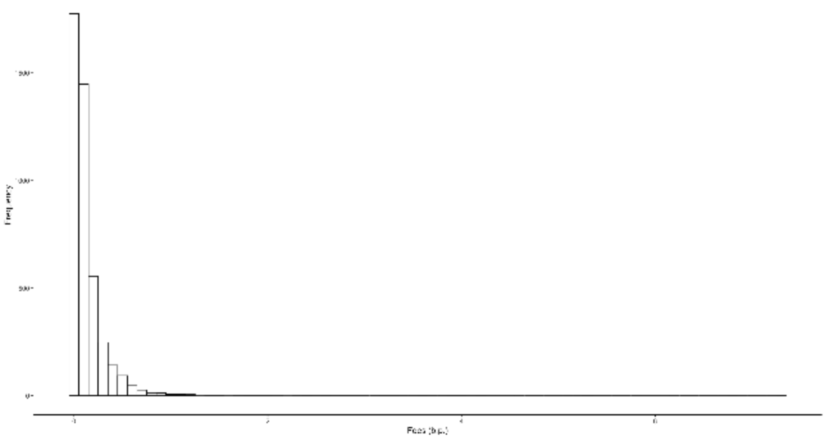
[11] We compute the daily realized variance of the fees using the `rRVar` function from the `highfrequency` package in `R`. Let $r_{i,t}$ be a vector of intraday returns over M transactions; then the daily Realized Variance is computed as $RVar_t = \sum_{i=1}^{M} r_{i,t}^2$.

Figure 2: Time Series of Fees



The Figure has time on the horizontal axis and the observed fee in basis points on the vertical.

Figure 3: Histogram of Fees

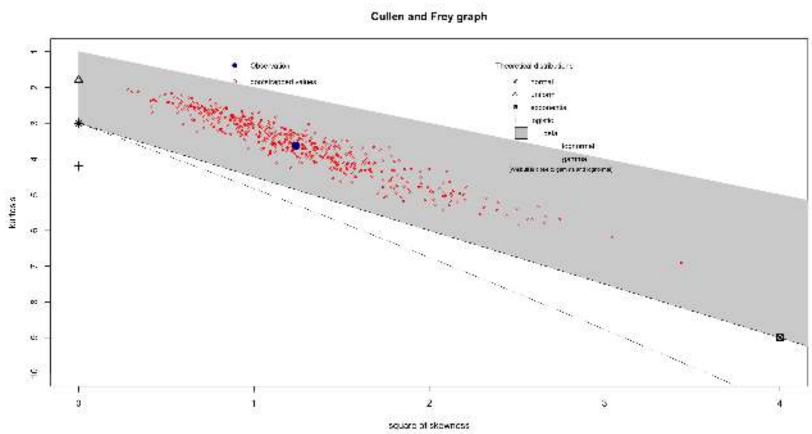


The Figure has buckets for the fees (the first starting at $\epsilon_i > 0$) on the horizontal axis and the corresponding frequencies on the vertical.

VARIABLE	minimum	maximum
daily fee minimum	1.289619e-07	2.143819e-06
daily fee maximum	5.359737e-05	0.000731819
average	4.579043e-06	4.105258e-05
realized variance	162.568	449.5017
skewness	1.101843	11.33901
kurtosis	3.567268	138.2703

In Figure 4 we deepen the analysis on the distribution of fees over a single day (here, June 3), through a Cullen Frey graph. Bootstrapping the returns one can infer that the range of couples square of skewness - kurtosis they admit is between zero and 3, 1 and 9 respectively. The graph also shows that both the uniform, Gaussian and logistic distribution are unable to capture such couples, while the lognormal, gamma and exponential underestimate the kurtosis with respect to the skewness intrinsic in the data. The beta distribution, varying its parameters, is instead able to reproduce all the couples in the grey zone, and therefore the couples corresponding to the data. Similar plots hold for the other days. We conclude that the best-fit distribution for the fees is a beta.

Figure 4: Cullen Frey Graph for fees - June 3



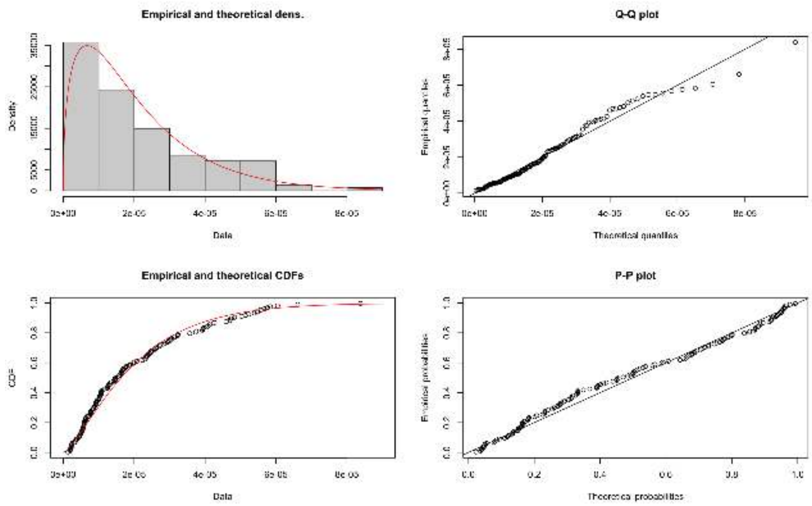
The Figure represents the kurtosis as a function of the squared asymmetry in correspondence to the observed fees and the bootstrapped ones, on June 3, as well as the couples kurtosis-squared-asymmetry for some fitted distributions. The latter are single points for all distributions but the beta, which encompasses a whole family of distributions, and is therefore represented by the grey zone, when its parameters vary.

Figure 5 reproduces the analysis carried over returns over the June 3 fees: it presents the histogram of fees, the theoretical beta distribution which best fits it (upper left), the comparison between the corresponding distributions (lower left), the qq-plot of the fees, which shows high departure from Gaussianity (upper right), and the probabilities of the beta best fit distribution compared to the empirical ones (lower right), the beta parameters are $m = 1.476931$ and $n = 70965.9$.

Let us now compare the intrinsic volatility, which is $\sigma/\sqrt(365)$, and represents the volatility of the price before the fees, S, with the one of the cum-fee price, $(1-\epsilon_i)S$, which depends also on the realized vol of ϵ_i. [12] The former is

[12] To compute the variance of $(1-\epsilon_i)S$, we assumed independence of S and ϵ_i and we used

Figure 5: Distribution fit for fees - June 3



The Figure presents the histogram of fees and the theoretical beta distribution which best fits it in the upper left plot, with the values of fees on the horizontal axis and frequencies or their density on the vertical; the comparison between the corresponding distributions, with the values of returns and the cumulative frequencies or density in the lower left plot; the qq-plot of fees, with empirical and theoretical quantiles on the axes,in the upper right plot; the pp-plot with empirical and theoretical best-fit probabilities in the lower right plot.

3.816%, while the latter ranges from 196 to 326%, 2 orders of magnitude greater. Volatility amplification is at work and makes overall volatility higher then the intrinsic one, as in Biais et al. (2022). While in Biais volatility amplification is generated by sunspots, in our model the market structure is modeled, the fees are endogenous and the market interaction in equilibrium is responsible for the volatility amplification.

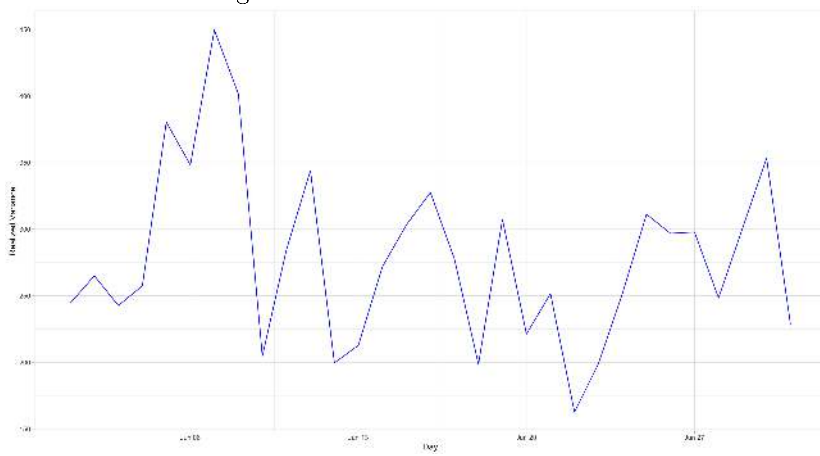
Also, Figure 6 shows that the realized volatility itself is highly unstable over time: not only there is volatility amplification, but it changes day after day in a remarkable way. In the Figure, the horizontal axis contains the dates bewteen June 1 and July 7, while the vertical axis plots the daily realized variance.

4.3 Transaction behavior and equilibrium

This Section studies the transaction behavior consistent with the BTC market data at hand, assuming for simplicity that all investors are equal, or that there exists a representative user, with a moderate risk aversion $\gamma_i = 4$. At th end of

either the max or min of the realized variance of ϵ_i and of the intraday average of $(1 - \epsilon_i)$ in all computations.

Figure 6: Realized Variance of Fees



The Figure represents the daily relaized variance of the fees as a function of time, over the observation period.

the Section we increase it to 10, to perform a robustness check. In the absence of fees, the optimal allocation to BTC with respect to dollars of the representative user, θ^*, would be $\alpha/(\gamma_i\sigma^2 - \alpha) = 0.058459$. Because the risk tolerance is high, the intrinsic vol is so high that the optimal exposure to the BTC would be low even without costs.

To perform our analysis, we then compute the departure from the optimal allocation above, λ, in correspondence to all realizations of ϵ in the high-frequency time series. We use the first order approximation[13]

$$\gamma^{2/3}\sigma^2 \left(\frac{3}{4}\right)^{1/3} K^{*2/3}(1-K^*)^{2/3}\epsilon_i^{1/3} \tag{18}$$

where, in correspondence to the base risk aversion parameter, $K^* = \alpha/(\gamma_i\sigma^2) = 0.05523$. The statistics of the departure are as follows:

VARIABLE	minimum	maximum
daily departure minimum	0.0008907159	0.002273255
daily departure maximum	0.006647116	0.01588742
average	0.0024426	0.005179375
realized variance	18.06312	49.94463

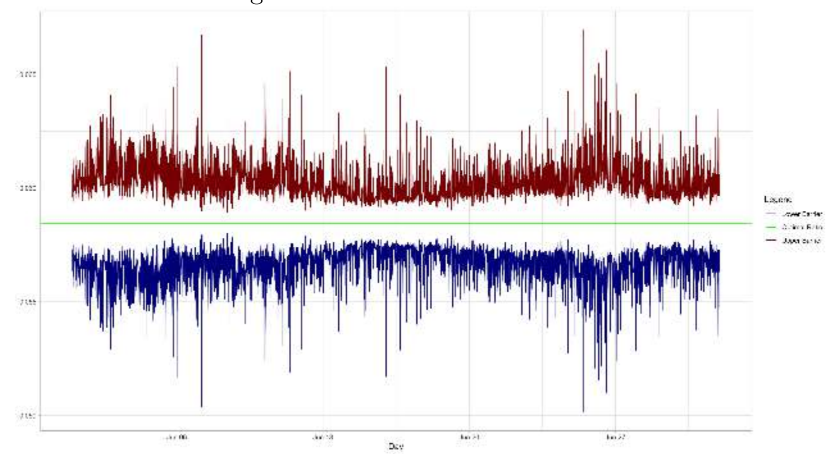
The departure is again very volatile, since it depends on the fees.

We compute the corresponding transaction barriers l_i and u_i using the formulas

[13]The second order approximation is hardly distinguishable from the first, given the order of magnitude of the fees in the sample.

in Theorem 2. Figure 7 presents their time series, which immediately shows that the barriers are quite stable in value, around the optimal ratio:

Figure 7: Time Series of Barriers



The Figure has time on the horizontal axis, the upper and lower barrier implicit in each transaction on the vertical, separated by an horizontal line representing the optimal allocation without costs.

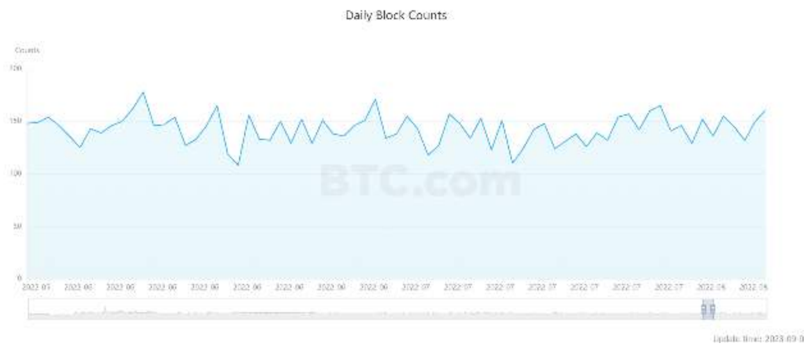
We get the following statistics for them and for the distance from the no-cost allocation:

VARIABLE	minimum	maximum
l_i minimum	0.05015355	0.05496866
l_i maximum	0.05726331	0.05799071
average	0.05573876	0.05717459
realized variance	0.01605102	0.106425
u_i minimum	0.05892946	0.05965927
u_i maximum	0.06197758	0.06694763
average	0.0597492	0.06120004
realized variance	0.01441195	0.085363

VARIABLE	minimum	maximum
$\theta^* - l_i$ minimum	0.05015355	0.05496866
$\theta^* - l_i$ maximum	0.003491207	0.008306324
average	0.001285275	0.002721112
$u_i - \theta^*$ minimum	0.0004695869	0.001199398
$u_i - \theta^*$ maximum	0.003517712	0.008487765
average	0.001289333	0.00274017

The vol of l_i stays within 12.6 and 32.6%, the one of u_i within 12 and 29%, only one order of magnitude bigger than the one of the price before the fees. These statistics confirm that the barriers, as a result of the optimizing behavior of the user, are much more stable - less volatile - than the fees and cum-fee price. As a result of the barriers staying quite constant, we can expect a number of daily transactions quite constant, and an expected time to next trade with low volatility, as the actual market shows. Figure 8 is indeed a time series of the number of block transactions executed per day, which stays quite flat in the observation period

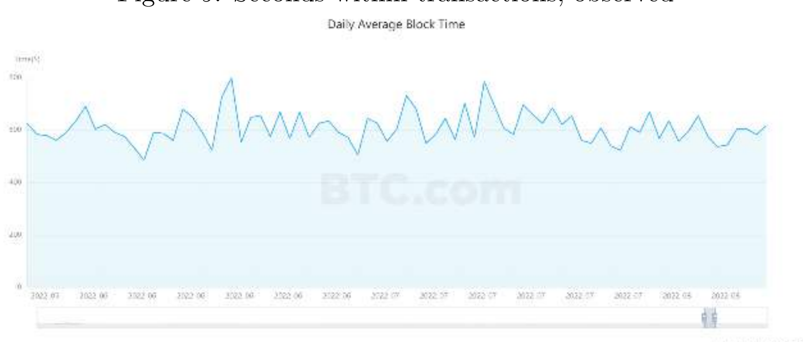
Figure 8: Number of transactions per day



The Figure is taken from BTC.com and represents the number of (block) transactions per day over the observation period.

Figure 9 confirms the message from Figure 8 and our motivation for that. It represents the time within the single transactions. It shows how long (in seconds) one has to wait to see the next trade, over the observation period

Figure 9: Seconds within transactions, observed



The Figure is taken from BTC.com and represents the time between successive (block) transactions over the observation period.

On top of the computations above for the barriers and their volatility, we can look at turnover measures. Indeed, we can measure the effect of the fees on the turnover of both the BTC and overall wealth.

The BTC turnover up to T for users - called Share-turnover or $ShTu$, is defined, similarly to Gerhold et al. for stocks, as the ratio of the traded number of BTCs to the total number of BTC held, up to T. Under our assumption of homogeneous investors, it depends on the fees because the barriers and trade do.

$$ShTu(\epsilon_i, T) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{dL(t) + dU(t)}{C(t)} dt$$

Similarly, the wealth turnover can be defined as the ratio of the traded value of wealth to the total liquidation value of wealth

$$WeTu(\epsilon_i, T) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{S(t)dL(t) + (1-\epsilon_i)\, S(t)dU(t)}{C^0(t)\exp(rt) + (1-\epsilon_i)C(t)S(t)} dt$$

It differs from the BTC turnover first of all because it is given in terms of values and not pure quantities, and it keeps into consideration that proportional fees have to be paid to liquidate also final wealth.

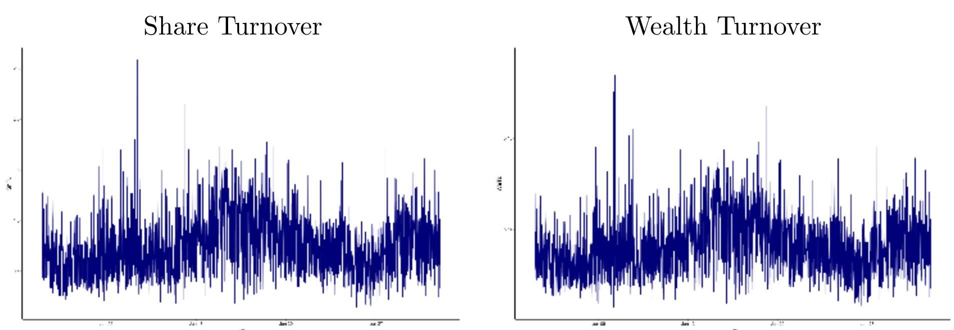
For small fees, the previous measures can be approximated by

$$ShTu = 2^{-1/3}\,(1-K^*)^{4/3}\, K^{*4/3} \sigma^2 \gamma_i^{1/3} \epsilon_i^{-1/3}$$

$$WeTu = K^* ShTu$$

Applying this formula as if any given realization of ϵ in the data would be replicated in the future, from the latter we have the time series of the turnovers in Figure 10:

Figure 10: Time Series of ShTu and WeTu



Each plot in the Figure has time on the horizontal axis and the turnover measure
for each single transaction on the vertical one

Computing the average on a single day we obtain for the turnovers the following statistics:

VARIABLE	minimum	maximum
daily $ShTu$ minimum	0.1448992	0.3463267
daily $ShTu$ maximum	1.012677	2.584521
average	0.7311611	
daily $ShTu$ minimum	0.008002935	0.01912799
daily $ShTu$ maximum	0.05593124	0.1427458
average	0.04038281	

Even though the minimum and maximum turnover vary with the fee, their average is quite low. This confirms that, even with highly volatile fees, the trade behavior is quite stable, no matter how you measure it. The optimizing behavior of agents seems to be at work, and they seem to react to volatile fees by adjusting trade not to incur in too high costs.

5 Equivalent safe rate

The last quantity we define and calibrate to data, under the symplifying assumption of homogeneous investors with moderate risk aversion, is the equivalent safe rate (ESR), namely the rate on an hypothetical safe asset which would give to users the same utility level as the equilibrium transaction policy in cryptos, with fees. This ESR has been shown by Gerhold et al. (2011) to be

$$r + \frac{\alpha^2 - \lambda^2}{2\gamma_i \sigma^2}$$

Considering the approximation (18) of λ, the ESR can be approximated by

$$r + \frac{\alpha^2}{2\gamma_i \sigma^2} - \frac{\gamma_i^{1/3}\sigma^2}{2^{-7/3}} 3^{2/3} \epsilon_i^{2/3} K^{*4/3}(1 - K^*)^{4/3}$$

By computing the ESR first for each transaction, and then taking the average over the observation period, we get 0.03504638. As a combined result of quite stable trading policies, and fees low in value but high in variance, we have quite a modest ESR. Users would give away BTCs, whose expected return in price is 14.8%, with a high volatility on top (σ) for a safe asset which returns a much more humble 3.5%.

As a final robustness check, let us increase the risk aversion of users from 4 to 10. Not only the ESR is almost unaffected, but also the trading behavior and its stability do remain, as the following statistics show.

(i) ESR, average of averages is now 0.03309855

(ii) min ranges $ShTu$ goes from 0.06068604 to 0.145047, $WeTu$ from 0.001340702 to 0.003204441

(iii) max ranges $ShTu$ goes from 0.4241251 to 1.082438, $WeTu$ from 0.009369953 to 0.02391368.

(iv) average $ShTu$ 0.3062217.

This is done as a robustness check for the analysis.

6 Summary and conclusions

We have formalized and studied a single equilibrium in a market for cryptocurrencies, of the Bitcoin type. The equilibrium is characterized through its optimal trading policies and fees. Trade is carried out discountinuously in time. Trading times either correspond to the hashing time or the maximum between it and the first time that the incoming user's ratio of BTC to riskless assets reaches the tolerance bounds corresponding to the declared fee, if he declares it. If the miner declares the fees, he adjusts it to the actual ratio of the user. We have calibrated the model to BTC.com high-frequency data and explained the volatility amplification from the prices without the fees to the ones with fees, which is two orders of magnitude greater. We have also explained the relative stability of the implied optimal policies. The equivalent safe rate is a modest 3.5%, since quiet trading with extremely volatile prices, once the fees are included, do erode utility from the expected return on BTCs of 14.8%.

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