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NASH EQUILIBRIA FOR DIVIDEND DISTRIBUTION WITH COMPETITION

TIZIANO DE ANGELIS, FABIEN GENSBITTEL, STÉPHANE VILLENEUVE

Abstract. We construct a Nash equilibrium in feedback form for a class of two-person stochastic games with absorption arising from corporate finance. More precisely, the paper focusses on a strategic dynamic game in which two financially-constrained firms operate in the same market. The firms distribute dividends and are faced with default risk. The strategic interaction arises from the fact that if one firm defaults, the other one becomes a monopolist and increases its profitability.

To determine a Nash equilibrium in feedback form, we develop two different concepts depending on the initial endowment of each firm. If one firm is richer than the other one, then we use a notion of control vs. strategy equilibrium. If the two firms have the same initial endowment (hence they are symmetric in our setup) then we need mixed strategies in order to construct a symmetric equilibrium.

1. Introduction

In this paper, we construct competitive Nash equilibria for a two-player nonzero-sum game of singular controls with an exogenous absorbing boundary for the state-dynamics of each player. Players' optimal strategies in Nash equilibria are obtained as a fixed point of so-called best reply maps but what constitutes an admissible map depends crucially on the game's formulation and on the class of players' admissible actions. We assume that players have complete information about the dynamics of the system (including the initial states), the class of admissible controls and the game's payoffs. A delicate point in our analysis concerns the rigorous mathematical formulation of the information available to each player regarding the strategy played by their opponent. In this respect we introduce in Definition 3.1 a notion of strategies in feedback form which we refer to as *control vs. strategy*. Remark 3.2 and Remark 3.3 offer a careful comparison of our strategies to standard game-theoretical notions.

From a modelling perspective, we build on the modern formulation of De Finetti's dividend problem [8] which is the most popular application of singular control theory in corporate finance. We consider two identically-efficient firms acting on a single-good market in which the demand for the good is random. Both firms have a capital evolution which is driven by the same arithmetic Brownian motion (aBm). Firms are not identical though, because they may have different initial endowments and they can choose different dividend policies. At time zero, the two firms are in duopoly but if/when one of the two firms defaults, the surviving firm becomes a monopolist with an increase of its profitability. Default of a firm occurs when its cash reserves are depleted, which corresponds to the firm's state process reaching zero value for the first time. We model the duopoly/monopoly transition with a change of drift in the aBm of the surviving firm. In this context the standard trade-off between exerting controls (paying dividends) and keeping a high level of cash reserves is exacerbated by the presence of a rival and the prospect of becoming monopolist. In this paper we focus on understanding how competition impacts on firms optimal dividend policies. In particular, we observe non-trivial deviations from the optimal dividend policy of the De Finetti's dividend model.

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In recent years, there is a concern, particularly in the digital economy, that cash-rich firms may engage in predatory strategies to drive poorer rivals out of the market and thus benefit from monopolistic profits. In the deep-pocket theory of predation (see [15] for a general presentation), predatory behavior may arise when a firm adopts a strategy intended to induce the exit from the market of a financially constrained competitor by depletion of its resources. Our paper contributes to this literature from a theoretical perspective by analyzing how a richer firm adapts its dividend strategy to ensure that its rival goes bankrupt.

There exists an abundant literature on single-agent singular control problems, dating back to seminal work by, e.g., Bather and Chernoff [2], Benes et al. [3], Karatzas [14] and many others. In terms of applications in corporate finance, the two papers Jeanblanc and Shiryaev [13] and Radner and Shepp [18] set the benchmark case for the analysis of corporate cash management in continuous time based on the work of De Finetti's [8]. In the pioneering papers [13] and [18], the cumulative net cash flow generated by the firm follows an arithmetic Brownian motion. The constant drift represents the firm's profitability per unit time, the Brownian motion carries the uncertainty and external financing is costly, which creates a precautionary demand for cash. Our paper is set within this simplified modelling framework but it extends the classical setup from [13] and [18] to one with two competing firms.

From a mathematical perspective, the literature on nonzero-sum stochastic games of singular control is still in its infancy. Kwon and Zhang [16] find Markov perfect equilibria in a game of competitive market share control, in which each player can make irreversible investment decisions via singular controls as well as deciding to strategically exit the market. De Angelis and Ferrari [7] and Dammann et al. [9] obtain Nash equilibria in the class of Skorokhod-reflection policies for a nonzero-sum game where two players control the same one-dimensional state dynamics. In [7] an equilibrium is found by establishing a connection between the nonzero-sum game of monotone controls and a nonzero-sum stopping game. In [9] an equilibrium is found by solving a free boundary problem. There are two important differences of our work compared to [7], [9] and [16]. The two players in those papers control the same dynamics and no default may occur (in [16] players may decide to exit the game and therefore the transition from duopoly to monopoly occurs *only* because of optimality considerations). In our paper instead each player controls her own dynamics and default may occur also in the absence of control (actually, controlling will increase the probability of default). Each player in [7] and [9] chooses a point on the real line and exerts control in order to reflect the dynamics at that point (one player pushes the dynamics upwards and the other one pushes it downwards). Equilibria in [16] are sought in a class of *barrier* strategies, which is close in spirit to the classical Skorokhod reflection. Another related paper is by Ekström and Lindensjö [11]. They study a N-player competitive game in an extraction problem from a common resource with Brownian dynamics. Ekström and Lindensjö find a Nash equilibrium for a class of regular Markov strategies of bang-bang type. That is, each player extracts at the maximum rate when the controlled dynamics is above a certain critical value. Since the game is symmetric, it turns out that all players act simultaneously (i.e., they all choose the same critical value). All players control the same dynamics and therefore they all default at the same time once the resources are depleted. Again, our setup is different because each player controls their own cash reserve and we allow for singular controls. Finally, we emphasise that [7], [9], [11] and [16] stipulate to look for equilibria in various classes of threshold policies. We do not make such an ansatz in our paper.

Now, we describe the structure of our solution and highlight its economic interpretation. When the two firms have different initial endowment our *control vs. strategy* Nash equilibrium is fully characterized by solving two interconnected free-boundary problems. The free-boundary for the "poorer" firm is constant whereas the free-boundary of the "richer" firm moves with the state-variable associated to the other firm's cash-reserves. Along the equilibrium trajectory, the poorer firm acts as if it were alone in the single-good market by following the classical optimal policy from [13] and [18] (we refer to it as the De Finetti's dividend policy). The richer firm instead controls the level of its cash reserves in order to stay ahead of its rival, making sure that the other firm defaults first. In doing so, the richer firm adapts its dividend policy whenever the poorer firm deviates from equilibrium. On the contrary, once the richer firm's strategy is given, the poorer firm's best response across all dividend policies (be it control or strategy) is to use the De Finetti's dividend policy. We emphasise that our equilibrium is sub-game Markov perfect, in the sense that the equilibrium strategies are Markovian (jointly with the controlled dynamics), they form an equilibrium for any starting point $Y_0 > X_0$ and guarantee that after time zero the controlled dynamics are always ordered as $Y_t > X_t$ (thus, optimality is preserved at any future time).

When the two firms have the same initial endowment our game is fully symmetric. As soon as one of the two firms distributes dividends, the symmetry is broken and the firms are back into the previous asymmetric situation. Therefore, the crucial step in the construction of an equilibrium is the choice of the time at which each firm starts paying dividends. In order to construct a symmetric equilibrium we allow firms to start making dividend payments at a randomized stopping time. This requires to characterize the equilibrium intensity of stopping as well as to rigorously define the actions of each player after the randomised stopping time within our context of *control vs. strategy* equilibria.

From an economic viewpoint, our paper highlights how a company adapts its dividend policy to the presence of a competitor. In a nutshell, we conclude that cash-rich firms are less likely to pay dividends, because its shareholders have more interest in waiting for their competitor to go bankrupt than in receiving early dividends.

The paper is organised as follows. In Section 2 we set up the problem and recall some useful facts about the classical dividend problem. In Section 3 we introduce our notions of equilibrium in *control vs. strategy* and we construct an equilibrium for the game with firms having different initial endowments. In Section 4 we introduce randomisation and construct a symmetric equilibrium for firms with the same initial endowment. A short Appendix with a small technical result completes the paper.

2. Problem setting

On a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ we consider a standard 1-dimensional Brownian motion $(B_t)_{t\geq 0}$ and denote $(\mathcal{F}_t)_{t\geq 0}$ its filtration augmented with P-null sets. We have two firms operating on the same market and whose cash reserves increase at a rate $\mu_0 > 0$ but are subject to a volatility $\sigma > 0$. The firms could default if their cash reserve drops below zero. In that case, the surviving firm becomes a monopolist, resulting in an higher rate of increase of its cash reserve, i.e., $\hat{\mu} > \mu_0$.

We denote by $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ the cash reserve dynamics of the first and second firm, respectively. Then, for $t \geq 0$ and $x, y \in [0, \infty)$, we have

$$\begin{aligned} X_t &= x + \mu_0 t + \sigma B_t - L_t \\ Y_t &= y + \mu_0 t + \sigma B_t - D_t \end{aligned} \tag{2.1}$$

where L_t is the cumulative amount of dividends paid by the first firm up to time t, and D_t is the analogue for the second firm. We will often use X^L and Y^D to emphasise the dependence of the processes on their controls. The processes L and D are drawn from the admissible class defined next:

Definition 2.1 (Admissible dividend policies). *A pair of processes $(L_t, D_t)_{t\geq 0}$ is called a* pair of admissible dividend policies *if L and D are non-decreasing, adapted to $(\mathcal{F}_t)_{t\geq 0}$ and right-continuous. Moreover, letting $(x)^+ := \max\{0, x\}$, it must hold*

$$L_t - L_{t-} \leq (X^L_{t-})^+ \quad \text{and} \quad D_t - D_{t-} \leq (Y^D_{t-})^+, \quad \text{for all } t \geq 0,\ \mathsf{P}\text{-a.s.} \tag{2.2}$$

Condition (2.2) ensures that the firms cannot pay dividends in excess of their cash reserve[1]. We denote default times by γ_X and γ_Y, with

$$\gamma_X = \inf\{t \ge 0 : X^L_t \le 0\} \quad \text{and} \quad \gamma_Y = \inf\{t \ge 0 : Y^D_t \le 0\}. \tag{2.3}$$

Finally, we denote $\mathsf{P}_{x,y}(\,\cdot\,) = \mathsf{P}(\,\cdot\,|X_{0-} = x, Y_{0-} = y)$.

Given a pair (L, D) of admissible dividend policies the expected payoffs $\mathcal{J}^1$ and $\mathcal{J}^2$, for the first and second firm, respectively, read

$$\begin{aligned}
\mathcal{J}^1_{x,y}(L, D) &:= \mathsf{E}_{x,y}\Big[\int_{[0,\gamma_X\wedge\gamma_Y]} \mathrm{e}^{-rt}\mathrm{d}L_t + 1_{\{\gamma_Y<\gamma_X\}}\mathrm{e}^{-r\gamma_Y}\hat{v}(X^L_{\gamma_Y})\Big],\\
\mathcal{J}^2_{x,y}(D, L) &:= \mathsf{E}_{x,y}\Big[\int_{[0,\gamma_X\wedge\gamma_Y]} \mathrm{e}^{-rt}\mathrm{d}D_t + 1_{\{\gamma_X<\gamma_Y\}}\mathrm{e}^{-r\gamma_X}\hat{v}(Y^D_{\gamma_X})\Big].
\end{aligned} \tag{2.4}$$

Here, $r > 0$ is a discount rate and $\hat{v}$ is the value function of the classical dividend problem with cash reserves growing at the rate $\hat{\mu}$. That is

$$\hat{v}(x) := \sup_{\xi} \mathsf{E}_x\Big[\int_{[0,\gamma_C]} \mathrm{e}^{-rt}\mathrm{d}\xi_t\Big], \tag{2.5}$$

with underlying dynamics given by

$$C^\xi_t = x + \hat{\mu}t + \sigma B_t - \xi_t, \quad t \ge 0, \tag{2.6}$$

and with $\gamma_C = \inf\{t \ge 0 : C^\xi_t \le 0\}$. The supremum is taken over all admissible dividend policies, according to Definition 2.1. Here we emphasise the dependence of the problem's structure on the drift of the underlying process, because we will later use results for the dividend problem when the drift is either μ_0 or $\hat{\mu}$. In particular, we will use the notation $\hat{v}(x) = w(x; \hat{\mu})$ and $C^\xi = C^{\hat{\mu};\xi}$, when convenient. An account of useful facts about the classical dividend problem will be provided below in Section 2.1.

The integral term in each one of the two payoffs in (2.4) is the discounted value of the cumulative dividends paid by the firm until both firms are active. At the (random) time $\gamma_X \wedge \gamma_Y$ one of the two firms goes bankrupt and the surviving firm is a monopolist[2] with a larger cashflow rate $\hat{\mu}$. At this point in time, say $\gamma_X < \gamma_Y$, the remaining firm (i.e., firm 2) is faced with the classical dividend problem but with an initial cash reserve $Y^D_{\gamma_X}$. Hence, the payoff at time γ_X reads $\hat{v}(Y^D_{\gamma_X})$.

Using admissible dividend policies, the two firms' managers are faced with the optimisation problems:

$$v_1(x, y; D) = \sup_L \mathcal{J}^1_{x,y}(L, D) \quad \text{and} \quad v_2(x, y; L) = \sup_D \mathcal{J}^2_{x,y}(D, L), \tag{2.7}$$

where v_1 refers to the optimal expected payoff of the first firm and v_2 to the second one. Our problem is a nonzero-sum game of singular controls with absorption and our aim is to construct a Nash equilibrium.

Since we deal with a game, we will unambiguously refer to the first and second firm as first and second player, respectively. By the symmetry of the set-up it is clear that the player with the largest initial cash reserve has an advantage on her opponent. We will show that this allows a rather explicit construction of a Nash equilibrium in control vs. strategy (Definition 3.1). Furthermore, in the completely symmetric situation in which $x = y$, we will construct a symmetric equilibrium in which the players use randomised strategies.

[1]Condition (2.2) could be replaced by the weaker condition $X^L_{\gamma_X} = 0$ on $\{\gamma_X < \infty\}$, or equivalently $L_t - L_{t-} \le X^L_{t-}$ for all $t \le \gamma_X$, as only the trajectory up to γ_X is relevant. Our choice is motivated by Definition 3.1 of controls and strategies as functionals on the canonical space, in which for simplicity we avoid to define controls and strategies only up to a stopping time.

[2]If $\gamma_X = \gamma_Y$ then no firm survives and the continuation payoff for both players is zero. One may equivalently notice that the monopolist's payoff is equal to zero when the initial cash reserve is zero.

2.1. **Useful facts about the classical dividend problem.** Here we recall a few well-known results concerning the classical dividend problem for a generic drift μ of the cash reserve. In the notation of (2.5) and (2.6) we consider a generic value function $w(x;\mu)$ when the underlying dynamics $C^{\mu;\xi}$ has drift μ (so that for (2.5) we have $\hat{v}(x) = w(x;\hat{\mu})$). All the results listed here can be found, for instance, in [20, Ch. 2.5.2].

It is well-known that the optimal dividend policy in the classical dividend problem is of the form

$$\xi^*_t = \xi^*_t(\mu) := \sup_{0\le s\le t}\Big(x - a_* + \mu s + \sigma B_s\Big)^+, \quad \xi^*_{0-} = 0, \tag{2.8}$$

where $a_* = a_*(\mu)$ is an optimal boundary and dividends are paid so that the cash reserve process $(C^{\mu;\xi^*}_t)_{t\ge 0}$ is reflected downwards at a_*. The solution is generally constructed by showing that the value function w belongs to the class $C^2([0,\infty))$ and that the pair (w, a_*) is the unique solution of the free boundary problem

$$\begin{aligned}
&\tfrac{\sigma^2}{2}w''(x;\mu) + \mu w'(x;\mu) - rw(x;\mu) = 0, \quad x \in (0, a_*(\mu)),\\
&\tfrac{\sigma^2}{2}w''(x;\mu) + \mu w'(x;\mu) - rw(x;\mu) \le 0, \quad x \in [a_*(\mu), \infty),\\
&w'(x;\mu) \ge 1 \text{ for all } x \in [0,\infty),\\
&w'(x;\mu) > 1 \iff x \in (0, a_*(\mu)),\\
&w(0;\mu) = 0.
\end{aligned} \tag{2.9}$$

The value of the optimal boundary $a_*(\mu)$ is determined by the smooth-pasting condition

$$w''(a_*(\mu);\mu) = 0 \tag{2.10}$$

and it can be calculated explicitly. In order to simplify the notation, we omit the dependence on μ from w and a_* when no confusion shall arise.

For $x \in (0, a_*)$ the expression for w reads

$$w(x) = C\big(e^{\beta_1 x} - e^{\beta_2 x}\big), \tag{2.11}$$

where $C = C(\mu) > 0$ is a constant that can be determined explicitly, while $\beta_1 = \beta_1(\mu) > 0 > \beta_2(\mu) = \beta_2$ are the two roots of the equation $\frac{\sigma^2}{2}\beta^2 + \mu\beta - r = 0$.

Finally, we notice that the conditions $w'(a_*) = 1$ and $w''(a_*) = 0$ and the first equation in (2.9) imply $rw(a_*) = \mu$. Since w is non-decreasing, then

$$rw(x) > \mu \iff x \in (a_*, \infty). \tag{2.12}$$

Moreover, the condition $w'(x) = 1$ for $x \ge a_*$ leads to $w(x) = (x - a_*) + w(a_*)$ for $x \ge a_*$. Then, simple algebra yields, for $x \in [0,\infty)$,

$$\tfrac{\sigma^2}{2}w''(x) + \mu w'(x) - rw(x) = -r[x - a_*]^+ = -[rw(x) - \mu]^+, \tag{2.13}$$

where $[p]^+ := \max\{p, 0\}$.

There are two particular values of μ which will crop up in our analysis below, i.e., $\mu = \mu_0$ and $\mu = \hat{\mu}$, corresponding to the drift for the duopoly and for the monopoly, respectively. Then we denote

$$\begin{aligned}
\hat{v}(x) := w(x;\hat{\mu}), \quad v_0(x) := w(x;\mu_0), \quad \hat{a} := a_*(\hat{\mu}), \quad a_0 := a_*(\mu_0)\\
\hat{\xi}_t := \xi^*_t(\hat{\mu}) \quad \text{and} \quad \xi^0_t := \xi^*_t(\mu_0).
\end{aligned} \tag{2.14}$$

2.2. **Notation.** We close the section with some more notation which will be used throughout the paper. Given a set $A \subset \mathbb{R}^2$ we denote its closure by $\overline{A}$. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ and and open set $A \subset \mathbb{R}^2$ we write $f \in C^k(A)$ for $k \in \mathbb{N}$ to indicate that f is k times continuously differentiable in A. We write $f \in C^k(\overline{A})$ to indicate that the function f with all its k derivatives admit a continuous extension to the boundary ∂A. Given two open sets $A \subsetneq B$ in $\mathbb{R}^2$, letting $E := B \setminus \overline{A}$, for $k \in \mathbb{N}$ we use the notation $f \in C^k(\overline{A} \cup \overline{E})$ to indicate that $f \in C^k(\overline{A}) \cap C^k(\overline{E})$ with derivatives which may be discontinuous across the boundary ∂A.

3. Nash equilibrium with asymmetric initial endowment

Here we consider firms with different initial endowments. With no loss of generality we specifically address the case $y > x$. Moreover, we allow Player 2 to use a strategy (in the case $x > y$ it would be Player 1 using a strategy). For a proper definition of strategy we need to introduce two classes of functions. The class $C_0([0,\infty))$ represents continuous functions $\varphi : [0,\infty) \to \mathbb{R}$ with $\varphi(0) = 0$. The class $D_0^+([0,\infty))$ represents càdlàg non-decreasing functions $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0-) = 0$. We introduce the canonical space $C_0([0,\infty)) \times D_0^+([0,\infty))$ equipped with the Borel σ-algebra. The coordinate mapping on the canonical space is denoted $\mathbb{W}_t(\varphi,\zeta) := (\varphi(t),\zeta(t))$ for any $(\varphi,\zeta) \in C_0([0,\infty)) \times D_0^+([0,\infty))$ and $t \in [0,\infty)$. Its raw filtration is denoted $(\mathcal{F}_t^{\mathbb{W}})_{t\geq 0}$. We say that a mapping

$$\chi : [0,\infty) \times C_0([0,\infty)) \times D_0^+([0,\infty)) \to \mathbb{R}$$

is *non-anticipative* if $\chi(t,\varphi,\zeta)$ is $\mathcal{F}_t^{\mathbb{W}}$-measurable for all $t \geq 0$. In order to avoid further notation, in what follows we treat mappings $\Theta : [0,\infty) \times C_0([0,\infty)) \to \mathbb{R}$ as mappings defined on the canonical space but with no dependence on the coordinate process ζ.

Definition 3.1 (**Control and Strategy).** *A mapping*

$$\Phi : [0,\infty) \times C_0([0,\infty)) \to [0,\infty)$$

is an admissible control *for Player 1 with initial condition x if Φ is non-anticipative, $t \mapsto \Phi(t,\varphi)$ is càdlàg and non-decreasing for any $\varphi \in C_0([0,\infty))$, with the convention $\Phi(0-,\varphi) = 0$, and*

$$\Phi(t,\varphi) - \Phi(t-,\varphi) \leq \left(x + \mu_0 t + \sigma\varphi(t) - \Phi(t-,\varphi)\right)^+ , \tag{3.1}$$

for all $\varphi \in C_0([0,\infty))$ and $t \geq 0$.

A mapping

$$\Psi : [0,\infty) \times C_0([0,\infty)) \times D_0^+([0,\infty)) \to [0,\infty)$$

is an admissible strategy *for Player 2 with initial condition y if Ψ is non-anticipative, $t \mapsto \Psi(t,\varphi,\zeta)$ càdlàg and non-decreasing for any $(\varphi,\zeta) \in C_0([0,\infty)) \times D_0^+([0,\infty))$, with the convention $\Psi(0-,\varphi,\zeta) = 0$, and*

$$\Psi(t,\varphi,\zeta) - \Psi(t-,\varphi,\zeta) \leq \left(y + \mu_0 t + \sigma\varphi(t) - \Psi(t-,\varphi,\zeta)\right)^+ , \tag{3.2}$$

for all $(\varphi,\zeta) \in C_0([0,\infty)) \times D_0^+([0,\infty))$ and $t \geq 0$.

In the game with initial conditions (x,y), every pair (Φ,Ψ) induces a unique pair of admissible controls $(t,\varphi) \to (\Phi(t,\varphi),\Psi(t,\varphi,\Phi(t,\varphi))$, and thus a unique pair of dividend policies defined by

$$L_t = \Phi(t,B) \quad \text{and} \quad D_t = \Psi(t,B,L).$$

Notice that indeed conditions (3.1) and (3.2) guarantee (2.2). Similarly, a pair (L,Ψ) where L is an admissible dividend policy for Player 1 and Ψ an admissible strategy for Player 2, induces a unique pair of admissible dividend policies $(L_t,\Psi(t,B,L))$.

Remark 3.2 (**Controls).** *The set of* controls *is introduced only to relate our definitions with the classical definition of equilibria in "*control against strategy*" appearing in the (stochastic) differential game literature. Indeed, in all the definitions of equilibrium we use, we allow the players to deviate by using any* dividend policy *and not only controls.*

Formally, the set of admissible dividend policies (Definition 2.1) is larger than the set of controls. The latter can be identified with the dividend policies which are adapted with respect to the raw *filtration of B. However, one can prove*[3] *that any admissible dividend policy L is indistinguishable on $[0,\gamma_X(L)]$ from a process $\Phi(t,B)$ where $\Phi:[0,\infty)\times C_0([0,\infty))\to[0,\infty)$ is a non-anticipative map, such that, for all φ outside of a P-negligible set, the trajectories $t\to\Phi(t,\varphi)$ are right-continuous and non-decreasing and satisfy (3.1). Therefore, L and Φ induce the same payoff for Player 1 (against any control, strategy or dividend policy) in the game we consider.*

Remark 3.3 (Strategies). *Defining strategies as non-anticipative maps with respect to the raw filtration of the coordinate process on $D_0^+([0,\infty))$ is conceptually important. Indeed, the ζ variable in Player 2's strategy $\Psi(t,\varphi,\zeta)$ represents the realized trajectory of the dividend policy used by Player 1. As such it might depend on the Brownian motion and also on some exogenous randomisation variable (see Definition 4.1), but could as well be deterministic. Therefore, it is natural to use the raw filtration on $D_0^+([0,\infty))$ to model the information available to a player using the strategy Ψ.*

The payoffs in the game are well-defined if one of the players uses a strategy and the other one uses a dividend policy (or a control). We may thus define a notion of Nash equilibrium in "control vs. strategy" adopting a terminology from the literature on (stochastic) differential games. Let us emphasize that this definition is asymmetric among the players: Ψ produces a dividend policy which depends on Φ, while Φ produces a dividend policy which depends only on the Brownian trajectory.

Definition 3.4 (Nash equilibrium in control vs. strategy). *Given $(x,y)\in[0,\infty)^2$, a pair (Φ^*,Ψ^*) is a Nash equilibrium if and only if*

$$\begin{aligned}\mathcal{J}^1_{x,y}(L,\Psi^*(\cdot,B,L))&\le\mathcal{J}^1_{x,y}(\Phi^*(\cdot,B),\Psi^*(\cdot,B,\Phi^*(\cdot,B))),\\ \mathcal{J}^2_{x,y}(D,\Phi^*(\cdot,B))&\le\mathcal{J}^2_{x,y}(\Psi^*(\cdot,B,\Phi^*(\cdot,B)),\Phi^*(\cdot,B)),\end{aligned}\tag{3.3}$$

for all other pairs of admissible dividend policies (L,D).

Notice that we allow Player 2 to deviate from the equilibrium strategy with any admissible dividend policy D. In particular, if Ψ is another strategy, then Player 2's payoff induced by the pair (Φ^*,Ψ), is the same as the one induced by the pair (Φ^*,D) with $D=\Psi(\cdot,B,\Phi^*(\cdot,B))$. Similarly, we could consider Player 1's deviations from equilibrium using strategies for which there exists an outcome against the strategy Ψ^* of Player 2 (i.e. any strategy Ψ' for Player 1 such that there exist dividend policies (L,D) satisfying $L_t=\Psi'(t,B,D)$ and $D_t=\Psi^*(t,B,L)$).

The main result of this section is the construction of a Nash equilibrium in "control vs. strategy".

Theorem 3.5 (NE with asymmetric endowment). *Let $y>x$ and recall $\hat a$, a_0 and $(\xi^0_t)_{t\ge0}$ as in (2.14). Then there exists a continuous function $b:[0,\infty)\to[0,\infty)$ and a constant $\alpha>0$ with*

(i) *$b(0)=\hat a$,*
(ii) *$b\in C^1([0,a_0])$,*
(iii) *b strictly decreasing on $[0,a_0]$,*
(iv) *$b(x)=\alpha>0$ for $x\ge a_0$,*

such that, setting $L^*=\xi^0$, $X^*=X^{L^*}$ and $Y^*=Y^{D^*}$, with

$$D^*_t=\sup_{0\le s\le t}\Big(y-x+L^*_s-b(X^*_s)\Big)^+,\quad D^*_{0-}=0,\tag{3.4}$$

the pair (L^*,D^*) is a Nash equilibrium in control vs. strategy.

The proof of the theorem will be distilled in a series of intermediate results which will be illustrated in the rest of this section. Before addressing those results, some remarks are in order.

[3] This follows from classical arguments, as developed in greater generality in [10] Appendix I. Our particular claim follows from Lemma 5.2 and Lemma 5.3 in [6].

The function $x \mapsto b(x)$ is actually constructed explicitly as the unique solution of an algebraic equation in (3.11). In equilibrium Player 1 is "forced" by Player 2 to solve the classical dividend problem with drift μ_0. Therefore, Player 1's equilibrium payoff reads $v_1(x,y) := \mathcal{J}^1_{x,y}(L^*, D^*) = v_0(x)$. Moreover, it will be clear from our analysis below (Proposition 3.7) that the equilibrium payoff of Player 2 is a C^1 function $v_2(x,y) := \mathcal{J}^2_{x,y}(L^*, D^*)$ that solves a suitable free boundary problem.

Remark 3.6. *Recalling the notation in Definition 3.1 and a_0 in (2.14), we have*

$$L^*_t = \Phi^*(t, B) \quad \text{and} \quad D^*_t = \Psi^*(t, B, L^*), \tag{3.5}$$

with

$$\begin{aligned} \Phi^*(t,\varphi) &:= \sup_{0\le s\le t}\Big(x - a_0 + \mu_0 s + \sigma\varphi(s)\Big)^+, \\ \Psi^*(t,\varphi,\zeta) &:= \sup_{0\le s\le t}\Big(y - x + \zeta(s) - b\big(x + \mu_0 s + \sigma\varphi(s) - \zeta(s)\big)\Big)^+, \end{aligned} \tag{3.6}$$

for $t \ge 0$. Notice for later use that the above control and strategy depend in a measurable way on the initial positions x and (x,y), respectively. When necessary, in Section 4 we will denote them $\Phi^*(x,t,\varphi)$ and $\Psi^*(x,y,t,\varphi,\zeta)$ to emphasise that dependence. Throughout the current section x and y are fixed, with $y > x$, and we use notations as in (3.6).

Our analysis in the rest of the section is organised as follows: first, we construct the solution of a suitable free boundary problem (Proposition 3.7); second, we illustrate properties of the dividend policy D^* (Lemma 3.8) and show that it is best response against the dividend policy L^* (Proposition 3.9); third, we show that L^* is best response against D^* (Lemma 3.11); finally, combining these results we obtain the proof of Theorem 3.5.

It is convenient to change our reference system and consider the state variables (x,z) with $z = y - x$. Letting $Z^{L,D}_t := Y^D_t - X^L_t$ it is immediate to check that $Z^{L,D}_t = y - x + L_t - D_t$, for $t \ge 0$. Moreover, for the default times we have $\gamma_Y = \inf\{t \ge 0 : Z^{L,D}_t \le -X^L_t\} =: \gamma_Z$ and $Y_{\gamma_X} = Z_{\gamma_X}$, P-a.s. It should be clear that $\gamma_X = \gamma_X(L)$ and $\gamma_Z = \gamma_Z(D)$, therefore we omit the dependence on the dividend policies, for ease of notation.

Let us assume that Player 1 uses the control $L^* = \xi^0$ and we will focus the first part of our analysis on constructing Player 2's best response. It is well-known from the classical dividend problem (and it can be easily verified from (2.8)) that $L^*_{0-} = 0$ and L^* is continuous except for a possible jump at time zero of size $L^*_0 = (x - a_0)^+$. Moreover, the Skorokhod condition $\mathrm{d}L^*_t = 1_{\{X^*_t = a_0\}}\mathrm{d}L^*_t$ holds for all $t > 0$, P-a.s. Finally $X^*_t \le a_0$ for all $t \ge 0$, P-a.s. In summary, L^* reflects the dynamics of X^* at the boundary a_0 and it is well-known to be optimal for the dividend problem with drift μ_0.

Then, Player 2 must solve the singular control problem with value

$$v_2(x,y;L^*) = \sup_D \mathcal{J}^2_{x,y}(D, L^*). \tag{3.7}$$

Changing variables we define $u_2(x,z) := v_2(x, z+x; L^*)$ and we have

$$u_2(x,z) = \sup_D \mathsf{E}_{x,z}\Big[\int_{[0,\gamma_{X^*}\wedge\gamma_Z]} \mathrm{e}^{-rt}\mathrm{d}D_t + 1_{\{\gamma_{X^*}<\gamma_Z\}}\mathrm{e}^{-r\gamma_{X^*}}\hat{v}(Z^{L^*,D}_{\gamma_{X^*}})\Big]. \tag{3.8}$$

It suffices to define the function u_2 on the set

$$H = \{(x,z) \in [0,a_0] \times \mathbb{R} \,|\, z \ge -x\}.$$

Indeed, if $x > a_0$, the initial jump of the control process L^* shifts the x coordinate to the value $X^*_0 = a_0$. Then, we can simply extend the definition of u_2 as

$$u_2(x,z) := u_2(a_0,z), \quad \text{for } x > a_0. \tag{3.9}$$

We will characterise properties of the function u_2, along with an optimal dividend policy, using a verification approach.

In the next proposition we use three closed sets defined as follows: given a decreasing function $b \in C([0,a_0])$ with $\alpha := b(a_0) > 0$,

$$H_{\leq} := \{(x,z) \in H \mid z \leq 0\}, \quad H_{[0,\alpha]} := \{(x,z) \in H \mid z \in [0,\alpha]\},$$
$$H_{[\alpha,b]} := \{(x,z) \in H \mid \alpha \leq z \leq b(x)\}. \tag{3.10}$$

Proposition 3.7. *Recall $\hat{v}$, v_0, $\hat{a}$ and a_0 from (2.14). There is a unique pair (u,b) with the following properties:*

(i) *The function b is continuous on $[0,\infty)$, strictly decreasing on $[0,a_0]$ with $b \in C^1([0,a_0])$, $b(0) = \hat{a}$ and $b(a_0) = \alpha > 0$. Here α is the unique solution of $\hat{v}'(\alpha) = v_0'(0)$. Moreover, $b(\cdot)$ satisfies*

$$b(x) = [(\hat{v}')^{-1} \circ v_0'](a_0 - x), \quad \forall x \in [0,a_0]. \tag{3.11}$$

(ii) *Defining $\mathcal{C} := \{(x,z) \in H : z < b(x)\}$ and $\mathcal{S} = H \setminus \mathcal{C}$, it holds*

$$u \in C^1(H) \cap C^2(H_{\leq} \cup H_{[0,\alpha]} \cup H_{[\alpha,b]} \cup \mathcal{S}),$$

with u_{xz} continuous across the boundary $x \mapsto b(x)$.

(iii) *The function u solves the variational system*

$$\begin{cases} \left(\frac{\sigma^2}{2}\partial_{xx}u + \mu_0\partial_x u - ru\right)(x,z) = 0, & \textit{for } (x,z) \in \overline{\mathcal{C}}, \\ \left(\frac{\sigma^2}{2}\partial_{xx}u + \mu_0\partial_x u - ru\right)(x,z) \leq 0, & \textit{for } (x,z) \in \mathcal{S}, \\ \partial_z u(x,z) > 1, & \textit{for } (x,z) \in \mathcal{C}, \\ \partial_z u(x,z) = 1, & \textit{for } (x,z) \in \mathcal{S}, \\ \partial_{zx}u(x,b(x)) = 0, & \textit{for } x \in [0,a_0], \\ u(0,z) = \hat{v}(z), & \textit{for } z \in [0,\infty), \\ u(x,-x) = 0, & \textit{for } x \in (0,a_0], \\ (\partial_z u - \partial_x u)(a_0,z) = 0, & \textit{for } z \in [-a_0,+\infty). \end{cases} \tag{3.12}$$

Proof. We proceed with the construction of the function u and of the boundary b in four steps, considering first the region $H_{[\alpha,b]}$, then the region $H_{[0,\alpha]}$ and finally the regions $H_{\leq}$ and $\mathcal{S}$. We emphasise that in steps 1–3 we will produce *candidates* for the function $u(x,z)$ (or its derivative $\partial_z u$) and the boundary $b(x)$. In Step 4 we will verify that such candidate pair solves the free boundary problem (3.12). With a slight abuse of notation, we use u and b to denote those candidates.

We begin with some basic preliminaries. Let $\beta_2 < 0 < \beta_1$ be the solutions of the quadratic characteristic equation $\frac{\sigma^2}{2}\beta^2 + \mu_0\beta - r = 0$. Since b must be strictly decreasing on $[0,a_0]$, we can equivalently consider its inverse $c(z) := b^{-1}(z)$, which must be continuous, decreasing on some interval $[b(a_0), b(0)]$, with $c(b(a_0))$ to be determined and $c(b(0)) = 0$. From the first equation in (3.12) we must have for $(x,z) \in \mathcal{C}$

$$u(x,z) = A(z)\mathrm{e}^{\beta_1 x} + B(z)\mathrm{e}^{\beta_2 x}, \tag{3.13}$$

for some maps $A(z)$ and $B(z)$ which we find it convenient to define (with a slight abuse of notation) as $A, B : \overline{\mathcal{C}} \to \mathbb{R}$. In particular, in order to guarantee that u satisfies the regularity required in (ii), it must be

$$A, B \in C^1(\overline{\mathcal{C}}) \cap C^2(H_{\leq} \cup H_{[0,\alpha]} \cup H_{[\alpha,b]}).$$

For $z \geq 0$, the boundary condition at $x = 0$ (sixth equation in (3.12)) leads to $A(z) + B(z) = \hat{v}(z)$, and by differentiation

$$A'(z) + B'(z) = \hat{v}'(z), \quad z \in [0, b(0)]. \tag{3.14}$$

We are now ready to construct the solution to the free-boundary problem.

Step 1. Here we take $(x,z) \in H_{[\alpha,b]}$.

The fourth equation in (3.12) yields $\partial_z u(x, b(x)) = 1$, which can be written equivalently as $\partial_z u(c(z), z) = 1$. That is,

$$A'(z)\mathrm{e}^{\beta_1 c(z)} + B'(z)\mathrm{e}^{\beta_2 c(z)} = 1, \quad z \in [b(a_0), b(0)]. \tag{3.15}$$

We deduce from (3.14) and (3.15) that:

$$A'(z)(\mathrm{e}^{\beta_1 c(z)} - \mathrm{e}^{\beta_2 c(z)}) = 1 - \hat{v}'(z)\mathrm{e}^{\beta_2 c(z)}. \tag{3.16}$$

The fifth equation in (3.12) (so-called smooth-pasting condition) can be written equivalently as $\partial_{zx} u(c(z), z) = 0$. That is,

$$A'(z)\beta_1\mathrm{e}^{\beta_1 c(z)} + B'(z)\beta_2\mathrm{e}^{\beta_2 c(z)} = 0, \quad z \in [b(a_0), b(0)]. \tag{3.17}$$

Combining (3.17) with (3.14) yields

$$A'(z)(\beta_1\mathrm{e}^{\beta_1 c(z)} - \beta_2\mathrm{e}^{\beta_2 c(z)}) = -\hat{v}'(z)\beta_2\mathrm{e}^{\beta_2 c(z)} \quad z \in [b(a_0), b(0)]. \tag{3.18}$$

Solving (3.18) and (3.16) for $A'(z)$ and equating the two expressions, we find

$$\hat{v}'(z) = \frac{\beta_1\mathrm{e}^{\beta_1 c(z)} - \beta_2\mathrm{e}^{\beta_2 c(z)}}{(\beta_1 - \beta_2)\mathrm{e}^{(\beta_1+\beta_2)c(z)}}. \tag{3.19}$$

Next we show that for each $z \in [b(a_0), b(0)]$ there is a unique $c(z)$ that solves (3.19). More precisely, we notice that (3.19) was derived only for $z \geq 0$ (because we used (3.14)). Therefore, as part of the proof we must show that $b(a_0) \geq 0$ (and even $b(a_0) > 0$).

Setting

$$\phi(\ell) := \frac{\beta_1\mathrm{e}^{\beta_1 \ell} - \beta_2\mathrm{e}^{\beta_2 \ell}}{(\beta_1 - \beta_2)\mathrm{e}^{(\beta_1+\beta_2)\ell}}, \quad \ell \in [0, \infty), \tag{3.20}$$

it is immediate to see that $\phi(0) = 1$ and simple algebra allows to check that $\phi' > 0$ on $(0, \infty)$ and that $\lim_{+\infty} \phi = +\infty$. Since $\hat{v}'(z) > 1$ for $z \in [0, \hat{a})$ and $\hat{v}'(\hat{a}) = 1$, we then obtain $c(\hat{a}) = 0$. Letting z decrease, starting from $z = \hat{a}$, the function $c(z)$ increases. That is, $z \mapsto c(z)$ is strictly decreasing in a left-neighbourhood of $\hat{a}$. Moreover, in such neighbourhood we have

$$c(z) = \phi^{-1}\big(\hat{v}'(z)\big). \tag{3.21}$$

For the inverse function $b(x) = c^{-1}(x)$ we have $b(0) = \hat{a}$ and $x \mapsto b(x)$ (strictly) decreasing on a right-neighbourhood of 0. Now we want to show that $b(x) > 0$ for $x \in [0, a_0]$.

Recall from (2.11) that $v_0'(x) = C_0(\beta_1\mathrm{e}^{\beta_1 x} - \beta_2\mathrm{e}^{\beta_2 x})$, that $v_0'(a_0) = 1$ and $v_0''(a_0) = 0$. From those expressions we deduce

$$C_0 = \frac{1}{\beta_1\mathrm{e}^{\beta_1 a_0} - \beta_2\mathrm{e}^{\beta_2 a_0}} \quad \text{and} \quad \beta_1^2\mathrm{e}^{\beta_1 a_0} = \beta_2^2\mathrm{e}^{\beta_2 a_0}. \tag{3.22}$$

In particular, plugging the second expression into the first one yields $C_0 = \beta_2/[\beta_1(\beta_2 - \beta_1)]\mathrm{e}^{-\beta_1 a_0}$. We can thus rewrite (3.20), for $\ell \in [0, a_0]$, as

$$\begin{aligned}\phi(\ell) &= \frac{\big(\beta_1\mathrm{e}^{\beta_1 \ell} - \beta_2\mathrm{e}^{\beta_2 \ell}\big)\,\mathrm{e}^{-(\beta_1+\beta_2)\ell}}{(\beta_1 - \beta_2)} \\ &= \frac{\beta_2}{\beta_1(\beta_2 - \beta_1)}\mathrm{e}^{-\beta_1 a_0}[\beta_1 e^{\beta_1(a_0-\ell)} - \beta_2\mathrm{e}^{\beta_2(a_0-\ell)}] = v_0'(a_0 - \ell).\end{aligned}$$

The equation (3.19) is therefore equivalent to

$$\hat{v}'(z) = v_0'(a_0 - c(z)) \iff \hat{v}'\big(b(x)\big) = v_0'\big(a_0 - x\big). \tag{3.23}$$

We notice that $\hat{v} \geq v$ by definition of the problem: any admissible dividend policy for the problem with drift μ_0 is admissible for the problem with drift $\hat{\mu}$ and it gives a weakly larger payoff in the latter because $\hat{\mu} > \mu_0$. Therefore $\hat{v}'(0+) \geq v_0'(0+)$ since $\hat{v}(0) = v_0(0) = 0$.

Arguing by contradiction, let us assume $x_0 := \inf\{x > 0 : b(x) = 0\} \in [0, a_0]$. Then $\hat{v}'(0+) = v_0'(a_0 - x_0) \le v_0'(0+)$, where the final inequality is by concavity of v_0. Combining with the inequality from the paragraph above it must be $\hat{v}'(0+) = v_0'(0+) > 0$. Recalling the ODE solved by the value function of the optimal dividend problem (cf. (2.9)) and using $\hat{v}(0) = v_0(0) = 0$ and $\hat{v}'(0+) = v_0'(0+) > 0$, we deduce

$$\hat{v}''(0+) - v_0''(0+) = \frac{2}{\sigma^2}(\mu_0 - \hat{\mu})v_0'(0+) < 0.$$

The latter implies $\hat{v}'(x) - v_0'(x) < 0$ for $x \in (0, \varepsilon)$ and sufficiently small $\varepsilon > 0$. Thus $\hat{v}(x) < v_0(x)$ for $x \in (0, \varepsilon)$, which contradicts $\hat{v} \ge v_0$. Then we conclude $x_0 \notin [0, a_0]$, as needed. The latter also implies that $\alpha = b(a_0)$ is the unique solution of $\hat{v}'(\alpha) = \phi(a_0) = v_0'(0)$.

By construction (cf. (3.21)), $b(x) = [(\hat{v}')^{-1} \circ \phi](x)$ is continuous for $x \in [0, a_0]$. Moreover, using that $\hat{v} \in C^2([0, \infty))$ with $\hat{v}''(z) < 0$ for $z \in [0, \hat{a})$ we have

$$b'(x) = \frac{\phi'(x)}{(\hat{v}'' \circ (\hat{v}')^{-1} \circ \phi)(x)} = \frac{\phi'(x)}{(\hat{v}'' \circ b)(x)} \in (-\infty, 0), \quad \text{for } x \in (0, a_0].$$

Letting $x \downarrow 0$ we have $b(0) = \hat{a}$ and $\hat{v}''(\hat{a}) = 0$ in the denominator of the equation above. However, also the numerator vanishes. Then, using De L'Hopital's rule, in the limit as $x \downarrow 0$ we have

$$b'(0+) \approx \frac{\phi''(0)}{\hat{v}'''\big(b(0)\big)b'(0+)} \implies \big[b'(0+)\big]^2 = \frac{\phi''(0)}{\hat{v}'''(\hat{a})}. \tag{3.24}$$

Simple algebra yields $\phi'''(0) = -\beta_1\beta_2$. Differentiating once the ODE for $\hat{v}$ and imposing the boundary conditions $\hat{v}'(\hat{a}) = 1$ and $\hat{v}''(\hat{a}) = 0$ yields $\hat{v}'''(\hat{a}) = 2r/\sigma^2$. Hence, from (3.24) we conclude $b'(0+) = \sigma\sqrt{|\beta_1\beta_2|/(2r)}$ and $b \in C^1([0, a_0])$ as claimed.

Note that (3.18) (or (3.16)) together with the explicit expression of c in (3.21) determine A', on the interval $[\alpha, \hat{a}]$. They also determine B' on $[\alpha, \hat{a}]$ by (3.14) and, finally, $\partial_z u$ on $H_{[\alpha,b]}$. More precisely, (3.18) and (3.17) yield

$$\begin{aligned} A'(z) &= \frac{-\beta_2 \mathrm{e}^{\beta_2 c(z)}}{\beta_1 \mathrm{e}^{\beta_1 c(z)} - \beta_2 \mathrm{e}^{\beta_2 c(z)}} \hat{v}'(z) > 0, \\ B'(z) &= (-\beta_1/\beta_2)\mathrm{e}^{(\beta_1 - \beta_2)c(z)} A'(z) = \frac{\beta_1 \mathrm{e}^{\beta_1 c(z)}}{\beta_1 \mathrm{e}^{\beta_1 c(z)} - \beta_2 \mathrm{e}^{\beta_2 c(z)}} \hat{v}'(z) > 0. \end{aligned} \tag{3.25}$$

For future reference notice that

$$A'(\alpha) = -\frac{\beta_2}{\beta_1 - \beta_2} \mathrm{e}^{-\beta_1 a_0}. \tag{3.26}$$

Combining the above we have an explicit expression for $\partial_z u(x, z) = A'(z)e^{\beta_1 x} + B'(z)e^{\beta_2 x}$ for $(x, z) \in H_{[\alpha,b]}$. We use that to compute $\partial_{xz} u(x, z)$ for $(x, z) \in H_{[\alpha,b]}$. In particular, for $x < c(z)$, $z \in [\alpha, \hat{a}]$,

$$\begin{aligned} \partial_{xz} u(x, z) &= A'(z)\beta_1 \mathrm{e}^{\beta_1 x} + B'(z)\beta_2 \mathrm{e}^{\beta_2 x} \\ &< A'(z)\beta_1 \mathrm{e}^{\beta_1 c(z)} + B'(z)\beta_2 \mathrm{e}^{\beta_2 c(z)} = 0, \end{aligned}$$

where the final equality holds due to (3.17). Since $\partial_z u(c(z), z) = 1$ (c.f. (3.15)), then $\partial_z u(x, z) > 1$ for $x < c(z)$, $z \in [\alpha, \hat{a}]$. This verifies the third condition in (3.12) in the set $\mathcal{C} \cap H_{[\alpha,b]}$. Finally, it is a matter of simple algebra to check that $\partial_{xx} u$ and $\partial_{zz} u$ are also continuous on $H_{[\alpha,b]}$.

In this step we have obtained formulae for the coefficients $A'(z)$, $B'(z)$ and the boundary $b(x)$ (or its inverse $c(z)$). Then we have a *candidate* expression for $\partial_z u(x, z)$ and for $b(x)$. To emphasise that these are just candidates for now, we adopt the notation

$$Q_1(x, z) := A'(z)\mathrm{e}^{\beta_1 x} + B'(z)\mathrm{e}^{\beta_2 x} = \partial_z u(x, z), \quad (x, z) \in H_{[\alpha,b]}. \tag{3.27}$$

Step 2. Here we take $(x, z) \in H_{[0,\alpha]}$.

For $z \in [0,\alpha)$, recall the reflection condition at (a_0, z) as given in the final equation in (3.12). Differentiating that condition with respect to z yields $\partial_{zz}u(a_0,z) = \partial_{xz}u(a_0,z)$. Using (3.13), the latter reads

$$A''(z)\mathrm{e}^{\beta_1 a_0} + B''(z)\mathrm{e}^{\beta_2 a_0} = A'(z)\beta_1\mathrm{e}^{\beta_1 a_0} + B'(z)\beta_2\mathrm{e}^{\beta_2 a_0}. \tag{3.28}$$

Differentiating also (3.14) we obtain $B''(z) = \hat{v}''(z) - A''(z)$, which we plug in the equation above to obtain

$$A''(z)\mathrm{e}^{\beta_1 a_0} + (\hat{v}''(z) - A''(z))\mathrm{e}^{\beta_2 a_0} = A'(z)\beta_1\mathrm{e}^{\beta_1 a_0} + (\hat{v}'(z) - A'(z))\beta_2\mathrm{e}^{\beta_2 a_0}.$$

Thus, A' satisfies a linear non-homogeneous ODE on the interval $[0,\alpha]$ given by

$$A''(z)(\mathrm{e}^{\beta_1 a_0} - \mathrm{e}^{\beta_2 a_0}) = A'(z)(\beta_1\mathrm{e}^{\beta_1 a_0} - \beta_2\mathrm{e}^{\beta_2 a_0}) + \hat{v}'(z)\beta_2\mathrm{e}^{\beta_2 a_0} - \hat{v}''(z)\mathrm{e}^{\beta_2 a_0}. \tag{3.29}$$

We impose the boundary condition at $z = \alpha$ given by (3.26). That ensures that

$$A' \text{ is continuous in } [0, b(x)] \text{ for every } x \in [0, a_0] \text{ (hence in } H_{[0,\alpha]} \cup H_{[\alpha,b]}). \tag{3.30}$$

Knowledge of A' yields also the function B' thanks to (3.14). Therefore, by continuity of A' we deduce that A'' is also continuous by (3.29) and, thanks to (3.28) and (3.13),

$$\text{functions } B',\ B'',\ \partial_z u,\ \partial_{zz}u,\ \partial_{zx}u \text{ and } \partial_{zxx}u \text{ are continuous in } H_{[0,\alpha]} \cup H_{[\alpha,b]}. \tag{3.31}$$

Now we analyse properties of A' in more detail. Plugging the expression for $A'(\alpha)$ from (3.26) into (3.29) and recalling that $c(\alpha) = a_0$, we have

$$A''(\alpha-) = -\frac{\hat{v}''(\alpha)\mathrm{e}^{\beta_2 a_0}}{\mathrm{e}^{\beta_1 a_0} - \mathrm{e}^{\beta_2 a_0}} > 0. \tag{3.32}$$

We want to prove that $A''(z) > 0$ on $[0,\alpha]$. Arguing by contradiction we assume that A'' vanishes on $[0,\alpha)$ and we let $\tilde{z}$ denote the largest zero of A''. Clearly $\tilde{z} < \alpha$ by (3.32). By differentiating (3.29) and taking $z = \tilde{z}$, we obtain

$$A'''(\tilde{z})(\mathrm{e}^{\beta_1 a_0} - \mathrm{e}^{\beta_2 a_0}) = g'(z), \tag{3.33}$$

where

$$g(z) := \hat{v}'(z)\beta_2\mathrm{e}^{\beta_2 a_0} - \hat{v}''(z)\mathrm{e}^{\beta_2 a_0}.$$

We claim (and we prove it later) that $g'(z) < 0$ for $z \in [0,\alpha]$. Thus $A'''(\tilde{z}) < 0$ because $\mathrm{e}^{\beta_1 a_0} > \mathrm{e}^{\beta_2 a_0}$. Since $A''(\tilde{z}) = 0$ and A'' does not change its sign on $(\tilde{z},\alpha)$ (by definition of $\tilde{z}$), it then follows that $A''(z) < 0$ for $z \in (\tilde{z},\alpha)$. That contradicts (3.32).

We want to show that g is strictly decreasing for $z \in [0,\alpha]$. With the notation from (2.14), recalling the expression for the value function of the classical dividend problem in (2.11) and setting $\hat{\beta}_j = \beta_j(\hat{\mu})$, $j = 1,2$ and $\hat{C} = C(\hat{\mu})$, we can express g as

$$\begin{aligned} g(z) &= \hat{C}\mathrm{e}^{\beta_2 a_0}\left(\beta_2(\hat{\beta}_1\mathrm{e}^{\hat{\beta}_1 z} - \hat{\beta}_2\mathrm{e}^{\hat{\beta}_2 z}) - \hat{\beta}_1^2\mathrm{e}^{\hat{\beta}_1 z} + \hat{\beta}_2^2\mathrm{e}^{\hat{\beta}_2 z}\right) \\ &= \hat{C}\mathrm{e}^{\beta_2 a_0}\left((\beta_2 - \hat{\beta}_1)\hat{\beta}_1\mathrm{e}^{\hat{\beta}_1 z} + \hat{\beta}_2(\hat{\beta}_2 - \beta_2)\mathrm{e}^{\hat{\beta}_2 z}\right), \end{aligned}$$

on the interval $z \in [0,\hat{a}) \supset [0,\alpha]$. Differentiating that expression yields

$$g'(z) = \hat{C}\mathrm{e}^{\beta_2 a_0}\left((\beta_2 - \hat{\beta}_1)\hat{\beta}_1^2\mathrm{e}^{\hat{\beta}_1 z} + \hat{\beta}_2^2(\hat{\beta}_2 - \beta_2)\mathrm{e}^{\hat{\beta}_2 z}\right),$$

Observing that $\hat{\mu} > \mu_0 \implies \hat{\beta}_2 < \beta_2$ we deduce that $g'(z) < 0$ from the expression above (recall that $\hat{C} > 0$).

Next, we want to check the third condition in (3.12), i.e., $\partial_z u(x,z) > 1$ for $(x,z) \in H_{[0,\alpha]}$. By direct calculation we have

$$\begin{aligned} \partial_{zz}u(x,z) &= A''(z)\mathrm{e}^{\beta_1 x} + (\hat{v}''(z) - A''(z))\mathrm{e}^{\beta_2 x}, \\ \partial_{zzx}u(x,z) &= A''(z)\beta_1\mathrm{e}^{\beta_1 x} + (\hat{v}''(z) - A''(z))\beta_2\mathrm{e}^{\beta_2 x}. \end{aligned} \tag{3.34}$$

As shown earlier, for $z \in [0,\alpha]$ we have $A''(z) > 0$. Moreover, $\hat{v}$ is concave and therefore $\hat{v}''(z) - A''(z) < 0$. We deduce for $(x,z) \in H_{[0,\alpha]}$

$$\begin{aligned} \partial_{zz} u(x,z) &= A''(z)\mathrm{e}^{\beta_1 x} + (\hat{v}''(z) - A''(z))\mathrm{e}^{\beta_2 x} \\ &\le A''(z)\mathrm{e}^{\beta_1 a_0} + (\hat{v}''(z) - A''(z))\mathrm{e}^{\beta_2 a_0} \\ &= \partial_{zz} u(a_0,z) = \partial_{zx} u(a_0,z) \le 0, \end{aligned} \tag{3.35}$$

where in the third equality we use the reflection condition from the final equation in (3.12) and it remains to prove the final inequality. For that, it is sufficient to observe that $\beta_2(\hat{v}''(z) - A''(z)) > 0$ implies

$$\partial_{zzx} u(a_0,z) = A''(z)\beta_1\mathrm{e}^{\beta_1 a_0} + (\hat{v}''(z) - A''(z))\beta_2\mathrm{e}^{\beta_2 a_0} > 0.$$

Then, thanks to the fact that A' is continuous at α, choosing $z = \alpha$ in (3.18) so that $c(\alpha) = a_0$ (recall $B'(\alpha) = \hat{v}'(\alpha) - A'(\alpha)$) we have

$$\partial_{zx} u(a_0,\alpha) = A'(\alpha)\beta_1\mathrm{e}^{\beta_1 a_0} + (\hat{v}'(\alpha) - A'(\alpha))\beta_2\mathrm{e}^{\beta_2 a_0} = 0.$$

Combining these last two expressions yields the final inequality in (3.35).

Finally, by construction $\partial_z u$ is continuous at (x,α) for all $x \in [0,a_0]$ and the value of $\partial_z u(x,\alpha) > 1$ was determined in Step 1 of this proof. Therefore, (3.35) implies that $\partial_z u(x,z) > 1$ for all $(x,z) \in H_{[0,\alpha]}$ as needed.

As in Step 1, also in this step we have obtained expressions for the coefficients $A'(z)$ an $B'(z)$. We use them to construct a candidate expression for $\partial_z u(x,z)$. To emphasise this fact we set

$$Q_2(x,z) := A'(z)\mathrm{e}^{\beta_1 x} + B'(z)\mathrm{e}^{\beta_2 x} = \partial_z u(x,z), \quad (x,z) \in H_{[0,\alpha]}. \tag{3.36}$$

Step 3. The set $(x,z) \in H_{\le}$ is analysed in this step.

The boundary condition $u(x,-x) = 0$ reads, for $z = -x$,

$$A(z)\mathrm{e}^{-\beta_1 z} + B(z)\mathrm{e}^{-\beta_2 z} = 0, \tag{3.37}$$

and by differentiation

$$B'(z) = [(\beta_1 - \beta_2)A(z) - A'(z)]\mathrm{e}^{(\beta_2-\beta_1)z}. \tag{3.38}$$

Note that in particular $B(0) = -A(0)$. The reflection condition $(\partial_z u - \partial_x u)(a_0,z) = 0$ (last equation in (3.12)) implies

$$A'(z)\mathrm{e}^{\beta_1 a_0} + B'(z)\mathrm{e}^{\beta_2 a_0} = A(z)\beta_1\mathrm{e}^{\beta_1 a_0} + B(z)\beta_2\mathrm{e}^{\beta_2 a_0}. \tag{3.39}$$

Combining (3.38) and (3.39), we obtain

$$A'(z)\mathrm{e}^{\beta_1 a_0} + [(\beta_1 - \beta_2)A(z) - A'(z)]\mathrm{e}^{(\beta_2-\beta_1)z}\mathrm{e}^{\beta_2 a_0} = A(z)\beta_1\mathrm{e}^{\beta_1 a_0} - A(z)\mathrm{e}^{(\beta_2-\beta_1)z}\beta_2\mathrm{e}^{\beta_2 a_0},$$

which leads to $A'(z) = \beta_1 A(z)$ and thus

$$A(z) = A(0)\mathrm{e}^{\beta_1 z} \quad \text{and} \quad B(z) = -A(0)\mathrm{e}^{\beta_2 z} \quad \text{for } z \in [-a_0, 0).$$

Plugging this expression back into (3.37) yields the form of $A(z)$ and $B(z)$, and then

$$u(x,z) = A(0)(\mathrm{e}^{\beta_1(x+z)} - \mathrm{e}^{\beta_2(x+z)}), \quad (x,z) \in H_{\le}. \tag{3.40}$$

In order to determine $A(0)$ we can, for example, impose $\partial_z u(a_0,0+) = \partial_z u(a_0,0-)$, noticing that $\partial_z u(a_0,0+) = Q_2(a_0,0+) > 1$ is obtained from Step 2 (see (3.36)). More precisely, we have

$$A(0) = \frac{Q_2(a_0,0+)}{\beta_1\mathrm{e}^{\beta_1 a_0} - \beta_2\mathrm{e}^{\beta_2 a_0}} > \frac{1}{\beta_1\mathrm{e}^{\beta_1 a_0} - \beta_2\mathrm{e}^{\beta_2 a_0}} = C_0, \tag{3.41}$$

where the last equality is (3.22). Therefore, for $z < 0$,

$$\partial_z u(x,z) = \frac{Q_2(a_0,0+)}{\beta_1\mathrm{e}^{\beta_1 a_0} - \beta_2\mathrm{e}^{\beta_2 a_0}}(\beta_1\mathrm{e}^{\beta_1(x+z)} - \beta_2\mathrm{e}^{\beta_2(x+z)}). \tag{3.42}$$

Taking one more derivative and observing that $\beta_1(x+z)\le\beta_1 a_0$, $\beta_2 z\ge 0$ and $\beta_2 x\ge\beta_2 a_0$ for $(x,z)\in H_{\le}$ we obtain

$$\begin{aligned}\partial_{zz}u(x,z)&=\frac{\partial_z u(a_0,0+)}{\beta_1\mathrm{e}^{\beta_1 a_0}-\beta_2\mathrm{e}^{\beta_2 a_0}}\big(\beta_1^2\mathrm{e}^{\beta_1(x+z)}-\beta_2^2\mathrm{e}^{\beta_2(x+z)}\big)\\&\le\frac{\partial_z u(a_0,0+)}{\beta_1\mathrm{e}^{\beta_1 a_0}-\beta_2\mathrm{e}^{\beta_2 a_0}}\big(\beta_1^2\mathrm{e}^{\beta_1 a_0}-\beta_2^2\mathrm{e}^{\beta_2 a_0}\big)\\&=\frac{\partial_z u(a_0,0+)}{\beta_1\mathrm{e}^{\beta_1 a_0}-\beta_2\mathrm{e}^{\beta_2 a_0}}\frac{v_0''(a_0)}{C(\mu_0)}=0,\end{aligned}$$

where the penultimate equality follows from (2.11), upon recalling that we are working with $\beta_1=\beta_1(\mu_0)$ and $\beta_2=\beta_2(\mu_0)$, and the final equality is by the smooth pasting condition (2.10). We then conclude that $\partial_z u(x,z)>1$ for $(x,z)\in H_{\le}$.

It is important to notice that, differently from Steps 1 and 2, here we have obtained an expression for the candidate value function $u(x,z)$, $(x,z)\in H_{\le}$, instead of a candidate for its z-derivative. We are going to use this fact in the fourth and final step of the proof.

Step 4. In this step we piece together the functions obtained in the previous steps and confirm that our candidate pair (u,b) solves the free boundary problem (3.12) as claimed.

Recalling Q_1 and Q_2 from (3.27) and (3.36), we define on $[0,a_0]\times[0,\infty)$

$$Q(x,z):=Q_1 1_{H_{[\alpha,b]}}(x,z)+Q_2 1_{H_{[0,\alpha]}}(x,z)+1_{\mathcal{S}}(x,z).$$

By construction and thanks to the smooth pasting condition $\partial_{zx}u(x,b(x))=0$, the functions Q and $\partial_x Q(x,z)$ are continuous on $[0,a_0]\times[0,\infty)$ (see (3.30) and (3.31)). Its derivative $\partial_{xx}Q$ is continuous on $H_{[0,\alpha]}\cup H_{[\alpha,b]}$ and it belongs to $L^\infty([0,a_0]\times[0,\infty))$. Next we define

$$u(x,z)=u(x,0-)+\int_0^z Q(x,\zeta)\mathrm{d}\zeta,\quad\text{for }(x,z)\in[0,a_0]\times[0,\infty),\tag{3.43}$$

where $u(x,0-)$ is given by (3.40) in Step 3. On the set $H_{\le}$ the function $u(x,z)$ is defined by (3.40).

It is then clear that $u\in C(H)$. By Step 3 we have $\partial_x u,\partial_{xx}u,\partial_z u\in C(H_{\le})$. By dominated convergence in $[0,a_0]\times[0,\infty)$ we have

$$\begin{aligned}\partial_x u(x,z)&=\partial_x u(x,0-)+\int_0^z\partial_x Q(x,\zeta)\mathrm{d}\zeta,\\\partial_{xx}u(x,z)&=\partial_{xx}u(x,0-)+\int_0^z\partial_{xx}Q(x,\zeta)\mathrm{d}\zeta.\end{aligned}$$

Moreover, it is clear that $\partial_z u(x,z)=Q(x,z)$ on $[0,a_0]\times[0,\infty)$. Then, thanks to the regularity of Q stated above, these derivatives paste continuously with the corresponding derivatives in $H_{\le}$. That is, $u\in C^1(H)$ and $\partial_{xx}u(x,z)\in C(\overline{\mathcal{C}}\cup\mathcal{S})$ as required in the free boundary problem.

By construction u solves the free boundary problem in $H_{\le}$. Next, for $(x,z)\in H_{[0,\alpha]}\cup H_{[\alpha,b]}\cup\mathcal{S}$ we can compute directly

$$\begin{aligned}&\big(\tfrac{\sigma^2}{2}\partial_{xx}u+\mu_0\partial_x u-ru\big)(x,z)\\&=\big(\tfrac{\sigma^2}{2}\partial_{xx}u+\mu_0\partial_x u-ru\big)(x,0-)+\int_0^z\big(\tfrac{\sigma^2}{2}\partial_{xx}Q+\mu_0\partial_x Q-rQ\big)(x,\zeta)\mathrm{d}\zeta.\end{aligned}$$

In particular, for $(x,z)\in H_{[0,\alpha]}\cup H_{[\alpha,b]}$ the right hand side of the equation above vanishes using the explicit expressions (3.40), (3.36) and (3.27). Instead, for $(x,z)\in\mathcal{S}$ we can compute

$$\begin{aligned}
&\big(\tfrac{\sigma^2}{2}\partial_{xx}u+\mu_0\partial_x u-ru\big)(x,z)\\
&=\big(\tfrac{\sigma^2}{2}\partial_{xx}u+\mu_0\partial_x u-ru\big)(x,0-)\\
&\ +\int_0^{b(x)}\big(\tfrac{\sigma^2}{2}\partial_{xx}Q+\mu_0\partial_x Q-rQ\big)(x,\zeta)\mathrm{d}\zeta-r(z-b(x))\\
&=-r(z-b(x))\le 0.
\end{aligned}$$

This shows validity of the first two equations in (3.12). The third, fourth, fifth and seventh condition in (3.12) hold by construction and by properties of (3.40), (3.36) and (3.27) illustrated in Steps 1–3. For the sixth condition in (3.12) we observe that $u(0,0)=0$ due to (3.40) and therefore

$$\begin{aligned}
u(0,z)&=u(0,0-)+\int_0^z Q(0,\zeta)\mathrm{d}\zeta\\
&=\int_0^{\hat a\wedge z}\big(A'(\zeta)+B'(\zeta)\big)\mathrm{d}\zeta+\int_{\hat a\wedge z}^z 1\mathrm{d}\zeta=\hat v(z),
\end{aligned}$$

where the final equality is due to (3.14), and the facts that $\hat v'=1$ on $[\hat a,\infty)$ and $\hat v(0)=0$. The final condition in (3.12) can be checked as follows: for $z\in[-a_0,0]$, $\big(\partial_z u-\partial_x u\big)(a_0,z)=0$ by (3.39); for $z\in[0,\alpha]$

$$\begin{aligned}
\big(\partial_z u-\partial_x u\big)(a_0,z)&=Q_2(a_0,z)-\partial_x u(a_0,0-)-\int_0^z\partial_x Q_2(a_0,\zeta)\mathrm{d}\zeta\\
&=Q_2(a_0,z)-\partial_x u(a_0,0-)-\int_0^z\partial_z Q_2(a_0,\zeta)\mathrm{d}\zeta\\
&=Q_2(a_0,0)-\partial_x u(a_0,0-)=0
\end{aligned}$$

where the second equality is by (3.28) and the final one by (3.42), upon recalling that $\partial_x u(a_0,0-)=\partial_z u(a_0,0-)$ by construction; finally, for $z\in[\alpha,\infty)$, we have

$$\begin{aligned}
\big(\partial_z u-\partial_x u\big)(a_0,z)&=Q(a_0,z)-\partial_x u(a_0,0-)-\int_0^z\partial_x Q(a_0,\zeta)\mathrm{d}\zeta\\
&=1-\partial_x u(a_0,0-)-\int_0^\alpha\partial_z Q_2(a_0,\zeta)\mathrm{d}\zeta\\
&=Q_2(a_0,0)-\partial_x u(a_0,0-)+1-Q_2(a_0,\alpha)=0,
\end{aligned}$$

where in the second equality we use that $Q(a_0,z)=1$ for $z\in[\alpha,\infty)$ and in the final one we use $Q_2(a_0,\alpha)=1$ (thanks to (3.30) and (3.31)) and $Q_2(a_0,0)=\partial_x u(a_0,0-)$ as above. □

The pair (u,b) constructed in the previous proposition is going to be used in a verification result to determine u_2 (cf. (3.8)) and to prove optimality of D^* for Player 2 (cf. (3.4)). Let us start by noticing that for $t\ge 0$

$$D_t^*=\sup_{0\le s\le t}\Big(Y_s^0-X_s^*-b(X_s^*)\Big)^+,$$

where $Y_s^0=y+\mu_0 s+\sigma B_s$. As a result the controlled pair (X^*,Y^*) introduced in Theorem 3.5 can be expressed as

$$\left\{\begin{aligned}
X_t^*&=X_t^0-\sup\nolimits_{0\le s\le t}\Big(X_s^0-a_0\Big)^+,\\
Y_t^*&=Y_t^0-\sup\nolimits_{0\le s\le t}\Big(Y_s^0-X_s^*-b(X_s^*)\Big)^+,
\end{aligned}\right. \tag{3.44}$$

with $X_s^0=x+\mu_0 s+\sigma B_s$.

It is useful to determine some properties of (L^*, D^*) relatively to the pair (X^*, Z^*), where $Z^* := Y^* - X^*$. Let us set $\gamma_* := \gamma_{X^*} \wedge \gamma_{Z^*} = \inf\{t \geq 0 : Z_t^* \leq -X_t^* \text{ or } X_t^* \leq 0\}$.

Lemma 3.8. *For b constructed in Proposition 3.7 and (L^*, D^*) defined as in Theorem 3.5, the following hold:*

(i) *D^* is continuous except for a possible jump at time zero of size*

$$D_0^* = \big(y - x + L_0^* - b(X_0^*)\big)^+ = \big(y - x \wedge a_0 - b(x \wedge a_0)\big)^+.$$

Therefore, Z^* is continuous except for a possible jump at time zero of size

$$Z_0^* - Z_{0-}^* = L_0^* - D_0^* = (x - a_0)^+ - \big(y - x \wedge a_0 - b(x \wedge a_0)\big)^+.$$

(ii) $\mathsf{P}(Z_{t\wedge\gamma_*}^* \leq b(X_{t\wedge\gamma_*}^*),\ \forall t \geq 0) = 1$.

(iii) *For any $T > 0$, we have P-a.s.*

$$\int_{(0,T\wedge\gamma_*]} 1_{\{Z_t^* < b(X_t^*)\}} \mathrm{d}D_t^* = 0.$$

Proof. Continuity of the mapping $t \mapsto D_{t\wedge\gamma_*}^*$ on $(0,\infty)$ is clear because of continuity of the mapping $t \mapsto L_{t\wedge\gamma_*}^* - b(X_{t\wedge\gamma_*}^*)$ on $(0,\infty)$. The expression for D_0^* is also clear by the explicit formula (3.4) and thus the one for $Z_0^* - Z_{0-}^*$ follows immediately.

The condition in item (ii) can be verified easily upon noticing that $D_{t\wedge\gamma_*}^* \geq z + L_{t\wedge\gamma_*}^* - b(X_{t\wedge\gamma_*}^*)$ for every $t \geq 0$. It remains to check the condition in item (iii). The argument is carried out path-wise. Fix $\omega \in \Omega$ and assume $Z_t^*(\omega) < b(X_t^*(\omega))$ (with no loss of generality it suffices to consider $t < \gamma_*(\omega)$). Then

$$Z_t^*(\omega) = z + L_t^*(\omega) - D_t^*(\omega) < b\big(X_t^*(\omega)\big) \implies D_t^*(\omega) > z + L_t^*(\omega) - b\big(X_t^*(\omega)\big).$$

Therefore, by continuity there is $\delta_\omega > 0$ such that

$$D_t^*(\omega) > \sup_{t\leq s\leq t+\delta_\omega} \Big(z + L_s^*(\omega) - b\big(X_s^*(\omega)\big)\Big)^+.$$

The latter implies $D_s^*(\omega) - D_t^*(\omega) = 0$ for $s \in [t, t+\delta_\omega)$ which proves the claim. □

Next we state our verification result.

Proposition 3.9. *Let (u, b) be the pair constructed in Proposition 3.7 and recall u_2 from (3.8) and (3.9). Set $u(x, z) = u(a_0, z)$ for $x > a_0$ and $z \geq -x$. Then,*

$$u(x, z) = u_2(x, z), \quad \text{for } x \in [0, \infty) \text{ and } z \geq -x.$$

Moreover, D^* as in (3.4) is an optimal dividend policy, in the sense that $u_2(x, z+x) = v_2(x, y; L^*) = \mathcal{J}_{x,y}^2(D^*, L^*)$ for all $(x, y) \in [0,\infty)^2$.

Proof. In Proposition 3.7 we showed that u solves the free boundary problem (3.12). Take an arbitrary dividend policy D and take the admissible pair (D, L^*). Denote $Z^D = Z^{L^*,D}$, $\gamma = \gamma_Z \wedge \gamma_{X^*}$ and $\mathcal{A} := \frac{\sigma^2}{2}\partial_{xx} + \mu_0\partial_x - r$, for simplicity. Fix $(x, z) \in H$. We wish to apply Dynkin's formula to calculate $\mathsf{E}_{x,z}[\mathrm{e}^{-r(t\wedge\gamma)} u(X_{t\wedge\gamma}^*, Z_{t\wedge\gamma}^D)]$. This can be done, for example, invoking [4, Thm. 2.1]. We must notice that

$$(X_{s\wedge\gamma}^*, Z_{s\wedge\gamma}^D) \in H, \quad \text{for all } s \in [0, \infty),$$

and, for any $T > 0$,

$$\begin{aligned} &\int_0^T \mathsf{P}_{x,z}\big(Z_s^D = b(X_s^*), Z_s^D > \alpha\big)\mathrm{d}s = \int_0^T \mathsf{P}_{x,z}\big(X_s^* = c(Z_s^D), Z_s^D > \alpha\big)\mathrm{d}s \\ &= \int_0^T \mathsf{P}_{x,z}\big(X_s^0 = c(Z_s^D) + L_s^*, Z_s^D > \alpha\big)\mathrm{d}s = 0, \end{aligned} \tag{3.45}$$

where the final equality holds due to Lemma A.1. Equation (3.45) guarantees Assumption A.1 in [4, Thm. 2.1], while the other assumptions are satisfied in our set-up thanks to the regularity of u and b.

Applying Dynkin's formula, up to a standard localisation argument that makes the stochastic integral a martingale, we obtain

$$
\begin{aligned}
&\mathsf{E}_{x,z}\Big[\mathrm{e}^{-r(t\wedge\gamma)}u(X^*_{t\wedge\gamma},Z^D_{t\wedge\gamma})\Big]\\
&=u(x,z)+\mathsf{E}_{x,z}\Big[\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}(\mathcal{A}u)(X^*_s,Z^D_s)\mathrm{d}s\Big]\\
&\quad+\mathsf{E}_{x,z}\Big[\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\big(\partial_z u-\partial_x u\big)(X^*_s,Z^D_s)\mathrm{d}L^{*,c}_s-\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\partial_z u(X^*_s,Z^D_s)\mathrm{d}D^c_s\Big]\\
&\quad+\mathsf{E}_{x,z}\Big[\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\big(u(X^*_s,Z^D_s)-u(X^*_{s-},Z^D_{s-})\big)\Big],
\end{aligned}
\tag{3.46}
$$

where $(L^{*,c},D^c)$ is the continuous part of the pair (L^*,D). Thanks to (3.45) we have for a.e. $s\in[0,\gamma(\omega)]$

$$
(\mathcal{A}u)\big(X^*_s(\omega),Z^D_s(\omega)\big)=(\mathcal{A}u)\big(X^*_s(\omega),Z^D_s(\omega)\big)1_{\{(X^*_s(\omega),Z^D_s(\omega))\in\mathcal{S}\}}\le 0,
\tag{3.47}
$$

where the first equality is by the first equation in (3.12) and the inequality by the second equation therein.

Jumps of D do not affect the dynamics of X^* and therefore, by definition of L^* and the fact that $X^*_{0-}\le a_0$ we have $L^*_t=L^{*,c}_t$ for $t\ge 0$. The sum of jumps then reads

$$
\begin{aligned}
&\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\big(u(X^*_s,Z^D_s)-u(X^*_{s-},Z^D_{s-})\big)\\
&=\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\big(u(X^*_s,Z^D_s)-u(X^*_s,Z^D_{s-})\big)\\
&=-\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\int_0^{|\Delta Z^D_s|}\partial_z u(X^*_s,Z^D_{s-}-\zeta)\mathrm{d}\zeta\le-\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\Delta D_s,
\end{aligned}
$$

where we use that $\Delta Z^D_s=-\Delta D_s\le 0$ and the third and fourth equation in (3.12). By the same two conditions we also obtain

$$
\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\partial_z u(X^*_s,Z^D_s)\mathrm{d}D^c_s\ge\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\mathrm{d}D^c_s.
$$

Combining these terms we obtain

$$
\begin{aligned}
&\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\big(u(X^*_s,Z^D_s)-u(X^*_{s-},Z^D_{s-})\big)-\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\partial_z u(X^*_s,Z^D_s)\mathrm{d}D^c_s\\
&\le-\sum_{s\in[0,t\wedge\gamma]}\mathrm{e}^{-rs}\Delta D_s-\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\mathrm{d}D^c_s=-\int_{[0,t\wedge\gamma]}\mathrm{e}^{-rs}\mathrm{d}D_s.
\end{aligned}
\tag{3.48}
$$

Finally, the last equation in (3.12) gives

$$
\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\big(\partial_z u-\partial_x u\big)(X^*_s,Z^D_s)\mathrm{d}L^{*,c}_s=\int_0^{t\wedge\gamma}\mathrm{e}^{-rs}\big(\partial_z u-\partial_x u\big)(a_0,Z^D_s)\mathrm{d}L^*_s=0,
\tag{3.49}
$$

where we used that $\mathrm{d}L^{*,c}_s=\mathrm{d}L^*_s=1_{\{X^*_s=a_0\}}\mathrm{d}L^*_s$ for $s>0$ in the first equality.

Combining (3.46), (3.47), (3.48) and (3.49) we obtain

$$u(x,z)\ge \mathsf{E}_{x,z}\Big[\int_{[0,t\wedge\gamma]}\mathrm{e}^{-rs}\mathrm{d}D_s+e^{-r(t\wedge\gamma)}u(X^*_{t\wedge\gamma},Z^D_{t\wedge\gamma})\Big].$$

On the event $\{\gamma\le t,\gamma_Z>\gamma_X\}$ we have $u(X^*_{t\wedge\gamma},Z^D_{t\wedge\gamma})=u(0,Z^D_{\gamma_X})=\hat v(Z^D_{\gamma_X})$ by the sixth equation in (3.12). On the event $\{\gamma\le t,\gamma_Z\le\gamma_X\}$ we have $u(X^*_{t\wedge\gamma},Z^D_{t\wedge\gamma})=u(X^*_{\gamma_Z},-X^*_{\gamma_Z})=0$ by the seventh equation in (3.12). Therefore

$$\begin{aligned}&\mathrm{e}^{-r(t\wedge\gamma)}u(X^*_{t\wedge\gamma},Z^D_{t\wedge\gamma})=1_{\{\gamma\le t,\gamma_Z>\gamma_X\}}\mathrm{e}^{-r\gamma_X}\hat v(Z^D_{\gamma_X})+1_{\{\gamma>t\}}\mathrm{e}^{-rt}u(X^*_t,Z^D_t)\\&\ge 1_{\{\gamma\le t,\gamma_Z>\gamma_X\}}\mathrm{e}^{-r\gamma_X}\hat v(Z^D_{\gamma_X}),\end{aligned}$$

where the final inequality uses $u\ge 0$ which is due to $u(x,-x)=0$ and $\partial_z u\ge 1$. Then

$$u(x,z)\ge \mathsf{E}_{x,z}\Big[1_{\{t\ge\gamma\}}\Big(\int_{[0,\gamma_X\wedge\gamma_Z]}\mathrm{e}^{-rs}\mathrm{d}D_s+1_{\{\gamma_Z>\gamma_X\}}\mathrm{e}^{-r\gamma_X}\hat v(Z^D_{\gamma_X})\Big)\Big].$$

Letting $t\to\infty$ and using Fatou's lemma, yields

$$u(x,z)\ge \mathsf{E}_{x,z}\Big[\int_{[0,\gamma_X\wedge\gamma_Z]}\mathrm{e}^{-rs}\mathrm{d}D_s+1_{\{\gamma_Z>\gamma_X\}}\mathrm{e}^{-r\gamma_X}\hat v(Z^D_{\gamma_X})\Big],\tag{3.50}$$

upon noticing that $\mathsf{P}_{x,z}(\gamma_X\wedge\gamma_Z<\infty)=1$.

Let us now consider the pair (L^*,D^*) with $(D^*_t)_{t\ge0}$ as in (3.4). Denote the controlled dynamics $(X^*,Z^*)=(X^{L^*},Z^{L^*,D^*})$ and the associated stopping time $\gamma_*:=\gamma_{Z^*}\wedge\gamma_{X^*}$. The controlled process is bound to evolve in $\overline{\mathcal{C}}$ in the random time-interval $(\!(0,\gamma_*]\!]$, thanks to Lemma 3.8. Then, we can repeat the arguments from above based on Dynkin's formula but now (3.47) reads $(\mathcal{A}u)(X^*_s(\omega),Z^*_s(\omega))=0$ for $s\in(0,\gamma_*(\omega)]$ and the inequality in (3.48) becomes an equality, due to Lemma 3.8-(iii) and $\partial_z u(x,b(x))=1$ (see the fourth equation in (3.12)). Thus we obtain

$$u(x,z)=\mathsf{E}_{x,z}\Big[\int_{[0,t\wedge\gamma_*]}\mathrm{e}^{-rs}\mathrm{d}D^*_s+1_{\{\gamma_*\le t,\gamma_{Z^*}>\gamma_{X^*}\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Z^D_{\gamma_{X^*}})+1_{\{\gamma_*>t\}}\mathrm{e}^{-rt}u(X^*_t,Z^*_t)\Big].$$

Now we let $t\to\infty$. Since $(X^*_t,Z^*_t)\in\overline{\mathcal{C}}$ for all $t\in(\!(0,\gamma_*]\!]$ and $\overline{\mathcal{C}}$ is compact, then u is bounded on $\overline{\mathcal{C}}$ and clearly

$$\lim_{t\to\infty}\mathsf{E}_{x,z}\Big[1_{\{\gamma_*>t\}}\mathrm{e}^{-rt}u(X^*_t,Z^*_t)\Big]=0.$$

Monotonce convergence also implies

$$\begin{aligned}u(x,z)&=\lim_{t\to\infty}\mathsf{E}_{x,z}\Big[\int_{[0,t\wedge\gamma_*]}\mathrm{e}^{-rs}\mathrm{d}D^*_s+1_{\{\gamma_*\le t,\gamma_{Z^*}>\gamma_{X^*}\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Z^D_{\gamma_{X^*}})\Big]\\&=\mathsf{E}_{x,z}\Big[\int_{[0,\gamma_{X^*}\wedge\gamma_{Z^*}]}\mathrm{e}^{-rs}\mathrm{d}D^*_s+1_{\{\gamma_{Z^*}>\gamma_{X^*}\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Z^D_{\gamma_{X^*}})\Big],\end{aligned}\tag{3.51}$$

where, in particular, we used that for each $\omega\in\Omega$ there is $T_\omega>0$ sufficiently large that $\gamma_*(\omega)<t$ for all $t\ge T_\omega$ and therefore

$$\int_{[0,t\wedge\gamma_*]}\mathrm{e}^{-rs}\mathrm{d}D^*_s(\omega)=\int_{[0,\gamma_*]}\mathrm{e}^{-rs}\mathrm{d}D^*_s(\omega),\quad\text{for all } t\ge T_\omega.$$

Combining (3.50) and (3.51) we obtain that $u=u_2$ on H and optimality of D^*. Notice that the equivalence $u(x,z)=u_2(x,z)$ extends to all $x\in[0,\infty)$ and $z\ge -x$ because of (3.9). □

Remark 3.10. *Recalling that $v_0(a_0)=C_0(\mathrm{e}^{\beta_1 a_0}-\mathrm{e}^{\beta_2 a_0})$, it follows from (3.40) and (3.41), that*

$$v_2(a_0,a_0)=u_2(a_0,0)>v_0(a_0).$$

Using that $\partial_z u_2(a_0,z)\ge 1$ for $z\ge 0$ and $v_0'(y)=1$ for $y\ge a_0$, we deduce that

$$v_2(a_0,y)=u_2(a_0,y-a_0)>v_0(y),\quad \textit{for all } y\ge a_0.$$

The proposition above has established that when Player 1 uses the control $L^*_t = \Phi^*(t, B)$ from (3.5), the best response of Player 2 is the strategy $\Psi^*(t, B, L^*)$. Now we want to show the viceversa, i.e., when Player 2 uses the strategy map $\Psi^*(t, B, L)$ against any dividend policy L of Player 1's, then Player 1's best action is to use L^*. That will establish the Nash equilibrium in the game with asymmetric endowment.

Lemma 3.11. *Assume Player 2 uses the strategy map Ψ^*. Then Player 1's best-response is the control map Φ^*, i.e., for every $y > x \geq 0$,*

$$v_1(x, y; \Psi^*) = \sup_L \mathcal{J}^1_{x,y}(L, \Psi^*(\cdot, B, L)) = \mathcal{J}^1_{x,y}(L^*, \Psi^*(\cdot, B, L^*)), \tag{3.52}$$

with $L^* = \Phi^*(\cdot, B)$.

Proof. Recalling the expression of Ψ^* in Remark 3.6 and using that $b(x) \geq \alpha > 0$, for all $x \geq 0$, then $\Psi^*(t, B, L) \leq (y - x + L_t - \alpha)^+$ for all $t \geq 0$ and any dividend policy L. That implies for Player 2's dynamics:

$$\begin{aligned} Y^*_t &= y + \mu_0 t + \sigma B_t - \Psi^*(t, B, L) \\ &= X^0_t + (y - x) - \Psi^*(t, B, L) \geq X^L_t + (y - x + L_t) \wedge \alpha > X^L_t, \end{aligned} \tag{3.53}$$

for all $t \geq 0$, any realisation $B(\omega) = (B_s(\omega))_{s \geq 0}$ of the Brownian path and any choice of Player 1's dividend policy $L = (L_s)_{s \geq 0}$. Therefore, $\gamma_{Y^*} > \gamma_{X^L}$, $\mathsf{P}_{x,y}$-a.s. and Player 1's expected payoff reduces to

$$\mathcal{J}^1_{x,y}(L, \Psi^*) = \mathsf{E}_{x,y}\Big[\int_{[0,\gamma_X]} \mathrm{e}^{-rt} \mathrm{d}L_t\Big].$$

Thus, Player 1 is faced with the classical dividend problem and L^* is optimal. □

Combining the results above we have a simple proof of Theorem 3.5.

Proof of Theorem 3.5. From Proposition 3.7 we have $\mathcal{J}^2_{x,y}(D^*, L^*) \geq \mathcal{J}^2_{x,y}(D, L^*)$ for any dividend policy D. From Lemma 3.11 we have $\mathcal{J}^1_{x,y}(L^*, D^*) \geq \mathcal{J}^1_{x,y}(L, D^*)$ for any dividend policy L. □

In the remainder of the paper we simplify our notation and adopt

$$v_1(x, y) = v_1(x, y; \Psi^*) \quad \text{and} \quad v_2(x, y) = v_2(x, y; \Phi^*), \tag{3.54}$$

for the equilibrium payoffs when $0 \leq x < y$.

Remark 3.12. *Notice that by continuity of the mappings $(x, y) \mapsto (v_1(x, y), v_2(x, y))$, we can actually extend the formulae in (3.54) to points on the diagonal $\{(x, y) \in [0, \infty)^2 : x = y\}$. For $x = y$ by exchanging the roles of the players, we obtain two different equilibria with asymmetric payoffs (i.e., (Φ^*, Ψ^*) and (Ψ^*, Φ^*) with payoffs $[v_1(x, x; \Psi^*), v_2(x, x; \Phi^*)]$ and $[v_1(x, x; \Phi^*), v_2(x, x; \Psi^*)]$, respectively). In these equilibria, despite the players' initial position being symmetric, one of the two players has an advantage and she is allowed to play a strategy; the other player can only play a control and obtains the smaller payoff $v_0(x)$ (notice that $v_1(x, x; \Psi^*) = v_2(x, x; \Psi^*) = v_0(x)$).*

These equilibria may appear unrealistic as none of the players would agree to be in the dominated position when starting from a symmetric position. Situations of this kind arise in classical "war of attrition" models (see [12] for a complete study of the deterministic model in continuous-time), where typically there exists also a symmetric equilibrium (i.e., with players using the same strategy) in randomised strategies. Equilibria of this type are constructed in [21] in a continuous-time stochastic model and an extension to a model with incomplete information is given in [17]. A complete characterization of the equilibria in randomised strategies in the continuous-time stochastic framework with complete information for one dimensional diffusions is given in [5].

In the next section we construct a symmetric equilibrium with randomised strategies for our game with symmetric initial endowment.

4. Nash equilibrium with symmetric initial endowment

When the two players have the same initial endowment, i.e., $x = y$, the game is fully symmetric. As soon as one of the players pays an arbitrarily small amount of dividends the symmetry is broken and the game falls back into the situation analysed in the previous section. From a game-theoretic point of view there is a second mover advantage and it is not clear whether a symmetric equilibrium can be found only using pure strategies. We allow players to use a randomised stopping time to decide the time of their first move. By symmetry, we only need to consider one function that describes the 'intensity of stopping' (in equilibrium) for both players. This function will be specified later.

We formally introduce the class of admissible mixed strategies for the problem starting at x. This definition follows the idea proposed by Aumann [1], in the sense that it is a family of strategies depending on an auxiliary randomisation variable u, which is jointly measurable in all its variables.

Definition 4.1 (Randomised Strategy). *A measurable mapping* $(u, t, \varphi, \zeta) \mapsto \Xi(u, t, \varphi, \zeta)$ *with*

$$\Xi : [0, 1] \times [0, \infty) \times C_0([0, \infty)) \times D_0^+([0, \infty)) \to [0, \infty)$$

is an admissible randomised strategy *with initial condition x if:*

(i) $\Xi(u, \cdot, \cdot, \cdot)$ *is non-anticipative for each* $u \in [0, 1]$,
(ii) $t \mapsto \Xi(u, t, \varphi, \zeta)$ *is càdlàg and non-decreasing for any* $(u, \varphi, \zeta) \in [0, 1] \times C_0([0, \infty)) \times D_0^+([0, \infty))$,
(iii) *For all* $(u, \varphi, \zeta) \in [0, 1] \times C_0([0, \infty)) \times D_0^+([0, \infty))$ *and* $t \geq 0$.

$$\Xi(u, t, \varphi, \zeta) - \Xi(u, t-, \varphi, \zeta) \leq (x + \mu_0 t + \sigma\varphi(t) - \Xi(u, t-, \varphi, \zeta))^+ ,$$

with the convention $\Xi(u, 0-, \varphi, \zeta) = 0$.

Since we are looking for a symmetric equilibrium, we will consider players using strategy against strategy (either randomised or not). We say that a pair of (non-randomised) strategies (Ψ_1, Ψ_2) (cf. Definition 3.1) induces a pair of dividend policies (L, D) if $L(t, \varphi)$ and $D(t, \varphi)$ are measurable maps from $[0, \infty) \times C_0([0, \infty))$ to $[0, \infty)$ such that for all $(t, \varphi) \in [0, \infty) \times C_0([0, \infty))$, the pair

$$L(t, \varphi) = \Psi_1(t, \varphi, D(\cdot \wedge t, \varphi)) \text{ and } D(t, \varphi) = \Psi_2(t, \varphi, L(\cdot \wedge t, \varphi)),$$

is well-defined. Here we use the notations $D(\cdot \wedge t, \varphi)$ and $L(\cdot \wedge t, \varphi)$ to indicate dependence on the whole path of the dividend policies (L, D) up to time t. Similarly, we say that a pair of randomised strategies (Ξ_1, Ξ_2) induces a pair of dividend policies $(\bar{L}, \bar{D})$ if $\bar{L}(u_1, u_2, t, \varphi)$ and $\bar{D}(u_1, u_2, t, \varphi)$ are jointly measurable maps from $[0, 1]^2 \times [0, \infty) \times C_0([0, \infty))$ to $[0, \infty)$ such that for Lebesgue a.e. $(u_1, u_2) \in [0, 1]^2$ and any $(t, \varphi) \in [0, \infty) \times C_0([0, \infty))$, the pair

$$\bar{L}(u_1, u_2, t, \varphi) = \Xi_1(u_1, t, \varphi, \bar{D}(u_1, u_2, \cdot \wedge t, \varphi)) \text{ and } \bar{D}(u_1, u_2, t, \varphi) = \Xi_2(u_2, t, \varphi, \bar{L}(u_1, u_2, \cdot \wedge t, \varphi)),$$

is well-defined. The pair $(\bar{L}, \bar{D})$ is said to be unique if it is unique up to a set of pairs (u_1, u_2) with zero Lebesgue measure.

With a slight abuse of notation, we identify the unique pair of dividend policies $(\bar{L}, \bar{D})$ induced by a pair of mixed strategies (Ξ_1, Ξ_2) with the pair of mixed strategies itself. With the notation $\bar{L}^{u_1,u_2} = \bar{L}(u_1, u_2, \cdot, \cdot)$ and $\bar{D}^{u_1,u_2} = \bar{D}(u_1, u_2, \cdot, \cdot)$, we define the associated payoff

$$\mathcal{J}^1_{x,y}(\Xi_1, \Xi_2) = \mathcal{J}^1_{x,y}(\bar{L}, \bar{D}) := \int_0^1 \int_0^1 \mathcal{J}^1_{x,y}(\bar{L}^{u_1,u_2}, \bar{D}^{u_1,u_2}) \mathrm{d}u_1 \mathrm{d}u_2,$$

and similarly for $\mathcal{J}^2_{x,y}(\bar{D}, \bar{L}) = \mathcal{J}^2_{x,y}(\Xi_2, \Xi_1)$.

Note also that a pair (L, Ξ_2) with L an admissible dividend policy and Ξ_2 a randomised strategy induces a pair of admissible dividend policies $(L, \bar{D})$ with $\bar{D}(u_2, t, B) = \Xi_2(u_2, t, B, L)$ for all $t \geq 0$

(recall that $L = \Phi(t, \varphi)$, by Remark 3.2). The associated payoff of Player 1, reads

$$\mathcal{J}^1_{x,y}(L, \Xi_2) = \mathcal{J}^1_{x,y}(L, \bar{D}) := \int_0^1 \mathcal{J}^1_{x,y}(L, \bar{D}^{u_2})\mathrm{d}u_2,$$

and we use a similar notation for the payoff of Player 2 associated to a pair $(\bar{L}, D) = (\Xi_1, D)$.

Definition 4.2 (**Nash Equilibrium in mixed strategies).** *Given $(x, y) \in [0, \infty)^2$, a pair of admissible mixed strategies (Ξ^*_1, Ξ^*_2) is a Nash equilibrium if and only if it induces a unique pair of dividend policies and*

$$\mathcal{J}^1_{x,y}(L, \Xi^*_2) \le \mathcal{J}^1_{x,y}(\Xi^*_1, \Xi^*_2) \quad \text{and} \quad \mathcal{J}^2_{x,y}(D, \Xi^*_1) \le \mathcal{J}^2_{x,y}(\Xi^*_2, \Xi^*_1),$$

for all other pairs of admissible dividend policies (L, D).

Remark 4.3. *Existence and uniqueness of a pair of dividend policies induced by a pair of strategies must be part of the definition of a Nash equilibrium in* strategy-against-strategy*, as in Definition 4.2. That is because, in continuous-time games, existence and uniqueness of a pair (L, D), induced by an arbitrary pair of strategies (Ψ_1, Ψ_2), are not guaranteed. Indeed, simultaneous reaction may cause some trouble.*

For example, let us consider the pure strategies $\Psi^\dagger(t, \varphi, \zeta) = 1_{\{\zeta(0)>0\}}$ and $\Psi^\sharp(t, \varphi, \zeta) = 1_{\{\zeta(0)=0\}}$, which do not depend on φ. It is easy to verify that the pair $(\Psi^\dagger, \Psi^\dagger)$ induces both the constant processes $(0, 0)$ and $(1, 1)$. Hence, uniqueness is lost. Instead, there exists no pair (L, D) induced by $(\Psi^\dagger, \Psi^\sharp)$. Hence, existence is also lost.

In preparation for the proof of the main result of this section we need to introduce some new objects. For $\zeta \in D^+_0([0, \infty))$, set $\sigma(\zeta) = \inf\{t \ge 0 : \zeta_t > 0\}$. Then, $\sigma(\zeta)$ is an $(\mathcal{F}^{\mathbb{W}}_t)$-optional time because it is an entry time to an open set for a càdlàg process (recall that $(\mathcal{F}^{\mathbb{W}}_t)_{t\ge0}$ is the raw filtration of the canonical process $\mathbb{W}_t(\varphi, \zeta) = (\varphi(t), \zeta(t))$).

We also extend the definition of Φ^* to account for an activation at an arbitrary time $s \in [0, \infty)$:

$$\Phi^*_s(t, \varphi) := \sup_{s\le u\le t} \Big(x - a_0 + \mu_0 u + \sigma\varphi(u)\Big)^+, \quad \text{for } \varphi \in C_0([0, \infty)).$$

In order to construct equilibrium randomised strategies, we let $\ell : [0, \infty) \to [0, \infty)$ be a measurable function to be determined at equilibrium and we define, for $\varphi \in C_0([0, \infty))$

$$\bar{\Gamma}^\ell_t(\varphi) := \int_0^t \mathrm{e}^{-\int_0^s \ell\big(x+\mu_0 u+\sigma\varphi(u)\big)\mathrm{d}u} \ell\big(x + \mu_0 s + \sigma\varphi(s)\big)\mathrm{d}s.$$

Then, for $u \in [0, 1]$ we also introduce

$$\bar{\gamma}^\ell(\varphi, u) := \inf\{t \ge 0 : \bar{\Gamma}^\ell_t(\varphi) \ge u\}.$$

Notice that $\bar{\gamma}^\ell(\varphi, u)$ is a $(\mathcal{F}^{\mathbb{W}}_t)$-stopping time as an entry time of a continuous process to a closed set. In particular, for each $\omega \in \Omega$ we denote

$$\Gamma^\ell_t(\omega) := \bar{\Gamma}^\ell_t(B(\omega)) = \int_0^t \mathrm{e}^{-\int_0^s \ell(X^0_u(\omega))\mathrm{d}u} \ell\big(X^0_s(\omega)\big)\mathrm{d}s = 1 - \mathrm{e}^{-\int_0^t \ell(X^0_s(\omega))\mathrm{d}s}, \tag{4.1}$$

where $X^0_t = x + \mu_0 t + \sigma B_t$. Further, given two random variables $U_i \sim \mathrm{Unif}(0, 1)$, $i = 1, 2$, defined on $(\Omega, \mathcal{F}, \mathsf{P})$, mutually independent and independent of the Brownian motion B, we define the randomised stopping times for the raw Brownian filtration

$$\gamma^\ell_i := \bar{\gamma}^\ell(B, U_i) = \inf\{t \ge 0 : \Gamma^\ell_t \ge U_i\}, \quad \text{for } i = 1, 2. \tag{4.2}$$

In order to find an equilibrium the two players need to find a function ℓ^* that generates a pair of optimal randomised stopping times (γ^*_1, γ^*_2). At equilibrium, on the event $\{\gamma^*_1 < \gamma^*_2\}$ Player 1 makes the first move and gives her opponent the second mover advantage. After the first move, the game can be analysed with the arguments from Section 3. Indeed, we will show that Player 1 is going to adopt the control L^* as in Theorem 3.5, while Player 2 is going to use the strategy D^*. On the event $\{\gamma_1^* > \gamma_2^*\}$ the first move is made by Player 2 and the situation is analogous but symmetric.

In particular, an important role will be played by the map Ξ^* defined below. Let

$$\ell^*(x) := \frac{[rv_0(x) - \mu_0]^+}{v_2(a_0, x) - v_0(x)}, \quad \text{for } x \geq 0,$$

with v_0 from (2.14). Notice that $\ell^*(x) > 0 \iff x \in (a_0, \infty)$ by (2.13) and since $v_2(a_0, x) > v_0(x)$ for $x \in [a_0, \infty)$ by Remark 3.10. Given $(u, \varphi, \zeta) \in [0,1] \times C_0([0,\infty)) \times D_0^+([0,\infty))$, set $\bar{\gamma}^*(\varphi, u) := \bar{\gamma}^{\ell^*}(\varphi, u)$ and simplify the notation to $\bar{\gamma}^* = \bar{\gamma}^*(\varphi, u)$, $\sigma = \sigma(\zeta)$. The mixed strategy map Ξ^* is defined as

$$\Xi^*(u, t, \varphi, \zeta) := 1_{\{t \geq \bar{\gamma}^* \wedge \sigma\}} \Big[1_{\{\sigma \geq \bar{\gamma}^*\}} \Phi^*_{\bar{\gamma}^*}(t, \varphi) + 1_{\{\sigma < \bar{\gamma}^*\}} \Psi^*(t, \varphi, \zeta) \Big], \tag{4.3}$$

where Ψ^* is defined as in Remark 3.6 with the initial condition (x, x). The fact that Ξ^* satisfies the non-anticipative property in Definition 4.1 is not straightforward, because σ is only a $(\mathcal{F}_t^{\mathbb{W}})$-optional time of the canonical filtration. Checking non-anticipativity of the map will be part of the proof of Theorem 4.4.

The next theorem provides a symmetric equilibrium in randomised strategies in the symmetric set-up and it is the main result of this section.

Theorem 4.4 (**NE with symmetric endowment**). *Set $\gamma_i^* := \gamma_i^{\ell^*}$, $i = 1, 2$, as in (4.2). There exists a unique admissible pair $(\bar{L}^*, \bar{D}^*)$ of dividend policies which satisfy for all $(u_1, u_2) \in [0,1]^2$*

$$\bar{L}^*(u_1, u_2, t, B) := \Xi^*(u_1, t, B, \bar{D}^*) \quad \textit{and} \quad \bar{D}^*(u_1, u_2, t, B) := \Xi^*(u_2, t, B, \bar{L}^*), \quad \textit{for } t \geq 0.$$

The pair (Ξ^*, Ξ^*) is a Nash equilibrium in randomised strategies and the equilibrium payoffs for the two players read

$$\mathcal{J}^1_{x,x}(\bar{L}^*, \bar{D}^*) = \mathcal{J}^2_{x,x}(\bar{D}^*, \bar{L}^*) = v_0(x), \quad x \in [0, \infty),$$

with v_0 as in (2.14).

A few remarks are in order before we proceed with the proof of the theorem. Due to the symmetry of the set-up, all the considerations that we make for one player's strategy also hold for the other player's strategy.

Remark 4.5 (**War of attrition and indifference principle**). *Our equilibrium shares some important features with stopping games with a second mover advantage, and especially with the so called "war of attrition" games. Once one of the players —say Player 1— activates her control (i.e., she pays dividends), we reach a position $X^L < Y^D$. At that point both players play an asymmetric equilibrium in which Player 1 is in a dominated position. Thus, a player's activation of her control can be thought of as "conceding", in the sense that the first mover accepts to be in a dominated position. Assuming that, when reaching a position $X^L \neq Y^D$ after the first move, these continuation equilibria will be played, none of the players wants to be the first to activate her control. Because of discounting and the risk of default, waiting is costly for both players. Then, there is a trade-off —say for Player 2— between the potential gains at the time Player 1 activates her control (i.e., $v_2(X,Y) - v_0(Y) > 0$) and the effect of discounting and default risk. The strategy Ξ^* for Player 2 can be described as follows: Player 2 will wait for a random amount of time γ_2^* and one of two mutually exclusive outcomes is possible:*

(a) *Player 1 activates her control strictly before γ_2^* and concedes (that corresponds to the event $\{\sigma < \bar{\gamma}^*\}$ in (4.3)); Then the players start playing a continuation equilibrium in which Player 1 is in a dominated position;*
(b) *Player 1 does not activate her control before γ_2^* (that corresponds to the event $\{\sigma \geq \bar{\gamma}^*\}$ in (4.3)) and Player 2 concedes; Then at γ_2^* players play a continuation equilibrium in which Player 2 is in a dominated position.*

The function ℓ^* is constructed in order to make each player indifferent *between conceding and waiting at any moment of time when $X_t^0 > a_0$ (the so-called* indifference principle*). That guarantees that waiting a random time γ^* before conceding is a best reply. As this is true for both players, it is then a Nash equilibrium. In particular, we notice that γ^* is the first jump time of a Poisson process with a stochastic intensity $\ell^*(X_t^0)$, which is positive if and only if $X_t^0 > a_0$.*

Remark 4.6 (**Observable quantities for the two players).** *The random time γ_2^* depends on the realization of a private randomization device U_2 used by Player 2. As such, it is not directly observable by Player 1. Then, the definition of Player 1's strategy cannot depend explicitly on γ_2^*. Instead, Player 1's strategy is allowed to depend on the trajectories of B and $\bar{D}^*$ and on γ_1^*, which are observable quantities for Player 1. Actually, the former claim remains valid for any admissible dividend policy D chosen by Player 2. Since D may depend on a randomised stopping time γ_2 of Player 2's, then the Player 1's strategy depends indirectly on γ_2.*

Remark 4.7 (**Our notion of equilibrium).** *The definition of Nash equilibrium we use is non-standard (Definition 4.2). Indeed, both players use strategies. The pair of dividend policies induced by (Ξ^*, Ξ^*) is unique and well-defined (we check this carefully in Step 1 of the proof below), and no player can do better by choosing any other admissible* ***dividend policy*** *(Step 2 in the proof below). It implies that no player can do better by choosing another (randomised) strategy Ξ such that the pair (Ξ^*, Ξ) induces a pair of admissible dividend policies (uniqueness is not necessary here). However, for some strategies Ξ, the pair (Ξ^*, Ξ) may not induce any pair of admissible dividend policies (see Remark 4.3) and such strategies cannot be considered in our definition. In this sense, our notion of equilibrium lies in between the "strategy vs. strategy" (which would require delays or other constraints for example) and "strategy vs. control".*

Proof. The proof is divided into five main steps. In the first step, we show that Ξ^* satisfies the conditions of Definition 4.1. In the second step we show that the pair

$$\bar{L}_t^* = \Xi^*(U_1, t, B, \bar{D}^*) \quad \text{and} \quad \bar{D}_t^* = \Xi^*(U_2, t, B, \bar{L}^*), \tag{4.4}$$

is well-defined. In the third step we calculate the players' payoffs associated to the pair $(\bar{L}^*, \bar{D}^*)$. Then we show optimality of such pair in two subsequent steps.

Step 1. Notice first that $(u, t, \varphi, \zeta) \mapsto \Xi^*(u, t, \varphi, \zeta)$ is jointly measurable. For every $u \in [0, 1]$, the trajectory

$$t \to \Xi^*(u, t, \varphi, \zeta) = 1_{\{t \geq \bar{\gamma}^* \wedge \sigma\}} \Big[1_{\{\sigma \geq \bar{\gamma}^*\}} \Phi_{\bar{\gamma}^*}^*(t, \varphi) + 1_{\{\sigma < \bar{\gamma}^*\}} \Psi^*(t, \varphi, \zeta) \Big],$$

is non-decreasing, right-continuous and it satisfies the admissibility condition in Definition 4.1(iii) thanks to analogous properties of Φ^* and Ψ^*. It only remains to check that $t \to \Xi^*(u, t, \varphi, \zeta)$ is non-anticipative. Using that $b(x) \geq \alpha > 0$ for $x \in [0, \infty)$, we deduce that for all $(t, \varphi, \zeta) \in [0, \infty) \times C_0([0, \infty)) \times D_0^+([0, \infty))$

$$\Psi^*(t, \varphi, \zeta) = \sup_{0 \leq s \leq t} \Big(\zeta(s) - b\big(x + \mu_0 s + \sigma\varphi(s) - \zeta(s)\big) \Big)^+ = 1_{\{t \geq \tau_\alpha(\zeta)\}} \Psi^*(t, \varphi, \zeta),$$

where we set $\tau_\alpha(\zeta) = \inf\{t \geq 0 : \zeta_t \geq \alpha\}$ for $\zeta \in D_0^+([0, \infty))$. It follows that, denoting $\tau_\alpha = \tau_\alpha(\zeta)$

$$\begin{aligned} \Xi^*(u, t, \varphi, \zeta) &= 1_{\{t \geq \bar{\gamma}^* \wedge \sigma\}} \Big[1_{\{\sigma \geq \bar{\gamma}^*\}} \Phi_{\bar{\gamma}^*}^*(t, \varphi) + 1_{\{\sigma < \bar{\gamma}^*\}} 1_{\{t \geq \tau_\alpha\}} \Psi^*(t, \varphi, \zeta) \Big] \\ &= 1_{\{t \geq \bar{\gamma}^* \wedge \sigma\}} \Big[1_{\{\zeta_{\bar{\gamma}^*-} = 0\}} \Phi_{\bar{\gamma}^*}^*(t, \varphi) + 1_{\{\zeta_{\bar{\gamma}^*-} > 0\}} 1_{\{t \geq \tau_\alpha\}} \Psi^*(t, \varphi, \zeta) \Big] \\ &= 1_{\{t \geq \bar{\gamma}^* \wedge \tau_\alpha\}} \Big[1_{\{\zeta_{\bar{\gamma}^*-} = 0\}} \Phi_{\bar{\gamma}^*}^*(t, \varphi) + 1_{\{\zeta_{\bar{\gamma}^*-} > 0\}} \Psi^*(t, \varphi, \zeta) \Big], \end{aligned}$$

where we used that $\tau_\alpha \geq \sigma$, and that $\{\zeta_{\bar{\gamma}^*-} = 0\} = \{\sigma \geq \bar{\gamma}^*\}$. The final expression guarantees the non-anticipativity property because $\bar{\gamma}^*(u, \cdot)$ and τ_α are $(\mathcal{F}_t^{\mathbb{W}})$-stopping times.

Step 2. Here we show that there is a unique solution of (4.4). By independence of (U_1,U_2) from the Brownian motion, we can work on a product space

$$(\Omega,\mathcal{F}) := (\Omega_0\times[0,1]^2,\mathcal{F}^0\times\mathcal{B}([0,1]^2))$$

equipped with the product measure $\mathsf{P}:=\mathsf{Q}\times\lambda\times\lambda$. Here λ denotes the Lebesgue measure and $(\Omega_0,\mathcal{F}^0,\mathsf{Q})$ denotes a probability space on which the Brownian motion B is defined. Fix a treble $(\omega,u_1,u_2)\in\Omega_0\times[0,1]^2$ so that $(U_1,U_2)=(u_1,u_2)$ and the trajectory of the Brownian motion $t\mapsto B_t(\omega)$ is fixed (so is the trajectory $t\mapsto X^0_t(\omega)$). Then, the random times γ^*_i, $i=1,2$, from (4.2) are uniquely determined. Here we should use the notation $\gamma^*_i(u_i,\omega)$, for $i=1,2$, $\bar L^*(\omega,u_1,u_2)$ and $\bar D^*(\omega,u_1,u_2)$ to stress the dependence of these quantities on (ω,u_1,u_2). This is rather cumbersome, so we drop the explicit dependence on the treble (ω,u_1,u_2) but we emphasise that the rest of the construction in this step is performed *pathwise*.

First we show that (4.4) admits at most one solution. Let us assume that $(\bar L^*,\bar D^*)$ is a solution pair for (4.4). Then, we show that $t<\gamma^*_1\wedge\gamma^*_2\implies\bar L^*_t=\bar D^*_t=0$ (actually $\bar L^*_s=\bar D^*_s=0$ for $s\in[0,t]$, by monotonicity). Arguing by contradiction, assume $\bar L^*_t>0$ and $t<\gamma^*_1\wedge\gamma^*_2$. Then from the definition of $\bar L^*$ and (4.4), it must be $\sigma(\bar D^*)<t$ and $\bar L^*_t=\Psi^*(t,B,\bar D^*)$. Moreover, using the definition of Ψ^* and recalling that $b(x)\ge\alpha$ for $x\in[0,\infty)$, yields $\bar L^*_t=\Psi^*(t,B,\bar D^*)>0\implies\bar D^*_t>\alpha$. Then,

$$X^{\bar L^*}_t=X^0_t-\Psi^*(t,B,\bar D^*)\ge X^0_t-(\bar D^*_t-\alpha)^+=X^0_t-\bar D^*_t+\alpha=Y^{\bar D^*}_t+\alpha. \tag{4.5}$$

However, since $t<\gamma^*_1\wedge\gamma^*_2$ and $\bar L^*_t>0$ imply $\bar D^*_t>\alpha$, then we should also have $t<\gamma^*_1\wedge\gamma^*_2$ and $\bar D^*_t>0$. Therefore, we can repeat the argument above swapping the roles of the two players, and obtain

$$Y^{\bar D^*}_t\ge X^{\bar L^*}_t+\alpha. \tag{4.6}$$

Combining (4.5) and (4.6) leads to a contradiction and it must be $t<\gamma^*_1\wedge\gamma^*_2\implies\bar L^*_t=\bar D^*_t=0$, as claimed.

Next, we notice that $\gamma^*_1\le\gamma^*_2\implies\sigma(\bar D^*)\ge\gamma^*_1$, because Player 2 starts paying dividends only after $\gamma^*_2\wedge\sigma(\bar L^*)$. Therefore $\bar L^*_t=\Phi^*_{\gamma^*_1}(t,B)1_{\{t\ge\gamma^*_1\}}$ and $\bar D^*_t=\Psi^*(t,B,\bar L^*)1_{\{t\ge\gamma^*_1\}}$ (notice that $\gamma^*_1=\sigma(\bar L^*)$ in this case). Similarly, $\gamma^*_2\le\gamma^*_1\implies\bar D^*_t=\Phi^*_{\gamma^*_2}(t,B)1_{\{t\ge\gamma^*_2\}}$ and $\bar L^*_t=\Psi^*(t,B,\bar D^*)1_{\{t\ge\gamma^*_2\}}$. Then, if a solution of (4.4) exists, it is uniquely determined by the properties above.

The existence of the pair $(\bar L^*,\bar D^*)$ is now easy. Indeed, the pair (γ^*_1,γ^*_2) is exogenously determined by the realisation of the pair (U_1,U_2) and the trajectory of B. Moreover, the definition of ℓ^* and (2.12) imply $\ell^*(x)>0\iff x\in(a_0,\infty)$. Therefore

$$\begin{aligned}&t\mapsto\Gamma^{\ell^*}_t(\omega)\text{ defines a measure on }[0,\infty)\\&\text{which is supported by the set }\{t\ge0:X^0_t(\omega)\in[a_0,\infty)\}.\end{aligned} \tag{4.7}$$

That implies that γ^*_1 and γ^*_2 can only occur during excursions of the process X^0 into the half-line $[a_0,\infty)$, and thus that $X^0_{\gamma^*_i}\in[a_0,\infty)$ for all (u_i,ω), $i=1,2$. Hence, it follows that P-almost surely $\Phi^*_{\gamma^*_i}(t,B)>0$ for all $t\ge\gamma^*_i$, $i=1,2$. We deduce that if $\gamma^*_1<\gamma^*_2$, then P-almost surely $\sigma(\bar L^*)=\gamma^*_1$ and, viceversa, if $\gamma^*_1>\gamma^*_2$, then P-almost surely $\sigma(\bar D^*)=\gamma^*_2$.

Then, the solution of (4.4) is given by

$$(\bar L^*_t,\bar D^*_t)=\begin{cases}1_{\{t\ge\gamma^*_1\}}\Big(\Phi^*_{\gamma^*_1}(t,B),\Psi^*\big(t,B,\Phi^*_{\gamma^*_1}(\cdot,B)\big)\Big)&\text{on }\{\gamma^*_1<\gamma^*_2\},\\1_{\{t\ge\gamma^*_2\}}\Big(\Psi^*\big(t,B,\Phi^*_{\gamma^*_2}(\cdot,B)\big),\Phi^*_{\gamma^*_2}(t,B)\Big)&\text{on }\{\gamma^*_2<\gamma^*_1\}.\end{cases} \tag{4.8}$$

Notice that $\gamma^*_1(u_1,\omega)\neq\gamma^*_2(u_2,\omega)\iff u_1\neq u_2$ and that $(\lambda\times\lambda)(U_1=U_2)=0$, so that (4.8) characterises the pair $(\bar L^*,\bar D^*)$ up to a P-null set.

Step 3. Here we calculate the players' payoffs under the strategy pair $(\bar L^*,\bar D^*)$. We denote $X^*=X^{\bar L^*}$, $Y^*=Y^{\bar D^*}$ with the associated default times γ_{X^*} and γ_{Y^*}. We also denote $\gamma_0=\inf\{t\ge$

$0 : X^0_t = 0\}$. In order to keep track of randomisation, for any realisation $(U_1, U_2) = (u_1, u_2)$ we use the notation

$$(X^*, Y^*) = (X^{*;u_1u_2}, Y^{*;u_1u_2}) \quad \text{and} \quad (\bar{D}^*, \bar{L}^*) = (\bar{D}^{*;u_1,u_2}, \bar{L}^{*;u_1,u_2}).$$

These maps are measurable in (u_1, u_2) by construction.

Player 2's payoff reads

$$\mathcal{J}^2_{x,x}(\bar{D}^*, \bar{L}^*) = \int_0^1 \int_0^1 \mathcal{J}^2_{x,x}(\bar{D}^{*;u_1,u_2}, \bar{L}^{*;u_1,u_2}) \mathrm{d}u_1 \mathrm{d}u_2. \tag{4.9}$$

For a fixed pair (u_1, u_2) we have

$$\begin{aligned} &\mathcal{J}^2_{x,x}(\bar{D}^{*;u_1,u_2}, \bar{L}^{*;u_1,u_2}) \\ &= \mathsf{E}_{x,x}\Big[\int_{[0,\gamma_{X^{*;u_1,u_2}} \wedge \gamma_{Y^{*;u_1,u_2}}]} \mathrm{e}^{-rt} \mathrm{d}\bar{D}^{*;u_1,u_2}_t \\ &\qquad + 1_{\{\gamma_{X^{*;u_1,u_2}} < \gamma_{Y^{*;u_1,u_2}}\}} \mathrm{e}^{-r\gamma_{X^{*;u_1,u_2}}} \hat{v}(Y^{*;u_1,u_2}_{\gamma_{X^{*;u_1,u_2}}})\Big]. \end{aligned} \tag{4.10}$$

The expression under expectation is zero on the event $\{\gamma^*_1(u_1) \wedge \gamma^*_2(u_2) \geq \gamma_0\}$, because default for both firms occurs before they actually start paying any dividends.

On the complementary event, we consider separately the cases $\gamma^*_1(u_1) < \gamma^*_2(u_2)$ and $\gamma^*_1(u_1) > \gamma^*_2(u_2)$. Here it is convenient to recall the notation from Remark 3.6, i.e., $\Phi^*(x, t, \varphi)$ and $\Psi^*(x, y, t, \varphi, \zeta)$, in order to keep track of the position of the process X^0 at the time when the strategies of the two players are activated. We also introduce a shift for the trajectories in the canonical space, defined as $\theta_t(\varphi(\cdot)) = \varphi(t + \cdot) - \varphi(t)$ for $\varphi \in C_0([0, \infty))$. Finally, from the definition of Φ^*_s and Ψ^* it is not hard to see that for $t \geq \gamma^*_i(u_i)$, $i = 1, 2$,

$$\begin{aligned} \Phi^*_{\gamma^*_i(u_i)}(t, B) &= \Phi^*\big(X^0_{\gamma^*_i(u_i)}, t - \gamma^*_i(u_i), \theta_{\gamma^*_i(u_i)}(B.)\big) =: \tilde{\Phi}^*\big(t, \theta_{\gamma^*_i(u_i)}(B.)\big) \\ \Psi^*\big(t, B, \Phi^*_{\gamma^*_i(u_i)}\big) &= \Psi^*\Big(X^0_{\gamma^*_i(u_i)}, X^0_{\gamma^*_i(u_i)}, t - \gamma^*_1(u_i), \theta_{\gamma^*_i(u_i)}(B.), \tilde{\Phi}^*\big(t, \theta_{\gamma^*_i(u_i)}(B.)\big)\Big) \\ &=: \tilde{\Psi}^*\Big(t, \theta_{\gamma^*_i(u_i)}(B.), \tilde{\Phi}^*\Big). \end{aligned}$$

Notice that, for example, on the event $\{\gamma^*_1(u_1) < \gamma^*_2(u_2)\}$ we have $\bar{L}^*_t = \tilde{\Phi}^*(t, \theta_{\gamma^*_1(u_1)}(B.))$ for $t \geq \gamma^*_1(u_1)$, which is the optimal dividend policy in the classical dividend problem starting at $X^0_{\gamma^*_1(u_1)}$. Then $\bar{D}^*_t = \tilde{\Psi}^*(t, \theta_{\gamma^*_1(u_1)}(B.), \tilde{\Phi}^*)$ for $t \geq \gamma^*_1(u_1)$ is Player 2's response in the game starting at $X^0_{\gamma^*_1(u_1)}$, when Player 1's concedes. From this dicusssion it follows that

$$\{\gamma^*_1(u_1) < \gamma^*_2(u_2)\} \subset \{\gamma_{X^{*;u_1,u_2}} < \gamma_{Y^{*;u_1,u_2}}\}.$$

A symmetric situation occurs on the event $\{\gamma^*_1(u_1) > \gamma^*_2(u_2)\}$.

Continuing from (4.10), on the event $\{\gamma^*_1(u_1) < \gamma^*_2(u_2)\}$ we have

$$\begin{aligned} &\mathsf{E}_{x,x}\Big[1_{\{\gamma^*_1(u_1)<\gamma_0\}}1_{\{\gamma^*_1(u_1)<\gamma^*_2(u_2)\}}\Big(\int_{[\gamma^*_1(u_1),\gamma_{X^{*;u_1,u_2}}]} \mathrm{e}^{-rt}\mathrm{d}\bar{D}^{*;u_1,u_2}_t + \mathrm{e}^{-r\gamma_{X^{*;u_1,u_2}}} \hat{v}(Y^{*;u_1,u_2}_{\gamma_{X^{*;u_1,u_2}}})\Big)\Big] \\ &= \mathsf{E}_{x,x}\Big[1_{\{\gamma^*_1(u_1)<\gamma_0\}}1_{\{\gamma^*_1(u_1)<\gamma^*_2(u_2)\}}\mathsf{E}_{x,x}\Big[\int_{[\gamma^*_1(u_1),\gamma_{X^{*;u_1,u_2}}]} \mathrm{e}^{-rt}\mathrm{d}\bar{D}^{*;u_1,u_2}_t \\ &\qquad + \mathrm{e}^{-r\gamma_{X^{*;u_1,u_2}}} \hat{v}(Y^{*;u_1,u_2}_{\gamma_{X^{*;u_1,u_2}}})\Big|\mathcal{F}_{\gamma^*_1(u_1)}\Big]\Big]. \end{aligned}$$

Using the strong Markov property, on the event $\{\gamma^*_1(u_1) < \gamma_0\} \cap \{\gamma^*_1(u_1) < \gamma^*_2(u_2)\}$ we can write

$$\mathsf{E}_{x,x}\Big[\int_{[\gamma^*_1(u_1),\gamma_{X^{*;u_1,u_2}}]} \mathrm{e}^{-rt}\mathrm{d}\bar{D}^{*;u_1,u_2}_t + \mathrm{e}^{-r\gamma_{X^{*;u_1,u_2}}} \hat{v}(Y^{*;u_1,u_2}_{\gamma_{X^{*;u_1,u_2}}})\Big|\mathcal{F}_{\gamma^*_1(u_1)}\Big] = \mathcal{J}^2_{X^0_{\gamma^*_1(u_1)}, X^0_{\gamma^*_1(u_1)}}(\tilde{\Psi}^*, \tilde{\Phi}^*),$$

where the final expression is Player 2's expected payoff when the game starts from $(X^0_{\gamma_1^*(u_1)}, X^0_{\gamma_1^*(u_1)})$, Player 1 uses $\tilde\Phi^*$ and Player 2 uses $\tilde\Psi^*$. From the analysis in Section 3 we know that

$$\mathcal{J}^2_{X^0_{\gamma_1^*(u_1)},X^0_{\gamma_1^*(u_1)}}(\tilde\Psi^*,\tilde\Phi^*) = v_2\big(X^0_{\gamma_1^*(u_1)}, X^0_{\gamma_1^*(u_1)}\big),$$

with v_2 as in (3.54). Since $X^0_{\gamma_1^*(u_1)} \in (a_0,\infty)$ by (4.7) and, by construction, $v_2(x,y)=v_2(a_0,y)$ for $x\ge a_0$, then

$$\mathcal{J}^2_{X^0_{\gamma_1^*(u_1)},X^0_{\gamma_1^*(u_1)}}(\tilde\Psi^*,\tilde\Phi^*) = v_2\big(a_0, X^0_{\gamma_1^*(u_1)}\big).$$

That yields

$$\begin{aligned}&\mathsf{E}_{x,x}\Big[1_{\{\gamma_1^*(u_1)<\gamma_0\}}1_{\{\gamma_1^*(u_1)<\gamma_2^*(u_2)\}}\Big(\int_{[\gamma_1^*(u_1),\gamma_{X^{*;u_1,u_2}}]}\mathrm{e}^{-rt}\mathrm{d}\bar D_t^{*;u_1,u_2}+\mathrm{e}^{-r\gamma_{X^{*;u_1,u_2}}}\hat v(Y^{*;u_1,u_2}_{\gamma_{X^{*;u_1,u_2}}})\Big)\Big]\\&=\mathsf{E}_{x,x}\Big[1_{\{\gamma_1^*(u_1)<\gamma_0\}}1_{\{\gamma_1^*(u_1)<\gamma_2^*(u_2)\}}\mathrm{e}^{-r\gamma_1^*(u_1)}v_2\big(a_0,X^0_{\gamma_1^*(u_1)}\big)\Big].\end{aligned}\tag{4.11}$$

On the event $\{\gamma_2^*(u_2)<\gamma_1^*(u_1)\}$ the roles of the two players are reversed, in the sense that Player 2 adopts the strategy $\Phi^*_{\gamma_2^*(u_2)}$ and Player 1 uses Ψ^*. Continuing from (4.10), arguments analogous to the ones that yield (4.11) allow us to deduce

$$\begin{aligned}&\mathsf{E}_{x,x}\Big[1_{\{\gamma_2^*(u_2)<\gamma_0\}}1_{\{\gamma_1^*(u_1)>\gamma_2^*(u_2)\}}\int_{[\gamma_2^*(u_2),\gamma_{Y^{*;u_1,u_2}}]}\mathrm{e}^{-rt}\mathrm{d}\bar D_t^{*;u_1,u_2}\Big]\\&=\mathsf{E}_{x,x}\Big[1_{\{\gamma_2^*(u_2)<\gamma_0\}}1_{\{\gamma_1^*(u_1)>\gamma_2^*(u_2)\}}\mathrm{e}^{-r\gamma_2^*(u_2)}v_0\big(X^0_{\gamma_2^*(u_2)}\big)\Big],\end{aligned}\tag{4.12}$$

where we also use that $\gamma_{X^{*;u_1,u_2}}>\gamma_{Y^{*;u_1,u_2}}$ on the event $\{\gamma_1^*(u_1)>\gamma_2^*(u_2)\}$.

Combining (4.11) and (4.12) with (4.10) and (4.9) yields

$$\mathcal{J}^2_{x,x}(\bar D^*,\bar L^*)=\mathsf{E}_{x,x}\big[1_{\{\gamma_2^*<\gamma_1^*\wedge\gamma_0\}}\mathrm{e}^{-r\gamma_2^*}v_0(X^0_{\gamma_2^*})+1_{\{\gamma_1^*<\gamma_2^*\wedge\gamma_0\}}\mathrm{e}^{-r\gamma_1^*}v_2(a_0,X^0_{\gamma_1^*})\big].\tag{4.13}$$

By the same arguments we obtain an analogous expression for $\mathcal{J}^1_{x,x}(\bar L^*,\bar D^*)$. Therefore the two players' payoffs are well-defined under the strategy pair $(\bar L^*,\bar D^*)$.

Step 4. In this step and in the next one we show that $\bar D^*$ is Player 2's best response to Player 1's playing $\bar L^*$. In particular, in this step we are going to show that

$$\mathcal{J}^2_{x,x}(D,\bar L^*)\le V(x),\quad \text{for any admissible dividend policy } D,\tag{4.14}$$

where

$$V(x)=\sup_\tau \mathsf{E}_x\Big[\mathrm{e}^{-r\tau}v_0(X^0_\tau)(1-\Gamma^*_\tau)1_{\{\tau<\gamma_0\}}+\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big],\tag{4.15}$$

with $\Gamma^*=\Gamma^{\ell^*}$ and the supremum is taken over $(\mathcal{F}_t)$-stopping times (recall that $(\mathcal{F}_t)$ is the Brownian filtration). By independence of U_1 from $\mathcal{F}_\infty$ the expected payoff in $V(x)$ can be rewritten as

$$\begin{aligned}&\mathsf{E}_x\Big[\mathrm{e}^{-r\tau}v_0(X^0_\tau)(1-\Gamma^*_\tau)1_{\{\tau<\gamma_0\}}+\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big]\\&=\mathsf{E}_x\Big[\mathrm{e}^{-r\tau}v_0(X^0_\tau)1_{\{\tau<\gamma_1^*\}}1_{\{\tau<\gamma_0\}}+1_{\{\gamma_1^*\le\tau\wedge\gamma_0\}}\mathrm{e}^{-r\gamma_1^*}v_2(a_0,X^0_{\gamma_1^*})\Big],\end{aligned}\tag{4.16}$$

which coincides with the right-hand side of (4.13) with τ instead of γ_2^*. There is no loss of generality in taking stopping times for the filtration $(\mathcal{F}_t)$, because it is a well-known fact that the value function in (4.15) does not change if we allow τ to be chosen from the class of randomised stopping times. Then, a priori it must be $\mathcal{J}^2_{x,x}(\bar D^*,\bar L^*)\le V(x)$.

In order to prove (4.14), we will work with the filtration $\mathcal{G}_t$ generated by $\mathcal{F}_t$ and the random variable U_1 so that γ_1^* is a $(\mathcal{G}_t)$-stopping time. It is well-known and it is not hard to prove that, thanks to independence of B and U_1, B is also a $(\mathcal{G}_t)$-Brownian motion. Given an admissible dividend policy D, Player 2's payoff reads

$$\begin{aligned}\mathcal{J}^2_{x,x}(D,\bar{L}^*) &= \mathsf{E}_{x,x}\Big[\int_{[0,\gamma_{X^*}\wedge\gamma_Y]} \mathrm{e}^{-rt}\mathrm{d}D_t + 1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat{v}(Y^D_{\gamma_{X^*}})\Big]\\ &= \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}}\int_{[0,\gamma_Y]}\mathrm{e}^{-rt}\mathrm{d}D_t\\ &\qquad + 1_{\{D_{\gamma_1^*-}=0\}}\Big(\int_{[0,\gamma_{X^*}\wedge\gamma_Y]}\mathrm{e}^{-rt}\mathrm{d}D_t + 1_{\{\gamma_X^*<\gamma_Y\}}\mathrm{e}^{-r\gamma_X^*}\hat{v}(Y^D_{\gamma_X^*})\Big)\Big],\end{aligned} \tag{4.17}$$

where in the second equality we use that on $\{D_{\gamma_1^*-}>0\}$ it must be $\gamma_Y\le\gamma_{X^*}$ by definition of $\bar{L}^*$ (recall that γ_Y only depends on D whereas γ_{X^*} also depends on D because of the structure of $\bar{L}^*$).

We now make two claims which we will prove separately after the end of this proof, for the ease of exposition. The claims are:

$$\begin{aligned}&\mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}}\Big(\int_{[0,\gamma_{X^*}\wedge\gamma_Y]}\mathrm{e}^{-rt}\mathrm{d}D_t + 1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat{v}(Y^D_{\gamma_{X^*}})\Big)\Big]\\ &\le \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}}1_{\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\mathrm{e}^{-r\gamma_1^*}v_2(a_0,X^0_{\gamma_1^*})\Big].\end{aligned} \tag{4.18}$$

and

$$\mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}}\int_{[0,\gamma_Y]}\mathrm{e}^{-rt}\mathrm{d}D_t\Big] \le \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y\ge\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0\big(X^0_{\sigma(D)}\big)\Big], \tag{4.19}$$

where we recall $\sigma(D)=\inf\{t\ge0: D_t>0\}$.

We substitute (4.18) and (4.19) into (4.17). Then we use $\{D_{\gamma_1^*-}>0\}=\{\sigma(D)<\gamma_1^*\}$ and $\{\gamma_Y\ge\rho\}\subset\{\gamma_0\ge\rho\}$ for any $(\mathcal{G}_t)$-stopping time ρ, to obtain

$$\begin{aligned}&\mathcal{J}^2_{x,x}(D,\bar{L}^*)\\ &\le \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}}1_{\{\gamma_Y\ge\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0\big(X^0_{\sigma(D)}\big) + 1_{\{D_{\gamma_1^*-}=0\}}1_{\{\gamma_{X^*}\wedge\gamma_Y\ge\gamma_1^*\}}\mathrm{e}^{-r\gamma_1^*}v_2\big(a_0,X^0_{\gamma_1^*}\big)\Big]\\ &\le \mathsf{E}_{x,x}\Big[1_{\{\sigma(D)<\gamma_1^*\}}1_{\{\gamma_0\ge\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0\big(X^0_{\sigma(D)}\big) + 1_{\{\sigma(D)\ge\gamma_1^*\}}1_{\{\gamma_0\ge\gamma_1^*\}}\mathrm{e}^{-r\gamma_1^*}v_2\big(a_0,X^0_{\gamma_1^*}\big)\Big]\\ &\le \sup_\tau \mathsf{E}_{x,x}\Big[1_{\{\tau<\gamma_1^*\}}1_{\{\gamma_0\ge\tau\}}\mathrm{e}^{-r\tau}v_0(X^0_\tau) + 1_{\{\tau\ge\gamma_1^*\}}1_{\{\gamma_0\ge\gamma_1^*\}}\mathrm{e}^{-r\gamma_1}v_2(a_0,X^0_{\gamma_1^*})\Big] = V(x),\end{aligned}$$

where the supremum ranges over all stopping times of the filtration $(\mathcal{F}_t)$ and we notice that

$$1_{\{\gamma_0=\tau\}}v_0(X^0_\tau) = 1_{\{\gamma_0=\tau\}}v_0(0)=0. \tag{4.20}$$

Thus, we have established (4.14).

Step 5. In this step we show that $\mathcal{J}^2_{x,x}(\bar{D}^*,\bar{L}^*)=V(x)$ so that optimality of $\bar{D}^*$ against $\bar{L}^*$ follows from (4.14). Thanks to the symmetry of the set-up, that will conclude the proof of the theorem and show that $(\bar{L}^*,\bar{D}^*)$ is a Nash equilibrium.

Our observation (4.20) and an application of Itô's formula yield for any stopping time τ (see (3.46) for the notation)

$$
\begin{aligned}
&\mathsf{E}_{x,x}\Big[\mathrm{e}^{-r\tau}v_0(X^0_\tau)\big(1-\Gamma^*_\tau\big)1_{\{\tau<\gamma_0\}}+\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big]\\
&=\mathsf{E}_x\Big[\mathrm{e}^{-r(\tau\wedge\gamma_0)}v_0(X^0_{\tau\wedge\gamma_0})\big(1-\Gamma^*_{\tau\wedge\gamma_0}\big)+\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big]\\
&=v_0(x)+\mathsf{E}_x\Big[\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rs}(1-\Gamma^*_s)\big(\mathcal{A}v_0\big)(X^0_s)\mathrm{d}s\Big]\\
&\quad+\mathsf{E}_x\Big[\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rs-\int_0^s\ell^*(X^0_u)\mathrm{d}u}\big(v_2(a_0,X^0_s)-v_0(X^0_s)\big)\ell^*(X^0_s)\mathrm{d}s\Big].
\end{aligned}
$$

By definition of Γ^* (see (4.1)) and (2.13) we see that

$$
\begin{aligned}
(1-\Gamma^*_s)\big(\mathcal{A}v_0\big)(X^0_s)&=-\mathrm{e}^{-\int_0^s\ell^*(X^0_u)\mathrm{d}u}\big[rv_0(X^0_s)-\mu_0\big]^+\\
&=-\mathrm{e}^{-\int_0^s\ell^*(X^0_u)\mathrm{d}u}\big(v_2(a_0,X^0_s)-v_0(X^0_s)\big)\ell^*(X^0_s)
\end{aligned}
$$

Therefore, for any τ

$$
\mathsf{E}_{x,x}\Big[\mathrm{e}^{-r\tau}v_0(X^0_\tau)\big(1-\Gamma^*_\tau\big)1_{\{\tau<\gamma_0\}}+\int_0^{\tau\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big]=v_0(x), \tag{4.21}
$$

which yields $V(x)=v_0(x)$ for all $x\in[0,\infty)$.

Starting from (4.13) and integrating out the randomisation device U_2 of Player 2, we have

$$
\begin{aligned}
&\mathcal{J}^2_{x,x}(\bar D^*,\bar L^*)\\
&=\int_0^1\mathsf{E}_{x,x}\big[1_{\{\gamma^*_2(u)<\gamma^*_1\wedge\gamma_0\}}\mathrm{e}^{-r\gamma^*_2(u)}v_0(X^0_{\gamma^*_2(u)})+1_{\{\gamma^*_1<\gamma^*_2(u)\wedge\gamma_0\}}\mathrm{e}^{-r\gamma^*_1}v_2(a_0,X^0_{\gamma^*_1})\big]\mathrm{d}u\\
&=\int_0^1\mathsf{E}_{x,x}\Big[\mathrm{e}^{-r\gamma^*_2(u)}v_0(X^0_{\gamma^*_2(u)})\big(1-\Gamma^*_{\gamma^*_2(u)}\big)1_{\{\gamma^*_2(u)<\gamma_0\}}+\int_0^{\gamma^*_2(u)\wedge\gamma_0}\mathrm{e}^{-rt}v_2(a_0,X^0_t)\mathrm{d}\Gamma^*_t\Big]\mathrm{d}u\\
&=v_0(x),
\end{aligned}
$$

where the second equality is due to (4.16) and the final one is by (4.21).

Therefore, we have shown $\mathcal{J}^2_{x,x}(\bar D^*,\bar L^*)=V(x)=v_0(x)$, as needed. □

It remains to prove the formulae in (4.18) and (4.19).

Proof of (4.18) and (4.19). Let us start with the proof of (4.18).

First we recall that by construction $v_2(x,y)=v_2(a_0,y)$ for $x\geq a_0$. Second we recall that $X^{\bar L^*}_{\gamma^*_1}=X^0_{\gamma^*_1}\geq a_0$ on $\{D_{\gamma^*_1-}=0\}$ by definition of $\bar L^*$ and γ^*_1 (c.f. (4.7)). Finally we recall that $v_2(x,y)=u_2(x,y-x)$ (see (3.8)) and that $Z^{\bar L^*,D}_t=Z^D_t=Y^D_t-X^{\bar L^*}_t$. Then, on $\{D_{\gamma^*_1-}=0\}$, we have

$$
\mathrm{e}^{-r\gamma^*_1}v_2(a_0,Y^D_{\gamma^*_1})=\mathrm{e}^{-r\gamma^*_1}v_2(X^0_{\gamma^*_1},Y^D_{\gamma^*_1})=\mathrm{e}^{-r\gamma^*_1}v_2(X^*_{\gamma^*_1},Y^D_{\gamma^*_1})=\mathrm{e}^{-r\gamma^*_1}u_2(X^*_{\gamma^*_1},Z^D_{\gamma^*_1}).
$$

For any $(\mathcal{G}_t)$-stopping time $\rho \le \gamma_{X^*} \wedge \gamma_Z$, Itô's formula yields (c.f. (3.46) for the notation)

$$
\begin{aligned}
\mathrm{e}^{-r\gamma_1^*} u_2(X^*_{\gamma_1^*}, Z^D_{\gamma_1^*}) = {}& \mathsf{E}_{x,x}\Big[\mathrm{e}^{-r(\gamma_1^*\vee\rho)} u_2(X^*_{\gamma_1^*\vee\rho}, Z^D_{\gamma_1^*\vee\rho}) - \int_{\gamma_1^*}^{\gamma_1^*\vee\rho} \mathrm{e}^{-rs}(\mathcal{A}u_2)(X^*_s, Z^D_s)\mathrm{d}s\Big|\mathcal{G}_{\gamma_1^*}\Big] \\
& - \mathsf{E}_{x,x}\Big[\int_{\gamma_1^*}^{\gamma_1^*\vee\rho} \mathrm{e}^{-rs}\big(\partial_z u_2 - \partial_x u_2\big)(X^*_s, Z^D_s)\mathrm{d}\bar{L}^{*,c}_s\Big|\mathcal{G}_{\gamma_1^*}\Big] \\
& + \mathsf{E}_{x,x}\Big[\int_{\gamma_1^*}^{\gamma_1^*\vee\rho} \mathrm{e}^{-rs}\partial_z u_2(X^*_s, Z^D_s)\mathrm{d}D^c_s\Big|\mathcal{G}_{\gamma_1^*}\Big] \\
& - \mathsf{E}_{x,x}\Big[\sum_{s\in(\gamma_1^*,\gamma_1^*\vee\rho]} \mathrm{e}^{-rs}\big(u_2(X^*_s, Z^D_s) - u_2(X^*_{s-}, Z^D_{s-})\big)\Big|\mathcal{G}_{\gamma_1^*}\Big],
\end{aligned}
\tag{4.22}
$$

where we removed the stochastic integral, which is a $(\mathcal{G}_t)$-(local)martingale (standard localisation arguments may be used if needed).

Now we recall from Step 5 in the proof of Proposition 3.7 that $(\mathcal{A}u_2)(X^*_s, Z^D_s) \le 0$ for a.e. $s \ge 0$, $\partial_z u_2(X^*_s, Z^D_s) \ge 1$ for all $s \ge 0$. Moreover, on the event $\{D_{\gamma_1^*-} = 0\}$ we have that $\bar{L}^*_s = \Phi_{\gamma_1^*}(s, B)$ for $s \ge \gamma_1^*$. That implies $\bar{L}^*_s = \bar{L}^{*,c}_s$ and $\mathrm{d}\bar{L}^*_s = 1_{\{X^*_s = a_0\}}\mathrm{d}\bar{L}^*_s$ for $s > \gamma_1^*$. Thus, for every $s > \gamma_1^*$ we have

$$
(\partial_z u_2 - \partial_x u_2)(X^*_s, Z^D_s)\mathrm{d}\bar{L}^{*,c}_s = (\partial_z u_2 - \partial_x u_2)(a_0, Z^D_s)\mathrm{d}\bar{L}^{*,c}_s = 0,
$$

and, using $\partial_z u_2 \ge 1$,

$$
u_2(X^*_s, Z^D_s) - u_2(X^*_{s-}, Z^D_{s-})\big) = u_2(X^*_s, Z^D_s) - u_2(X^*_s, Z^D_{s-})\big) \le -\Delta D_s.
$$

Combining these observations with (4.22) and rewriting u_2 in terms of v_2 leads us to

$$
\begin{aligned}
& 1_{\{D_{\gamma_1^*-}=0\}}\mathrm{e}^{-r\gamma_1^*} v_2(a_0, Y^D_{\gamma_1^*}) = 1_{\{D_{\gamma_1^*-}=0\}}\mathrm{e}^{-r\gamma_1^*} v_2(X^*_{\gamma_1^*}, Y^D_{\gamma_1^*}) \\
& \ge 1_{\{D_{\gamma_1^*-}=0\}}\mathsf{E}_{x,x}\Big[\mathrm{e}^{-r(\gamma_1^*\vee\rho)} v_2(X^*_{\gamma_1^*\vee\rho}, Y^D_{\gamma_1^*\vee\rho}) + \int_{(\gamma_1^*,\gamma_1^*\vee\rho]} \mathrm{e}^{-rs}\mathrm{d}D_s\Big|\mathcal{G}_{\gamma_1^*}\Big].
\end{aligned}
\tag{4.23}
$$

Choose $\rho = \gamma_{X^*} \wedge \gamma_Y$. On the event $\{D_{\gamma_1^*-} = 0\} \cap \{\gamma_1^* < \gamma_{X^*} \wedge \gamma_Y\}$ we have

$$
v_2(X^*_{\gamma_{X^*}\wedge\gamma_Y}, Y^D_{\gamma_{X^*}\wedge\gamma_Y}) = 1_{\{\gamma_{X^*}<\gamma_Y\}} v_2(X^*_{\gamma_{X^*}\wedge\gamma_Y}, Y^D_{\gamma_{X^*}\wedge\gamma_Y}) = 1_{\{\gamma_{X^*}<\gamma_Y\}}\hat{v}(Y^D_{\gamma_{X^*}}),
$$

where the first equality holds because $v_2(x, 0) = 0$ and the second one because $v_2(0, y) = \hat{v}(y)$. For $\omega \in \{D_{\gamma_1^*-} = 0\} \cap \{\gamma_1^* = \gamma_{X^*} \wedge \gamma_Y\}$ the process D has no jump at γ_1^* because of the admissibility condition (2.2). Hence it must be also $\gamma_{X^*}(\omega) = \gamma_Y(\omega) = \gamma_0(\omega)$ and $X^*_{\gamma_0}(\omega) = Y^D_{\gamma_0}(\omega) = X^0_{\gamma_0}(\omega) = 0$. Thus, on the event $\{D_{\gamma_1^*-} = 0\} \cap \{\gamma_1^* = \gamma_{X^*} \wedge \gamma_Y\}$ we have

$$
v_2(X^*_{\gamma_1^*}, Y^D_{\gamma_1^*}) + \int_{(\gamma_1^*,\gamma_1^*]} \mathrm{e}^{-rs}\mathrm{d}D_s = v_2(0, 0) = 0 = \hat{v}(Y^D_{\gamma_1^*}),
$$

where for the integral we use that $(\gamma_1^*, \gamma_1^*] = \varnothing$.

Since $\{\gamma_1^* \le \gamma_{X^*} \wedge \gamma_Y\} \in \mathcal{G}_{\gamma_1^*}$, combining the observation above with (4.23), we conclude that

$$
\begin{aligned}
& \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\mathrm{e}^{-r\gamma_1^*} v_2(a_0, Y^D_{\gamma_1^*})\Big] \\
& \ge \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\Big(\int_{(\gamma_1^*,\gamma_{X^*}\wedge\gamma_Y]} \mathrm{e}^{-rs}\mathrm{d}D_s + 1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat{v}(Y^D_{\gamma_{X^*}})\Big)\Big].
\end{aligned}
\tag{4.24}
$$

Notice that

$$
v_2(a_0, Y^D_{\gamma_1^*}) - v_2(a_0, Y^D_{\gamma_1^*-}) \le -\Delta D_{\gamma_1^*},
$$

because $\partial_y v_2 = \partial_z u_2 \ge 1$. Moreover, on the event $\{D_{\gamma_1^*-} = 0\} \cap \{\gamma_1^* \le \gamma_{X^*} \wedge \gamma_Y\}$

$$
\int_{(\gamma_1^*,\gamma_{X^*}\wedge\gamma_Y]} \mathrm{e}^{-rs}\mathrm{d}D_s + \mathrm{e}^{-r\gamma_1^*}\Delta D_{\gamma_1^*} = \int_{[0,\gamma_{X^*}\wedge\gamma_Y]} \mathrm{e}^{-rs}\mathrm{d}D_s.
$$

Then, adding on both sides of (4.24) the quantity $\mathrm{e}^{-r\gamma_1^*}\Delta D_{\gamma_1^*}$ we obtain

$$\begin{aligned}&\mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\mathrm{e}^{-r\gamma_1^*}v_2(a_0,Y^D_{\gamma_1^*-})\Big]\\&\ge \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\Big(\int_{[0,\gamma_{X^*}\wedge\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s+1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Y^D_{\gamma_{X^*}})\Big)\Big].\end{aligned}\tag{4.25}$$

Finally, we notice that

$$\begin{aligned}&1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*>\gamma_{X^*}\wedge\gamma_Y\}}\Big(\int_{[0,\gamma_{X^*}\wedge\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s+1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Y^D_{\gamma_{X^*}})\Big)\\&=1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*>\gamma_{X^*}\wedge\gamma_Y\}}1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(X^0_{\gamma_{X^*}})=0,\end{aligned}$$

where the final equality uses that for every $\omega\in\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*>\gamma_{X^*}\wedge\gamma_Y\}$ we have $D_t(\omega)=\bar L^*_t(\omega)=0$ for all $t\in[0,\gamma_{X^*}(\omega)\wedge\gamma_Y(\omega)]$, hence implying that $\gamma_{X^*}(\omega)=\gamma_Y(\omega)=\gamma_0(\omega)$[4]. Combining with (4.25) we arrive at

$$\begin{aligned}&\mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}\cap\{\gamma_1^*\le\gamma_{X^*}\wedge\gamma_Y\}}\mathrm{e}^{-r\gamma_1^*}v_2(a_0,Y^D_{\gamma_1^*-})\Big]\\&\ge \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}=0\}}\Big(\int_{[0,\gamma_{X^*}\wedge\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s+1_{\{\gamma_{X^*}<\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(Y^D_{\gamma_{X^*}})\Big)\Big].\end{aligned}$$

The expression in (4.18) is finally obtained upon noticing that on $\{D_{\gamma_1^*-}=0\}$ we have $Y^D_{\gamma_1^*-}=X^0_{\gamma_1^*}$.

Now we prove (4.19). Recall $\sigma(D)=\inf\{t\ge0:D_t>0\}$. For any $(\mathcal{G}_t)$-stopping time $\rho\le\gamma_Y$, Itô's formula yields

$$\begin{aligned}&\mathrm{e}^{-r\sigma(D)}v_0(Y^D_{\sigma(D)})\\&=\mathsf{E}_{x,x}\Big[\mathrm{e}^{-r(\rho\vee\sigma(D))}v_0\big(Y^D_{\rho\vee\sigma(D)}\big)-\int_{\sigma(D)}^{\rho\vee\sigma(D)}\mathrm{e}^{-rs}\big(\mathcal{A}v_0\big)(Y^D_s)\mathrm{d}s+\int_{\sigma(D)}^{\rho\vee\sigma(D)}\mathrm{e}^{-rs}\partial_yv_0(Y^D_s)\mathrm{d}D^c_s\\&\quad-\sum_{s\in(\sigma(D),\sigma(D)\vee\rho]}\mathrm{e}^{-rs}\big(v_0(Y^D_s)-v_0(Y^D_{s-})\big)\Big|\mathcal{G}_{\sigma(D)}\Big],\end{aligned}\tag{4.26}$$

where we removed the stochastic integral (using standard localisation if needed). From (2.9) we know that for all $s\ge0$

$$\big(\mathcal{A}v_0\big)(Y^D_s)\le0,\quad \partial_yv_0(Y^D_s)\ge1,\quad v_0(Y^D_s)-v_0(Y^D_{s-})\le\int_{Y^D_{s-}}^{Y^D_s}\partial_yv_0(u)du\le-(D_s-D_{s-}).$$

Combining these with (4.26) and taking $\rho=\gamma_Y$, we obtain

$$\begin{aligned}&1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y>\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0(Y^D_{\sigma(D)})\\&\ge1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y>\sigma(D)\}}\mathsf{E}_{x,x}\Big[\int_{(\sigma(D),\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s\Big|\mathcal{G}_{\sigma(D)}\Big].\end{aligned}$$

[4]Notice that $X^*_t(\omega)=Y^D_t(\omega)=X^0_t(\omega)$ for $t\in[0,\gamma_{X^*}(\omega)\wedge\gamma_Y(\omega)]$. So even if we consider the event $\{\gamma_{X^*}\le\gamma_Y\}$ instead of $\{\gamma_{X^*}<\gamma_Y\}$ we obtain $1_{\{\gamma_{X^*}\le\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(X^0_{\gamma_{X^*}})=1_{\{\gamma_{X^*}\le\gamma_Y\}}\mathrm{e}^{-r\gamma_{X^*}}\hat v(0)=0$.

Since $v_0(Y^D_{\sigma(D)}) - v_0(Y^D_{\sigma(D)-}) \le -\Delta D_{\sigma(D)}$, then adding $\mathrm{e}^{-r\sigma(D)}\Delta D_{\sigma(D)}$ on both sides in the expression above and recalling that $Y^D_t(\omega) = X^0_t(\omega)$ for $t \in [0, \sigma(D)(\omega))$, yields

$$\begin{aligned}
&1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y>\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0(X^0_{\sigma(D)})\\
&\ge 1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y>\sigma(D)\}}\mathsf{E}_{x,x}\Big[\int_{[\sigma(D),\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s\Big|\mathcal{G}_{\sigma(D)}\Big]\\
&= 1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y>\sigma(D)\}}\mathsf{E}_{x,x}\Big[\int_{[0,\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s\Big|\mathcal{G}_{\sigma(D)}\Big].
\end{aligned} \tag{4.27}$$

Now we notice that on $\{\gamma_Y = \sigma(D)\}$ we have $Y^D_{\sigma(D)} = Y^D_{\gamma_Y} = 0$ and $Y^D_{\sigma(D)-} = X^0_{\sigma(D)}$. It follows that $\Delta D_{\sigma(D)} = X^0_{\sigma(D)}$ and $v_0(Y^D_{\sigma(D)}) - v_0(Y^D_{\sigma(D)-}) = v_0(0) - v_0(X^0_{\sigma(D)}) = -v_0(X^0_{\sigma(D)})$. Since $v_0' \ge 1$ on $[0,\infty)$, then

$$-v_0(X^0_{\sigma(D)}) = v_0(Y^D_{\sigma(D)}) - v_0(Y^D_{\sigma(D)-}) = -\int_0^{\Delta D_{\sigma(D)}} v_0'(Y^D_{\sigma(D)-} - y)\mathrm{d}y \le -\Delta D_{\sigma(D)}.$$

We deduce that on the event $\{D_{\gamma_1^*-} > 0\} \cap \{\gamma_Y = \sigma(D)\}$

$$\int_{[0,\gamma_Y]} \mathrm{e}^{-rs}\mathrm{d}D_s = e^{-r\sigma(D)}\Delta D_{\sigma(D)} \le e^{-r\sigma(D)}v_0(X^0_{\sigma(D)}).$$

Notice also that on the event $\{D_{\gamma_1^*-} > 0\} \cap \{\gamma_Y < \sigma(D)\}$,

$$\int_{[0,\gamma_Y]} \mathrm{e}^{-rs}\mathrm{d}D_s = 0.$$

Taking expectation in (4.27) and using these observations we can conclude

$$\mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}}\int_{[0,\gamma_Y]}\mathrm{e}^{-rs}\mathrm{d}D_s\Big] \le \mathsf{E}_{x,x}\Big[1_{\{D_{\gamma_1^*-}>0\}\cap\{\gamma_Y\ge\sigma(D)\}}\mathrm{e}^{-r\sigma(D)}v_0\big(X^0_{\sigma(D)}\big)\Big],$$

as claimed in (4.19). □

Appendix A.

In this short appendix we recall a simple useful lemma (see, e.g., [17, Lem. 4.4]).

Lemma A.1. *Let $(\nu_t)_{t\ge 0}$ be a càdlàg process of bounded variation and let $(M_t)_{t\ge0}$ be a continuous semimartingale. Assume there is a positive, (locally) integrable process $(m_t)_{t\ge0}$ such that*

$$\langle M\rangle_t = \int_0^t m_s\mathrm{d}s, \quad t \ge 0,$$

and $m_t \ge \varepsilon$ for all $t \ge 0$, for some $\varepsilon > 0$. Then

$$\mathsf{E}\Big[\int_0^T 1_{\{M_s=\nu_s\}}\mathrm{d}s\Big] = 0.$$

Proof. Set $N := M - \nu$ and let $h_\delta(z) := 1_{(-\delta,\delta)}(z)$. By the occupation time formula (see, e.g., [19, Thm. IV.45.1]) we have

$$\int_0^T h_\delta(N_s)\mathrm{d}\langle N\rangle_s = \int_{\mathbb{R}} h_\delta(z)\ell^z_T\mathrm{d}z = \int_{-\delta}^{\delta}\ell^z_T\mathrm{d}z, \quad \mathsf{P}-a.s.$$

where $(\ell^z_t)_{t\ge0}$ is the local time at $z \in \mathbb{R}$ of the process N. The left-hand side of the expression above is finite and therefore $\ell^z_T < \infty$, P-a.s., for a.e. $z \in (-\delta,\delta)$. Letting $\delta \downarrow 0$, using the dominated convergence theorem on both sides of the expression, we obtain

$$\begin{aligned} 0 &= \int_0^T 1_{\{N_s=0\}} \mathrm{d}\langle N\rangle_s = \int_0^T 1_{\{M_s=\nu_s\}} \mathrm{d}\langle M\rangle_s \\ &= \int_0^T 1_{\{M_s=\nu_s\}} m_s \mathrm{d}s \ge \varepsilon \int_0^T 1_{\{M_s=\nu_s\}} \mathrm{d}s, \quad \mathsf{P}-a.s., \end{aligned}$$

where for the second equality we recall that ν is of bounded variation. The final expression yields the claim of the lemma. □

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