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Privatization under Political Ties*

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Abstract

I study a product differentiation model where a politically tied public firm competes with a private one in qualities and prices. I show that the public firm may mimic the median voter's preferences to maximize consensus. Then, I study the effects of partial privatization. Depending on the degree of privatization, (i) the private firm may enter the market or not; (ii) the public firm's profits may exceed those of its competitor. I also show that the socially optimal degree of privatization is interior.

Keywords Mixed Markets, Privatization, Product Differentiation.
JEL Classification L33, H44, D72.

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1 Introduction

State-owned enterprises (SOEs) compete with private ones in many markets. Many SOEs operate in both developed and developing countries.[1] A recent IMF study reports that: "*State-owned enterprises' assets are worth $45 trillion, equivalent to half of global GDP*" (IMF [2021]).

SOEs compete with private firms in the goodion of goods of different qualities. Previous work refers to these markets as mixed oligopolies. Benassi et al. [2016] show evidence that SOEs usually produce goods of lower quality than their private competitor. However, that is not necessarily the case. Brunello and Rocco [2008] report the case of education, where "*evidence suggests that private schools are heterogeneous, with some offering poorer academic quality and some others offering better quality than public schools.*"

How do SOEs affect welfare? On the one hand, political connections could negatively affect markets' efficiency in allocation and goodion (Boycko et al. [1996]). On the other hand, in the interests of consumers and voters, SOEs may reduce markups and tackle externalities (Willner [2001]). To answer this question, it is necessary to make assumptions about SOEs' objective functions. While it is reasonable to assume that private firms maximize profits, the objective function of SOEs is not straightforward.

Previous work assumes that SOEs maximize welfare. This assumption can be appropriate when benevolent technocrats run public firms.[2] However, it does not capture the incentives of politicians who may control public firms. Moreover, the public authority may own only a share of the stocks of a given company (partial privatization).

In this paper, I show that in a product differentiation (PD) model à la Tirole [1988], a politically tied public firm may care about the median voter because the policy preferred by them is always a Condorcet winner if consumers vote between two alternative combinations of qualities and prices. Then, I study the impact of partial privatization on market structure, welfare, and profits.

In my model, consumers are heterogeneous in their willingness to pay for quality and buy one unit of an indivisible good. Firms' marginal costs are increasing in quality. The timing of the game has four steps. First, firms choose whether to enter the market or not. Second, firms choose the quality of their

[1]See Smith and Trebilcock [2001], Heywood et al. [2021], and the references therein, for evidence about the presence of SOEs in modern economies.

[2]Throughout the paper, I refer to "public" firms as SOEs.

goods. Third, firms choose prices. Finally, consumers buy one of the two goods.

I capture partial privatization by assuming that the public firm maximizes a convex combination between its own profits and the median voter's utility. Intuitively, the partially privatized public firm's board comprises a share of $\lambda \in [0, 1]$ managers nominated by private shareholders aiming at profit maximization and a share of $(1 - \lambda)$ managers nominated by the incumbent politician seeking to increase the median voter's utility. This simple framework allows me to study the effect of the degree of privatization on market structure, welfare, and profits.

I start my analysis by providing a political economy micro-foundation of the public firm's objective function. I show that even if the policy space is bi-dimensional (quality and price), since consumers' preferences satisfy the single crossing property, the median voter theorem (MVT) applies (Gans and Smart [1996]). In particular, suppose that consumers are asked to vote between two alternative policies (two alternative pairs of qualities and prices). The median voter is always pivotal: a policy is preferred to another by a majority of voters if and only if the median voter prefers it. This result implies that welfare-maximizing policies may not be politically feasible. I proceed by solving for the market equilibrium.

I show that a Subgame Perfect Nash Equilibrium (SPNE) exists for any possible degree of privatization. In the price stage, the public firm wants to serve the median voter if and only if the price of the private firm's good is high enough. There exist three classes of Nash Equilibria (NE) in prices such that both firms face positive demands. In the first class, the median voter buys from the public firm. In the second one, they buy from the private firm. In the third class, the equilibrium is in mixed strategies. Duopoly NE in prices are always unique. Possibly, there also exists NE in prices where only the public firm is active. In general, there always exists a mapping from a pair of qualities to a pair of NE prices. In the quality stage, firms anticipate what NE price will be played. By solving for a NE in this stage, I obtain the following welfare results.

The degree of privatization determines the market structure. When privatization is low, the public firm produces a high quality and sets prices below marginal costs. The private firm has no incentive to enter, so the market is a public monopoly.[3] In the public monopoly SPNE, welfare can be negative because consumer surplus does not compensate for the losses of the public firm. Welfare

[3]I allow the public firm to run a deficit. I introduce a strict budget constraint as an extension.

increases with privatization. When privatization is high enough, the market is a duopoly. There are two classes of duopoly SPNE, depending on which firm produces for the majority of consumers. In the first (respectively, second) class, privatization is low (respectively, high), and the public (respectively, private) firm produces for the majority of consumers. Each class features a SPNE where the public firm produces the high-quality good and another where the private firm produces the high-quality good. Within each class, these two equilibria are payoffs- (and welfare-) equivalent.

The degree of privatization also determines market profits. If the degree of privatization is sufficiently low, the public firm makes negative profits. If the degree of privatization is neither too high nor too low, surprisingly, the public firm makes higher profits than its private competitor. The intuition is simple. The median voter's utility maximization is a commitment device. The public firm produces an "intermediate" quality level, attracting the majority of consumers. The private firm differentiates its good for the minority of consumers whose willingness to pay is far from the median (either very high or very low). The private firm suffers from fierce competition. Its clients would only have to sacrifice a little utility if they had to buy the public firm's good, and so cannot be charged a too high markup. On the contrary, most of the public firm's customers would have to sacrifice a lot of utility to buy the private firm's good. Then, the public firm can charge a higher markup. If privatization is high enough, the private firm faces higher demand and makes more profits than the public firm.

The welfare-maximizing degree of privatization is interior. When the public firm only cares about the median voter, the market is a public monopoly, and welfare is low. In the duopoly equilibria, if privatization is low, consumers enjoy low-quality levels, and firms make small profits. On the contrary, when privatization is too high, profits are higher, but the high markups harm consumer surplus. I show evidence of a trade-off between high-quality levels and low markups. When privatization is high, no firm wants to invest in quality without making consumers pay for the cost of the investment. When privatization is low, the public firm would be willing to do so, but this would come at the cost of large economic losses and a waste of economic resources.

I introduce some extensions to explore the robustness of my results. First, I introduce a strict budget constraint on the public firm's maximization problem. The constraint can improve welfare, but partial privatization is still socially optimal. Second, I allow consumers not to buy any good (partial market coverage).

In this case, I show that privatization decreases the share of buyers in equilibrium. Hence, privatization may raise inequality concerns, especially in the case of "essential" goods, like education or healthcare.

The remainder of the paper is organized as follows. In the next section, I provide a brief literature review. In Section 3, I present the model. In Section 4, I provide a micro-foundation of the public firm's objective function. In Section 5, I characterize the SPNE. In Section 6, I introduce some extensions. Finally, Section 7 concludes.

2 Related Literature

I contribute to the theory of competition in differentiated markets between public and private firms. Previous work separates into two streams of literature: Political Economy and Industrial Organization. My main contribution is to link these two approaches.

A broad literature in Political Economy presents voting models for the public provision of private goods when private competitors are in the market (see Stiglitz [1974], Epple and Romano [1996], Lülfesmann and Myers [2011], Dotti [2019], and the papers cited therein). Unlike this literature, my paper allows for market power and endogenous PD.

A growing literature in Industrial Organization (IO) studies public vs. private competition with endogenous PD. These papers usually employ the standard PD model of Tirole [1988] in a duopoly where a welfare-maximizing firm competes with a private one.[4] Contributions differ from each other in the following dimensions. On the supply side, one could assume different cost structures and welfare functions for the public firm's objective function; on the demand side, full or partial market coverage and different distributions for consumers' willingness to pay for quality. With uniform distributions of consumers' types, Grilo [1994] considers a cost function increasing in quality and linear in quantity. She shows that the mixed duopoly with a welfare-maximizing public firm can replicate the first best equilibrium outcome.[5] Delbono et al. [1991] extend this result to the partial market coverage case with a cost function quadratic

[4]See Shaked and Sutton [1982], Wauthy [1996], Motta [1993], among others, for models of vertical PD.

[5]Cremer and Thisse [1991] shows that the vertical differentiation model with quadratic costs for quality is equivalent to the Hotelling-type model with quadratic transportation costs. Cremer et al. [1991] prove that location choices in a mixed oligopoly are efficient when the public firm minimizes total transportation costs and the number of firms is 2 or greater or equal than 6.

in quality. Benassi et al. [2016], Benassi et al. [2017] and Laine and Ma [2017] relax the uniform distribution assumption for consumers' types and derive conditions for the social desirability of privatization. Some papers also relax the welfare-maximizing assumption. For instance, the public firm is subjected to price regulation in Estrin and De Meza [1995]. Klumpp and Su [2019] assume that public firms have a preference for re-distribution. In Benassi et al. [2016], the public firm cares about consumer surplus only. Inoue et al. [2009] assume that the public firm cares only about the welfare of its clients.

Some papers consider the case of partial privatization (Matsumura [1998], Lu and Poddar [2007], Ishibashi and Kaneko [2008]).[6] In most papers, partial privatization is captured by assuming that the public firm maximizes a convex combination between social welfare and its profits. Unlike these papers, I consider the median voter's utility.

3 Model

In this Section, I describe the model.

Players There are two firms. Firms are indexed by $i \in \{0, 1\}$, where 0 is the public firm, and 1 is the private one. Throughout the paper, I refer to good i as the good produced by firm i.

There is a continuum of consumers. Each consumer has a type θ, which captures their marginal willingness to pay (WTP) for quality. Types are uniformly distributed: $\theta \sim \mathcal{U}[\theta_h - 1, \theta_h]$, with $\theta_h > 1$.[7] The type θ proxies consumers' income (Motta [1993], Grilo [1994]). There is perfect and symmetric information.

Actions and Timing The timing of the game runs as follows.

Stage (1) Firm i chooses $a_i \in \{1, 0\}$, where $a_i = 1$ denotes the action of entering the market, and $a_i = 0$ the action of staying out. Firms that choose $a_i = 0$ do

[6] For example, Ishibashi and Kaneko [2008] show that partial privatization can be socially desirable in a vertically differentiated mixed duopoly if there are fixed costs of quality improvement and the public firm cares about social welfare. Nabin et al. [2014] proves that partial privatization can be optimal when the public firm cares about its customers' surplus and faces zero goodion costs.

[7] The uniform distribution assumption is standard in the PD literature. Some recent papers (Benassi et al. [2016], Benassi et al. [2017], Laine and Ma [2017]) relax this assumption in PD models where a welfare-maximizing firm competes with a profit-seeking one.

not move again. If $a_1 = a_0 = 0$, the game ends. If at least one firm enters the market, the game proceeds to the second stage.

Stage (2) Firms simultaneously choose the quality of their good $q_i \in [0, Q]$. The parameter $Q > 0$ represents the highest technologically feasible quality level.

Stage (3) Firms simultaneously choose the price of their good $p_i \in [0, \infty)$.

Stage (4) Each consumer chooses which good to buy.[8]

Payoffs The payoff of a consumer of type θ when they buy good i is:

$$u(q_i, p_i \mid \theta) = \theta q_i - p_i \ . \tag{1}$$

Firm i's profit function is:

$$\pi_i(x_i, q_i, p_i, a_i) = a_i \left[\left(p_i - \alpha q_i^2 \right) x_i - \Phi \right] \tag{2}$$

where x_i is the output produced by firm i, $\alpha > 0$, and $\Phi \geq 0$ is the cost of entry. Quality is costly. Firms' marginal cost of output is constant but quadratic in quality. Since consumers have unitary demand, then $x_i \in [0, 1]$.

Firm 1 maximizes profits. Firm 0's payoff is a convex combination between its profits and the utility of the median voter $\overline{u}$:

$$V_0(\pi_0, \overline{u}) = \lambda \pi_0 + (1 - \lambda) \overline{u} \ . \tag{3}$$

The median voter is the consumer with the median type.[9] The median voter's type is: $\overline{\theta} = \frac{2\theta_h - 1}{2}$. In *Stage* (4), the median voter chooses which good to buy. Then,

$$\overline{u} = \max \left\{ \overline{\theta} q_1 - p_1, \overline{\theta} q_0 - p_0 \right\} \ . \tag{4}$$

[8]I assume that consumers cannot avoid consumption. This assumption is appropriate when demand is highly inelastic, as for healthcare or education (Grilo [1994]). If consumers could also choose not to buy any good and get a payoff of zero, full market coverage would emerge endogenously if either of the following additional assumptions holds. First, I could assume that consumers' utility from good i is: $u(q_i, p_i \mid \theta) = y + \theta q_i - p_i$, where y is high enough. Second, all consumers would always buy if θ_h is high enough (Wauthy [1996]). Any of these assumptions would not change players' maximization problems. As an extension, I study the case of partial market coverage in Section 6.2.

[9]In Section 4, I show that this consumer would be pivotal in any majority voting competition where consumers had to vote between two alternative policies (price and quality of firm 0).

The parameter $\lambda \in [0, 1]$ captures the degree of partial privatization (Matsumura [1998], Kumar and Saha [2008]). There are at least two alternative interpretations of equation (3). First, the profits of the public firm may enter directly into consumers' (or the government's) utility functions as these allow for tax cuts. Second, firm 0 may be a politically connected private firm. For example, politicians may nominate or have friendships with managers (Akcigit et al. [2023]). Alternatively, the private firm may be concerned about the consensus over its action among voters and consumers, as postulated by the stakeholder theory of the firm (Donaldson and Preston [1995]).

4 The Political Economy Block

In this Section, I discuss why a politically-tied public firm may care about the median voter's payoff.

A main potential threat to the pivotality of the median voter is the bi-dimensionality of the policy space. Nonetheless, Gans and Smart [1996] provides a sufficient condition for applying the MVT in more than one dimension: the single-crossing property of preferences. To introduce this argument, let me define some additional notation. Consider the following two-dimensional policy space: $\Delta = [0, Q] \times [0, \infty)$. Any point in this space $\delta \in \Delta$ represents a combination of quality and price. Define the majority voting preference relation $\geq_{MV}$ as follows. For all $\delta', \delta \in \Delta$, $\delta' \geq_{MV} \delta$ if and only if $u\left(\delta' \mid \theta\right) > u\left(\delta \mid \theta\right)$ for all $\theta \in A$, where $A \subseteq [\theta_h - 1, \theta_h]$, $\mu(A) \geq \frac{1}{2}$, and μ is the Lebesgue measure. Suppose consumers are asked to vote between two alternative policies implementable by firm 0, δ and δ'. The following Proposition shows the pivotality of the median voter.

Proposition 1. *(Gans and Smart [1996])*
For all $\delta', \delta \in \Delta$,

$$\delta' \geq_{MV} \delta \Leftrightarrow u\left(\delta' \mid \overline{\theta}\right) > u\left(\delta \mid \overline{\theta}\right) \tag{5}$$

Proof. See Appendix A.1 □

The intuition behind this result is simple. Consumers' preferences (1) satisfy the single crossing property. Take two policy vectors: δ' and δ such that $\delta' > \delta$. Under the policy δ', all consumers receive a higher quality but also pay a higher price than under δ. The single crossing property requires that if a consumer with type θ prefers δ' over δ, then all consumers with type $\theta' > \theta$ also prefer δ' over δ. Since one could map consumers' type θ to their preferred policy δ, and this mapping would be monotonic, the MVT holds: δ' can be supported by a majority of voters against δ if and only if the median voter prefers δ' over δ. Proposition 1 is just an application of Theorems 1 and 2 in Gans and Smart [1996]. However, it provides a solid micro-foundation to the reduced-form approach I adopted for the public firm's objective function. Suppose that the public firm cares about its consensus. Then, the public firm knows that it can maximize the consensus by aligning its preferences with those of the median voter.

Proposition 1 is robust to a number of refinements. First, Proposition 1 does not require the distribution of types θ to be uniform. Second, it does not require payoffs to be linear. The only requirement is that payoffs satisfy single crossing, which is, to the best of my knowledge, always the case in PD models.[10] I now study the welfare implications of a partial privatization policy of firm 0.

5 Market Equilibrium

In this Section, I look for Subgame Perfect Nash Equilibria (SPNE) of the game described in Section 5.1. I proceed by Backward Induction (BI). If not specified, starred variables denote SPNE values.

5.1 Demand Functions

I start by solving consumers' maximization problem in *Stage* (4). Thus, I obtain the demand functions faced by the firms in *Stage* (3).

Suppose $a_0 = a_1 = 1$. In *Stage* (4), each consumer chooses which of the two goods to buy. Let $j \in \{0, 1\}$ and $j \neq i$. Suppose $q_i > q_j$ and $p_i > p_j$. The indifferent consumer $\hat{\theta}(p_0, p_1, q_0, q_1)$ is:

$$\hat{\theta}\left(p_i, p_j, q_i, q_j\right) = \frac{p_i - p_j}{q_i - q_j} . \tag{6}$$

By the single crossing property of preferences, all consumers with $\theta > \hat{\theta}\left(p_i, p_j, q_i, q_j\right)$

[10] In addition, given any possible subgame NE price pair as a function of qualities, the public firm's quality that maximizes the median voter's payoff is again a majority voting NE.

buy good i.[11] If $(q_i - q_j)(p_i - p_j) < 0$, then one firm is a monopolist.[12]

Demand functions are as follows. The demand for firm i is:

$$x_i\left(\hat{\theta}\left(p_i, p_j, q_i, q_j\right), p_i, p_j, q_i, q_j\right) =$$
$$\begin{cases} \theta_h - \hat{\theta}\left(p_i, p_j, q_i, q_j\right) & \text{if } p_i > p_j,\, q_i > q_j, \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) \in [\theta_h - 1, \theta_h] \\ \hat{\theta}\left(p_i, p_j, q_i, q_j\right) - (\theta_h - 1) & \text{if } p_i < p_j,\, q_i < q_j, \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) \in [\theta_h - 1, \theta_h] \\ 0 & \text{if } p_i > p_j,\, q_i > q_j \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) > \theta_h; \\ & \text{or } p_i < p_j,\, q_i < q_j \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) < \theta_h - 1\,; \\ & \text{or } q_i = q_j \text{ and } p_i > p_j; \\ & \text{or } p_i = p_j \text{ and } q_i < q_j \\ 1 & \text{if } p_i > p_j,\, q_i > q_j \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) < \theta_h - 1; \\ & \text{or } p_i < p_j,\, q_i < q_j \text{ and } \hat{\theta}\left(p_i, p_j, q_i, q_j\right) > \theta_h; \\ & \text{or } q_i = q_j \text{ and } p_i < p_j; \\ & \text{or } p_i = p_j \text{ and } q_i > q_j \\ \frac{1}{2} & \text{if } p_i = p_j \text{ and } q_i = q_j \end{cases} . \tag{7}$$

The demand for firm j is:

$$x_j\left(\hat{\theta}\left(p_i, p_j, q_i, q_j\right), p_i, p_j, q_i, q_j\right) = 1 - x_i\left(\hat{\theta}\left(p_i, p_j, q_i, q_j\right), p_i, p_j, q_i, q_j\right) . \tag{8}$$

Firms' demand functions are not continuous on Δ. To see this, fix $q_i = q_j$. Then,

$$\begin{aligned} x_i\left(p_j, p_j, q_j, q_j\right) &= \frac{1}{2} \\ \lim_{p_i \to p_{j-}} x_i\left(p_i, p_j, q_j, q_j\right) &= 1 \\ \lim_{p_i \to p_{j+}} x_i\left(p_i, p_j, q_j, q_j\right) &= 0\,. \end{aligned} \tag{9}$$

If $q_i = q_j$, firm i's demand function jumps discontinuously from one to one half as soon as p_i "hits" p_j from the left, and it jumps again to zero as soon as p_i "crosses" p_j. Therefore, if $q_i = q_j$, firms' demand functions are discontinuous at $p_i = p_j$.[13] Payoff functions are also not continuous on Δ.

[11]See appendix A.1 for a formal illustration of the single crossing property.

[12]Assume $q_i > q_j$ but $p_i < p_j$. In this case, all consumers buy good i.

[13]The discontinuity of the demand functions does not depend on the tie-breaking rule. I assume that if $q_i = q_j$ and $p_i = p_j$, half of the consumers break the tie in favor of good i, and the other half break in favor of good j. However, the demand functions are discontinuous under any possible tie-breaking rule. The tie-breaking rule does not affect the existence of a symmetric SPNE.

5.2 Price Stage

In this Section, I show the existence, uniqueness, and characterization of a NE in *Stage* (3).

There are several challenges to the identification of a NE. First, firms' payoff functions are discontinuous (see Section 5.1). Therefore, the standard Glicksberg [1952]'s result about the existence of a NE does not go through. Firms' payoffs are not even lower semi-continuous, so the theorems in Dasgupta and Maskin [1986] cannot be applied.

Second, because of the max operator in (4), firm 0's payoff is not quasi-concave and not single peaked. Firm 0 needs to anticipate the action of the median voter, generating a potential discontinuity in its best response function. Rewriting condition (4), the median voter buys good 0 if and only if:

$$p_0 \leq p_1 - \overline{\theta}(q_1 - q_0) \ . \tag{10}$$

For a given p_1, by moving p_0, firm 0 can determine whether to serve the median voter or not. To construct a NE, I need to characterize the set of all p_1 such that serving the median voter is optimal for firm 0.

Third, firms' payoffs also depend on the market structure. Let $v_i\left(p_i, p_j, q_i, q_j\right)$ be the payoff function of firm i. Let us define the following prices:

$$\begin{aligned} \underline{p}_i\left(p_j, q_i, q_j\right) &= \sup\left\{p : x_j\left(p, p_j, q_i, q_j\right) > 0\right\} \ ; \\ \overline{p}_i\left(p_j, q_i, q_j\right) &= \inf\left\{p : x_i\left(p, p_j, q_i, q_j\right) > 0\right\} \ . \end{aligned} \tag{11}$$

The set of p_i such that, given $\left(p_j, q_i, q_j\right)$, both firms have positive market shares is:

$$D_i\left(p_j, q_i, q_j\right) = \left(\underline{p}_i\left(p_j, q_i, q_j\right), \overline{p}_i\left(p_j, q_i, q_j\right)\right) \ . \tag{12}$$

Let also

$$\begin{aligned} \underline{D}_i\left(p_j, q_i, q_j\right) &= \left[0, \underline{p}_i\left(p_j, q_i, q_j\right)\right] \\ \overline{D}_i\left(p_j, q_i, q_j\right) &= \left[\overline{p}_i\left(p_j, q_i, q_j\right), \infty\right) \ . \end{aligned} \tag{13}$$

For a given p_j, by moving p_i, firm i can end up in three different market structures: i's monopoly, j's monopoly, and duopoly. Necessarily, any duopoly NE must be such that deviations towards monopoly are unattractive. Therefore, to construct the NE, FOCs are neither necessary nor sufficient. Suppose p_j^* is a candidate NE price of firm j. I need to compare the payoffs from three different strategies of firm i:

$$\begin{aligned}
p_i^{'} &\in \arg \max_{p_i \in D_i\left(p_j^*, q_i, q_j\right)} v_i\left(p_i, p_j^*, q_i, q_j\right) \\
p_i^{''} &\in \arg \max_{p_i \in \underline{D}_i\left(p_j^*, q_i, q_j\right)} v_i\left(p_i, p_j^*, q_i, q_j\right) \\
p_i^{'''} &\in \arg \max_{p_i \in \overline{D}_i\left(p_j^*, q_i, q_j\right)} v_i\left(p_i, p_j^*, q_i, q_j\right) .
\end{aligned} \tag{14}$$

FOCs only produce $p_i^{'}$. In the following Lemma, I tackle these challenges and show the main results of this Section.

Lemma 1. *(Subgame NE Prices)*
Suppose $a_1 = a_0 = 1$. Consider the subgame induced in Stage (3) by a pair of qualities (q_0, q_1).

(i) If $q_0 \neq q_1$, there exists $\hat{p}_1(q_0, q_1)$ such that firm 0 finds it optimal to serve the median voter if and only if:

$$p_1 \geq \hat{p}_1(q_0, q_1) \ . \tag{15}$$

(ii) There exists a NE.

(iii) If in the NE both firms have positive market shares with probability 1, the NE is unique.

(iv) Let $q_0 = q_1$ and consider the NE. If $\lambda \geq \frac{1}{2}$, both firms price at marginal costs. If $\lambda < \frac{1}{2}$, firm 0 is a monopolist.

(v) Let $\lambda \geq \frac{1}{2}$. In the NE, firm 0 has a positive market share. There exist $\left(q_1^a, q_1^b\right)$ such that firm 1 has a positive market share if and only if:

$$q_0 \leq q_1 \leq q_1^a \quad or \quad q_0 \geq q_1 \geq q_1^b. \tag{16}$$

Proof. All the proofs are in Appendix A.2. See Appendix A.2.1 for the proof of (i); Appendix A.2.2 for the proof of (ii); Appendix A.2.4 for the proof of (iv); and Appendix A.2.5 for the proof of (v). □

Lemma 1 is a useful result. It shows the existence and the properties of a

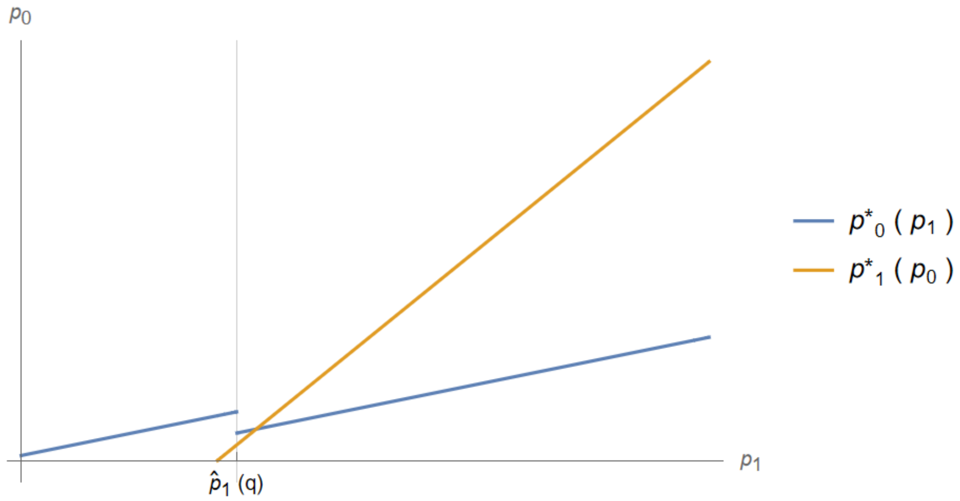

Figure 1: **(Duopoly) Best Responses in the Price Stage** This plot corresponds to the case where neither of the two firms wants to "push" its competitor out of the market (see Appendix A.2.5.2).

mapping from a pair of qualities to a pair of NE prices. Let us start from (i). This result shows that firm 0 wants to serve the median voter if and only if p_1 is high enough. When p_1 is low, it is (potentially) more profitable for the median voter to buy it, thereby increasing firm 0's payoff.[14] To obtain $\hat{p}_1(q_0, q_1)$, I compare the maximum payoff that firm 0 can obtain by serving the median voter and not serving them.

The floor $\hat{p}_1(q_0, q_1)$ is increasing in λ. Intuitively, as the degree of privatization increases, the set of p_1 such that firm 0 wants to attract the median voter shrinks. $\hat{p}_1(q_0, q_1)$ increases in α. The impact of θ_h on $\hat{p}_1(q_0, q_1)$ depends on the ordering of qualities, that is,

$$\frac{\partial \hat{p}_1(q_0, q_1)}{\partial \theta_h} > 0 \Leftrightarrow q_1 > q_0 \,. \tag{17}$$

When θ_h increases, the median voter's WTP increases. Then, it is more likely that they will buy the high-quality good.

To characterize the NE, I need to find firm 1's best response, which I obtain by maximizing its profits. In Figure 1, I show firms' best responses for a generic quality pair $q_0 < q_1$. Un-surprisingly, prices are strategic complements. Firm 0's best response function is discontinuous at $\hat{p}_1(q_0, q_1)$. Therefore, there may be two types of intersections: either $p_P < \hat{p}_1(q_0, q_1)$ or $p_P > \hat{p}_1(q_0, q_1)$; or there

[14]In Appendix A.2.1, I show the expression of $\hat{p}_1(q_0, q_1)$ and its comparative statics.

may not be an intersection.

Lemma 1 ((ii)) ensures the existence of a NE. The proof of existence follows from Reny [1999]. The game satisfies the following properties. The first property is reciprocal upper semi-continuity: when the payoff of one player "jumps" up, the payoff of the other player "jumps" down (Reny [1999], Simon [1987]). The second property is payoff-security.[15] The intersection of these two properties, by Corollary 5.2 of Theorem 3.1 in Reny [1999], guarantees the existence of a (mixed strategy) NE. Since firm 0's payoff is not quasi-concave in its own price, and its best response function is not continuous, the NE does not need to be in pure strategies. In (iii), I show that duopoly NE can be unique because of a local form of strict concavity of firm 1's payoff.

In (iv), I characterize the NE in the case of homogeneous goods ($q_0 = q_1$). When $\lambda \geq \frac{1}{2}$, Bertrand undercutting drives prices down to marginal costs. When $\lambda < \frac{1}{2}$, to please the median voter, firm 0 undercuts firm 1's price below marginal costs, thereby becoming a monopolist.

In (v), I provide a full characterization of the NE for $\lambda \geq \frac{1}{2}$. To obtain NE prices, I proceed as follows. First, I assume that firms are in a duopoly, and I intersect firms' (duopoly) best response functions. Then, I check that the intersection of best responses implies that firms are indeed in a duopoly. Finally, I check that firms do not want to "push" their competitor out of the market.

This result ensures the existence of a mapping from a given quality pair to a pair of NE prices. The mapping is unique when (q_0, q_1) maps to a duopoly NE. There are three classes of subgame price NE. In the first class, the median voter buys good 0 and $p_1 > \hat{p}_1(q_0, q_1)$. In the second class, the median voter buys good 1 and $p_1 < \hat{p}_1(q_0, q_1)$. In the third class, $p_1 = \hat{p}_1(q_0, q_1)$, and firm 0 plays a mixed strategy. So, the median voter buys either good with positive probability.

Holding qualities fixed, the NE where the median voter buys good 0 is characterized by lower prices. Then, I refer to this NE as the *low* prices NE (p^L). The NE such that the median voter buys good 1 is the *high* prices NE (p^H). I refer to the NE in mixed strategies as the *mixed* prices NE (p^M). With a slight abuse of notation, I refer to $p^K(q_0, q_1)$, or simply p^K, as the region of the space (q_0, q_1) where the $K = L, M, H$ price NE exists. Figure 2 shows these three regions, and it can be illustrated as follows.

First, let us define two thresholds, $\underline{q}_0$ and $\overline{q}_0$, such that $\underline{q}_0 \leq \overline{q}_0$.[16] Let us fix a generic q_0. There are three possible cases:

[15] See Appendix A.3.1, Reny [1999] and Edwards and Routledge [2023]

[16] The expressions for the different thresholds are shown in Appendix A.2.5.

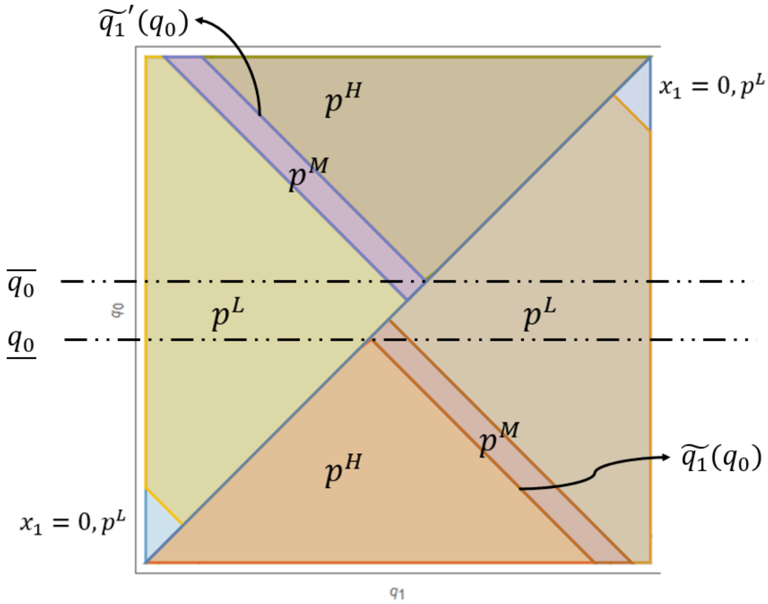

Figure 2: **Existence of Price NE in the** (q_1, q_0) **Space**. Firm 1 is out of the market in the top-right and bottom-left corners.

Case (a) $q_0 < \underline{q}_0$.

Case (b) $q_0 > \overline{q}_0$.

Case (c) $q_0 \in \left[\underline{q}_0, \overline{q}_0\right]$.

Let's start with *Case* (a) (low q_0). When q_1 is very low compared to q_0, the median voter buys good 0. As q_1 increases and approaches q_0, we reach the p^H region, where the median voter buys good 1. If we keep increasing q_1, at some point we will hit another threshold, $\tilde{q}_1(q_0)$, such that for all $q_1 > \tilde{q}_1(q_0)$, the price NE is either p^M or p^L. The median voter buys good 0 with some positive probability. The price of good 1 is too high because all this quality is expensive. A similar argument applies to *Case* (b) (high q_0). In this case, the median voter buys good 1 with probability one as long as q_1 is neither too high nor too low: $q_1 \in \left[\tilde{q}_1'(q_0), q_0\right]$. Interestingly, in *Case* (c), the median voter never buys good 1 with probability one, no matter what q_1 is. In this case, the quality of firm 0 is neither too low to be less attractive nor too high to be too costly than q_1.

According to intuition, the interval $\left[\underline{q}_0, \overline{q}_0\right]$ increases in λ. NE prices are also non-decreasing in the degree of privatization λ and increasing in α.

When $\lambda < \frac{1}{2}$, the optimal monopolistic pricing of firm 0 is $p_0 = 0$, and it seems complicated to show precisely for what values of p_1 firm 0 wants to become a monopolist. Therefore, it does not seem easy to provide a full characterization of the NE in this case. However, if it exists a duopoly price NE, it is characterized by the same expressions that characterize the NE if $\lambda \geq \frac{1}{2}$. In the following Section, I make use of Lemma 1 to characterize the SPNE and study their welfare implications.

5.3 SPNE

In this Section, I solve for the game SPNE. For this purpose, I impose the following assumptions.

Assumption 1. *Let* $\phi \to 0$ *and* $Q \geq \frac{\theta_h}{\alpha}$.

Assumption 1 has two implications. First, $\phi \to 0$ ensures that firms enter the market if and only if they expect positive payoffs. Second, $Q \geq \frac{\theta_h}{\alpha}$ implies that the upper bound of the quality spectrum Q is high enough never to be binding.

The main challenges to the identification of a SPNE is the following. Gsiven the mapping from qualities to NE prices described by Lemma 1, firms have a "technology" to move across the different price regions. At each region, firms' expected payoffs (albeit continuous at the "borders") differ, and many cases must be taken into account.[17] As a result, firms' payoff functions are not quasi-concave, and best responses are not continuous (and potentially not single-valued). In any SPNE, the following conditions must be satisfied. Let $v_i\left(q_i, q_j, p^k\left(q_i, q_j\right)\right)$ be the payoff function of firm $i \in \{0,1\}$ inside the p^k region, with $k \in \{L, M, H\}$. Let $\Psi_i^k(q_j)$ be the set of q_i such that, given q_j, the point (q_i, q_j) belongs to the p^k region. Suppose (q_i^*, q_j^*) are some candidate SPNE qualities. A necessary condition for the pair (q_i^*, q_j^*) to be a SPNE is:

$$q_i^* \in \arg \max_{q_i \in \Psi_i^k\left(q_j^*\right)} v_i\left(q_i, q_j, p^k\left(q_i, q_j\right)\right). \tag{18}$$

By combining FOCs and SOCs, I can ensure the pair of necessary conditions (18). However, firm i knows that increasing (or decreasing) further q_i may end up in another price region p^h, with $h \in \{P, M, L\}$, and $h \neq k$. Then, I need to

[17]See Figure 2 for an illustration of these regions.

check:

$$
\begin{aligned}
&v_i\left(q_i^*, q_j^*, p^k\left(q_i^*, q_j^*\right)\right) \geq v_i\left(q_i^h, q_j^*, p^h\left(q_i^h, q_j^*\right)\right) \\
&\text{where:} \\
&q_i^h = \arg \max_{q_i \in \Psi_i^h\left(q_j^*\right)} v_i\left(q_i, q_j^*, p^h\left(q_i, q_j^*\right)\right) .
\end{aligned}
\tag{19}
$$

Finally,

$$
v_i\left(q_i^*, q_j^*, p^k\left(q_i^*, q_j^*\right)\right) > 0 \,, \tag{20}
$$

guarantees that firms want to enter the market in *Stage* (1).

In the following Proposition, I show the existence of SPNE and how partial privatization affects the market structure.

Proposition 2. *(Market Structure in SPNE)*
For each (q_0, q_1), fix the NE described in Lemma 1. Consider Stage (2) and Stage (1).

(i) For all $\lambda \in [0,1]$, there exist a SPNE.

(ii) If $\lambda = 0$, in the unique SPNE, firm 0 is a monopolist ($a_1^* = 0$).

(iii) It exists $\overline{\lambda} \in (0, \frac{1}{2}]$ such that there is a public monopoly SPNE ($a_1^* = 0$) if and only if $\lambda < \overline{\lambda}$.

(iv) If $\lambda \geq \frac{1}{2}$, there exists (at least) two (duopoly) SPNE. In the first SPNE, $q_0 < q_1$. In the second, $q_0 > q_1$. The two SPNE are payoffs- (and welfare-) equivalent.

Proof. All the proofs are in Appendix A.3. See Appendix A.3.1 for the proof of (i); Appendix A.3.2 for the proof of (ii); Appendix A.3.3 for the proof of (iii); and Appendix A.3.4 for the proof of (iv). □

Proposition 2 features some important results. In (i), I show that since *Stage* (2) is a continuous game, a SPNE exists (Glicksberg [1952]).

In (ii), I show that when $\lambda = 0$, the market is a public monopoly. Firm 0 only cares about the median voter, and it has a dominant strategy: $q_0 = Q$ and $p_0 = 0$. Given this strategy, under Assumption 1, firm 1 does not want to enter the market.

For $\lambda \in \left(0, \frac{1}{2}\right)$, it does not seem easy to fully characterize the SPNE. However, in (iii), I show how partial privatization affects the market structure. If λ is low, NE prices are low. Anticipating this, firm 1 does not enter the market.

This result depends on the assumption that firm 0 does not have a budget constraint.[18] However, it may explain the persistence of public monopolies to the entry of private firms. Public firms may be considered credible to commit to charging low prices (below marginal costs) because of their political ties, thereby discouraging the entry of private firms. Proposition 2 shows the existence of natural (public) monopolies that do not stem from technology but rather from preferences. The proof of (iii) follows two steps. First, I show that if firm 1 wants to enter at some $\lambda = \overline{\lambda}$, it also wants to enter for any $\lambda > \overline{\lambda}$. When λ increases, the entry profits of firm 1 increase. Second, I show that if $\lambda = \frac{1}{2}$, it does not exist a public monopoly SPNE. Therefore, there exists a public monopoly SPNE if and only if $\lambda < \overline{\lambda}$.

When λ is high enough ($\lambda \geq \overline{\lambda}$), firm 1 enters the market((iv)).[19] I describe the different SPNE in Appendix A.4.[20] In the following Section, I discuss the welfare properties of the different SPNE described in Proposition 2.

5.3.1 Welfare

In this Section, I focus on two variables of interests: aggregate social welfare, and profits.

I adopt a specific but rather predominant (at least in the IO literature) welfare criterion: utilitarian welfare, the sum of players' cardinal (money-metric) utilities. With a slight abuse of notation, I define p_L (respectively, p_H) and q_L (respectively, q_H) as the price and quality of the lower (respectively, higher) quality good. Given (6), consumer surplus is:

$$CS\left(p_H, p_L, q_H, q_L\right) = \int_{\theta_h - 1}^{\hat{\theta}} \theta q_L - p_L, d\theta + \int_{\hat{\theta}}^{\theta_h} \theta q_H - p_H, d\theta \,. \tag{21}$$

[18] I relax this assumption in Section 6.1.

[19] In the proof of (iv), I report a full characterization of SPNE for specific $\theta_h = 2$ and $\alpha = \frac{1}{2}$. This is without loss of generality because the results on equilibria' existence and their welfare properties do not depend on θ_h and α. In *Stage* (2), θ_h and α are just multiplicative factors of firms' payoff functions. Therefore, varying θ_h and α does not alter firms' strategic behavior in the choice of quality. As in Motta [1993] and Lambertini [2006], the relation between SPNE qualities and θ_h, α is linear.

[20] While SPNE uniqueness seems too much to be expected from this setting, I can give some insights into the problem of equilibrium selection. First, I can claim that the SPNE in Proposition 2 are unique in the class of SPNE where both firms play pure strategies and at least one firm plays an "interior" strategy (a strategy resulting from its FOC). Second, I focus on mixed strategy SPNE only in the region where it does not exist a pure SPNE because of the discontinuity of best responses. Note that there might possibly exist other SPNE in mixed strategies, which, however, are complicated to construct and possibly not to be selected when players can play pure strategy SPNE.

Aggregate welfare is:

$$W(CS, \pi_H, \pi_L) = CS + \pi_L + \pi_H \ , \tag{22}$$

where π_L (respectively, p_H) are the profits of the firm producing the low quality (respectively, high quality) good. I now define the First Best allocation. The social planner problem is:

$$\begin{aligned} &\max_{q_H, q_L, p_H, p_L} W(CS, \pi_H, \pi_L) \\ &\textit{subject to} \\ &\qquad q_H \geq q_L \geq 0 \\ &\qquad p_H \geq p_L \geq 0 \end{aligned} \tag{23}$$

Any solution of problem (23) satisfies allocative efficiency (Grilo [1994], Laine and Ma [2017]). Allocative efficiency requires that the indifferent consumer (6) is the one whose switch from the low-quality good to the high-quality one equalizes social costs and social benefits:

$$\hat{\theta} = \frac{\alpha(q_H^2 - q_L^2)}{(q_H - q_L)} \ . \tag{24}$$

A sufficient (but not necessary) condition for (24) is that prices equal marginal costs. If the public firm maximizes welfare, equilibrium qualities and prices solve problem (23).[21] Welfare implications change when the public firm cares about the median voter. In the following Proposition, I discuss the welfare properties of the NE, from the point of view of aggregate welfare and profits.

[21]The proof for this result is omitted for the sake of brevity, but it is available upon request. Similar results are in Grilo [1994], Delbono et al. [1991]. Suppose the planner maximizes consumer surplus (21) and controls one firm. Then, the planner would set a price so that all consumers buy the higher quality good. Note that consumer surplus (21) coincides with consumers' average utility. Benassi et al. [2016] propose a slightly different approach. Their (gross) surplus function (if translated in the framework of this model) would be $\int_{\theta_h - 1}^{\hat{\theta}} \theta q_L d\theta + \int_{\hat{\theta}}^{\theta_h} \theta q_H d\theta$. The authors argue in a footnote of the paper that the median voter theorem could be invoked when the distributions of consumers' types is asymmetric. They claim that the median voter's utility coincides with the average consumer's utility and consumer surplus under symmetric distributions. However, consumer surplus coincides with the average gross surplus (in their specification, since they do not consider prices, or average utility, in my specification), which is different from the surplus of the consumer with the average (and median) type. The two values coincide if and only if surplus (and not types) is uniformly distributed in the population.

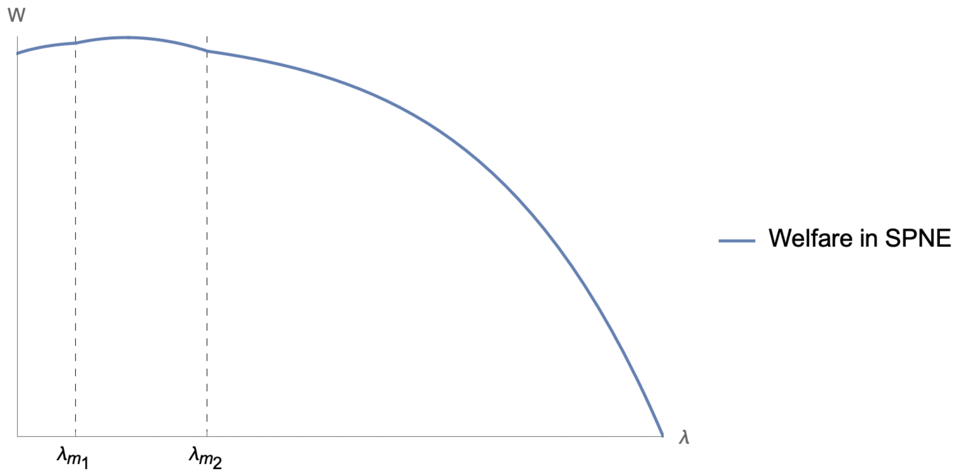

Figure 3: **Welfare in SPNE** for $\lambda \in \left[\frac{1}{2}, \lambda_{p_1}\right]$. The figure shows aggregate welfare in the SPNE where the median voter buys good 0. See Appendix A.8 for welfare in the other SPNE ($\lambda > \lambda_{p_1}$).

Proposition 3. *(Profits and Welfare in SPNE)*
Consider the SPNE described in Proposition 2.

(i) In the public monopoly SPNE, social welfare is monotonically increasing in λ.

(ii) The socially optimal level of privatization λ^* is interior: $\lambda^* \in (0, 1)$.

(iii) Consider the SPNE such that the median voter buys good 0. It exists $\hat{\lambda}$ such that:

$$\begin{aligned} &\pi_S^* < 0 < \pi_1^* \Leftrightarrow \lambda \in \left[\frac{1}{2}, \lambda_{m_1}\right); \\ &0 \le \pi_0^* < \pi_1^* \Leftrightarrow \lambda \in \left[\lambda_{m_1}, \hat{\lambda}\right); \\ &0 < \pi_1^* \le \pi_0^* \Leftrightarrow \lambda \in \left[\hat{\lambda}, \lambda_{p_2}\right] . \end{aligned} \tag{25}$$

Proof. See Appendix A.5. □

Proposition 3 contains the main results of my paper. First, zero privatization is not socially optimal. In that case, firm 0 commits to $q_0 = Q$ and $p_0 = 0$, realizing a big economic loss which is not compensated by consumer surplus.

Second, the socially optimal degree of privatization λ^* is interior. At $\lambda = 0$, there is a unique SPNE where $W^* \to -\infty$. At $\lambda = 1$, there is a unique SPNE, which is welfare-dominated by $\lambda = \lambda_{p_1}$. Possibly, $\lambda^* \neq \lambda_{p_1}$, but necessarily $\lambda^* \in (0, 1)$. Along the SPNE described in Proposition 2, welfare is maximized at $\lambda =$

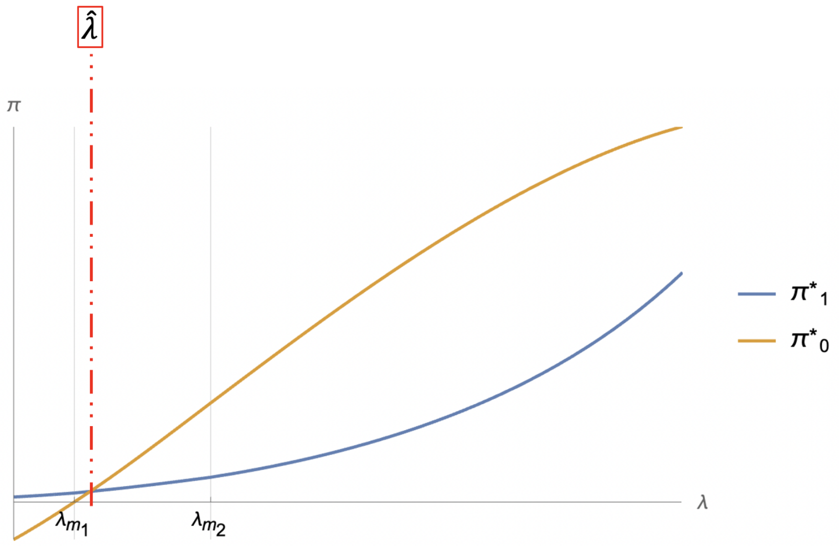

Figure 4: **Welfare in SPNE** for $\lambda \in \left[\frac{1}{2}, \lambda_{p_1}\right]$. The figure profits in the SPNE where the median voter buys good 0. See Appendix A.8 for profits in the other SPNE $\lambda > \lambda_{p_1}$.

λ_{p_1}, where the median voter buys good 1. In this SPNE, firms produce relatively similar quality levels. Therefore, they charge lower markups. However, since λ is high enough, firms are profitable. In the SPNE where the median voter buys good 0, welfare has an inverted-U shape in λ (Figure 3). There is a trade-off between high qualities and low markups. When λ is low, qualities and markups are low. Then, increasing λ can be welfare enhancing. However, if λ is too high, firms have incentives to invest in qualities, but they ask for high markups.

Third, the degree of privatization determines market profits. Figure 4 shows profits in the SPNE where the median voter buys good 0. When λ is low, the firm 0's profits are negative. When λ is neither too high nor too low, firm 0 makes more profits than firm 1. To the best of my knowledge, this result is new to the literature of mixed oligopoly with PD. The intuition behind it is simple. Firm 0 produces a larger output than firm 1. If λ is sufficiently high, it also charges a higher markup. This is an example of commitment power in games. If firm 1 raised the markup, it would lose its customers because the quality/price ratio of firm 0 is relatively close. On the contrary, most customers of firm 0 find the good of firm 1 too far from their WTP. These buyers would sacrifice a lot of utility if they had to change provider and then can be charged a higher markup. This commitment device requires the firm 0 to serve the majority of consumers.

In fact, in any SPNE such that the median voter buys good 1, $\pi_1^* > \pi_0^*$.[22] In the next Section, I propose some extensions to test the robustness of my results.

6 Extensions

6.1 Marginal Cost Pricing

Consider a "modified" version of the game in Section 3. Firm 0 now does not choose p_0, because it commits to $p_0 = \alpha q_0^2$ in the price stage. Then, the game runs as in the previous sections. I have the following result.

Proposition 4. *(Strict Budget Constraint)*
There exist two (duopoly) payoffs-equivalent SPNE. In the first SPNE, $q_0 < q_1$. In the second, $q_0 > q_1$. Welfare in the un-constrained SPNE described in Proposition 2 (if evaluated at $\lambda = \lambda^*$) is higher than in these constrained SPNE.

Proof. See Appendix A.6. □

Public firms may be constrained to make non-negative profits, and they may be legally obligated to price their goods at marginal costs. Proposition 4 shows that this commitment is not necessarily welfare improving, but rather, it depends on the degree of partial privatization.

6.2 Partial Market Coverage

In this Section, consumers are also allowed not to buy any good. The setup is the standard PD model with partial market coverage.[23] Unfortunately, in this case, it does not seem easy to solve for a SPNE. I have the following result for the price stage.

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[22] The empirical evidence about politically connected SOEs' profitability is still controversial. For instance, Agrawal and Knoeber [2001], Menozzi et al. [2012], among others, find negative effects of political ties on firms' profitability. Faccio [2006] and Goldman et al. [2009], among others, have opposite results. Proposition 3 suggests a potential explanation. For political ties to be profitable, they must be not too tight.

[23] I write down the setup in Appendix A.7. See Wauthy [1996] and Lambertini [2006], among many others, for private duopoly PD models with partial market coverage.

Proposition 5. *(Partial Market Coverage)*
Consider the subgame induced in the price stage by a pair of qualities (q_0, q_1). The share of consumers that do not buy any good is non-decreasing in λ.

Proof. See Appendix A.7. □

When λ increases, firm 0 charges higher prices. Because of the strategic complementarity of prices, the NE price of firm 1 is also non-decreasing in λ. Then, the share of consumers who do not buy any good increases accordingly. Proposition 5 shows that privatization may raise inequality concerns, especially in the case of "essential" goods, like education or healthcare.

7 Concluding Remarks

This paper proposes a model of quality competition between a public and a private firm. I contribute to the literature by providing a new definition for the public firm using the median voter theorem. I argue that this definition might be adopted when public firms have political ties. I study the impact of partial privatization on market structure, profits, and welfare.

As privatization increases, the market shifts from a monopoly to a duopoly, and the public firm can become more profitable than the private one. I also show that the socially optimal degree of privatization is interior.

My paper has several limitations. First, although I have existence and characterization results with interesting welfare properties, I cannot offer a complete characterization of the set of all SPNE for the whole parameters range. Second, I adopt restrictive assumptions such as a uniform distribution of consumers' type, a fixed number of firms, and symmetric cost functions. Future research may explore the robustness of the findings by relaxing some of these assumptions. Another natural avenue for future research is to employ and test the applicability of my definition of a politically-tied public firm in different models.

References

A. Agrawal and C. R. Knoeber. Do some outside directors play a political role? *The Journal of Law and Economics*, 44(1):179–198, 2001.

U. Akcigit, S. Baslandze, and F. Lotti. Connecting to power: political connections, innovation, and firm dynamics. *Econometrica*, 91(2):529–564, 2023.

C. Benassi, M. Castellani, and M. Mussoni. Price equilibrium and willingness to pay in a vertically differentiated mixed duopoly. *Journal of Economic Behavior & Organization*, 125:86–96, 2016.

C. Benassi, A. Chirco, and C. Colombo. Mixed spatial duopoly, consumers' distribution and efficiency. *Economics Letters*, 156:74–77, 2017.

M. Boycko, A. Shleifer, and R. W. Vishny. A theory of privatisation. *The Economic Journal*, 106(435):309–319, 1996.

G. Brunello and L. Rocco. Educational standards in private and public schools. *The Economic Journal*, 118(533):1866–1887, nov 2008.

H. Cremer and J.-F. Thisse. Location models of horizontal differentiation: a special case of vertical differentiation models. *The Journal of Industrial Economics*, pages 383–390, 1991.

H. Cremer, M. Marchand, and J.-F. Thisse. Mixed oligopoly with differentiated products. *International Journal of Industrial Organization*, 9(1):43–53, 1991.

P. Dasgupta and E. Maskin. The existence of equilibrium in discontinuous economic games, i: Theory. *The Review of economic studies*, 53(1):1–26, 1986.

F. Delbono, V. Denicolo, and C. Scarpa. Quality choice in a vertically differentiated mixed duopoly. Technical report, Quaderni-Working Paper DSE, 1991.

T. Donaldson and L. E. Preston. The stakeholder theory of the corporation: Concepts, evidence, and implications. *Academy of Management Review*, 20(1): 65–91, 1995.

V. Dotti. The political economy of public education. *Research in Economics*, 73 (1):35–52, 2019.

R. Edwards and R. Routledge. Existence and uniqueness of nash equilibrium in discontinuous bertrand games: a complete characterization. *International Journal of Game Theory*, 52(2):569–586, 2023.

D. Epple and R. E. Romano. Ends against the middle: Determining public service provision when there are private alternatives. *Journal of public economics*, 62(3):297–325, 1996.

S. Estrin and D. De Meza. Unnatural monopoly. *Journal of Public Economics*, 57 (3):471–488, 1995.

M. Faccio. Politically connected firms. *American Economic Review*, 96(1):369–386, 2006.

J. S. Gans and M. Smart. Majority voting with single-crossing preferences. *Journal of Public Economics*, 59(2):219–237, 1996.

I. L. Glicksberg. A further generalization of the kakutani fixed theorem, with application to nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1):170–174, 1952.

E. Goldman, J. Rocholl, and J. So. Do politically connected boards affect firm value? *The Review of Financial Studies*, 22(6):2331–2360, 2009.

I. Grilo. Mixed duopoly under vertical differentiation. *Annales d'Economie et de Statistique*, pages 91–112, 1994.

J. S. Heywood, S. Wang, and G. Ye. Partial privatization upstream with spatial price discrimination downstream. *Review of Industrial Organization*, 59(1):57–78, 2021.

IMF. State-owned enterprises in the time of covid-19. *IMF Blog*, 2021.

T. Inoue, Y. Kamijo, and Y. Tomaru. Interregional mixed duopoly. *Regional Science and Urban Economics*, 39(2):233–242, 2009.

K. Ishibashi and T. Kaneko. Partial privatization in mixed duopoly with price and quality competition. *Journal of Economics*, 95(3):213–231, 2008.

T. Klumpp and X. Su. Price–quality competition in a mixed duopoly. *Journal of Public Economic Theory*, 21(3):400–432, 2019.

A. Kumar and B. Saha. Spatial competition in a mixed duopoly with one partially nationalized firm. *Journal of Comparative Economics*, 36(2):326–341, 2008.

L. T. Laine and C.-t. A. Ma. Quality and competition between public and private firms. *Journal of Economic Behavior & Organization*, 140:336–353, 2017.

L. Lambertini. *The Economics of Vertically Differentiated Markets*. Edward Elgar Publishing, 2006.

Y. Lu and S. Poddar. Firm ownership, product differentiation and welfare. *The Manchester School*, 75(2):210–217, 2007.

C. Lülfesmann and G. M. Myers. Two-tier public provision: comparing public systems. *Journal of Public Economics*, 95(11-12):1263–1271, 2011.

T. Matsumura. Partial privatization in mixed duopoly. *Journal of Public Economics*, 70(3):473–483, 1998.

A. Menozzi, M. Gutiérrez Urtiaga, and D. Vannoni. Board composition, political connections, and performance in state-owned enterprises. *Industrial and Corporate Change*, 21(3):671–698, 2012.

M. Motta. Endogenous quality choice: price vs. quantity competition. *The Journal of Industrial Economics*, pages 113–131, 1993.

M. H. Nabin, X. Nguyen, P. M. Sgro, and C.-C. Chao. Strategic quality competition, mixed oligopoly and privatization. *International Review of Economics & Finance*, 34:142–150, 2014.

P. J. Reny. On the existence of pure and mixed strategy nash equilibria in discontinuous games. *Econometrica*, 67(5):1029–1056, 1999.

A. Shaked and J. Sutton. Relaxing price competition through product differentiation. *The Review of Economic Studies*, pages 3–13, 1982.

L. K. Simon. Games with discontinuous payoffs. *The Review of Economic Studies*, 54(4):569–597, 1987.

D. A. C. Smith and M. J. Trebilcock. State-owned enterprises in less developed countries: Privatization and alternative reform strategies. *European Journal of Law and Economics*, 12(3):217–252, 2001.

Stiglitz. The demand for education in public and private school systems. *Journal of Public Economics*, 3(4):349–385, 1974.

J. Tirole. *The Theory of Industrial Organization*. MIT press, 1988.

X. Wauthy. Quality choice in models of vertical differentiation. *The Journal of Industrial Economics*, pages 345–353, 1996.

J. Willner. Ownership, efficiency, and political interference. *European Journal of Political Economy*, 17(4):723–748, 2001.

A Appendix

The solution of the model can be replicated by downloading the *Mathematica* Replication Package.

A.1 Proof of Proposition 1

Take $\delta' > \delta$ and $\theta' > \theta$. Then, consumers' preferences satisfy the single crossing property:

$$u\left(\delta' \mid \theta\right) > u\left(\delta \mid \theta\right) \Rightarrow u\left(\delta' \mid \theta'\right) > u\left(\delta \mid \theta'\right) . \tag{26}$$

Since preferences satisfy (26), Theorems 1 and 2 in Gans and Smart [1996] prove the result.

A.2 Proof of Lemma 1

A.2.1 Proof of (i)

To obtain firm 0's best response, I compare the optimal payoff that firm 0 can obtain by serving the median voter and by not serving them. There are two possible cases: $q_0 < q_1$ and $q_0 > q_1$.

- $q_0 < q_1$. Let us rewrite firm 0's payoff:

$$\begin{aligned} & V_0\left(p_0, p_1, q_0, q_1\right) = \\ & \lambda\left(p_0 - \alpha q_0^2\right)\left(-\theta_h + \frac{p_1 - p_0}{q_1 - q_0} + 1\right) + \\ & (1-\lambda)\max\left\{\frac{1}{2}(2\theta_h - 1)q_0 - p_0, \frac{1}{2}(2\theta_h - 1)q_1 - p_1\right\}. \end{aligned} \tag{27}$$

If condition (10) is satisfied, so that:

$$\max\left\{\frac{1}{2}(2\theta_h - 1)q_0 - p_0, \frac{1}{2}(2\theta_h - 1)q_1 - p_1\right\} = \frac{1}{2}(2\theta_h - 1)q_0 - p_0, \tag{28}$$

firm 0's payoff is:

$$\lambda\left(p_0 - \alpha q_0^2\right)\left(-\theta_h + \frac{p_1 - p_0}{q_1 - q_0} + 1\right) + (1-\lambda)\left(\frac{1}{2}(2\theta_h - 1)q_0 - p_0\right). \tag{29}$$

Otherwise, firm 0's payoff rewrites as:

$$\lambda\left(p_0 - \alpha q_0^2\right)\left(-\theta_h + \frac{p_1 - p_0}{q_1 - q_0} + 1\right) + (1-\lambda)\left(\frac{1}{2}(2\theta_h - 1)q_1 - p_1\right). \tag{30}$$

I now compare the maxima of (29) and (30). To find the (interior) maximizers, FOCs are sufficient by quasi-concavity.[24] The FOC of (29) requires:

$$\begin{aligned} & \frac{\lambda(p_1 - 2p_0 + q_0(\theta_h + \alpha q_0 - 2)) - q_1((\theta_h - 2)\lambda + 1) + q_0}{q_1 - q_0} = 0 \Rightarrow \\ & p_0 = \frac{\lambda(p_1 + q_0(\theta_h + \alpha q_0 - 2)) - q_1((\theta_h - 2)\lambda + 1) + q_0}{2\lambda} = \underline{p}_0^{L^*}(p_1, q_0, q_1) \end{aligned} \tag{31}$$

[24] The function (27) is not quasi-concave in p_0. However, both functions (29) and (30) are quasi-concave in p_0.

The FOC of (30) requires:

$$\begin{aligned}
&\frac{\lambda(p_1 - 2p_0 - \theta_h q_1 + q_1 + q_0(\theta_h + \alpha q_0 - 1))}{q_1 - q_0} = 0 \Rightarrow \\
&p_0 = \frac{1}{2}(p_1 - \theta_h q_1 + q_1 + q_0(\theta_h + \alpha q_0 - 1)) = \underline{p}_0^{H^*}(p_1, q_0, q_1) \ .
\end{aligned} \tag{32}$$

Note that $\underline{p}_0^{H^*}(p_1, q_0, q_1) \geq \underline{p}_0^{L^*}(p_1, q_0, q_1)$. Then, the superscripts H, L refer, respectively, to *high* and *low* prices. Moreover,

$$\lim_{\lambda \to 1} \underline{p}_0^{L^*}(p_1, q_0, q_1) = \underline{p}_0^{H^*}(p_1, q_0, q_1) \ . \tag{33}$$

I now plug (31) into (29) to obtain:

$$\frac{1}{4}\left(2(\lambda - 1)(2p_1 - 2\theta_h q_1 + q_1) + \frac{\lambda(p_1 - \theta_h q_1 + q_1 + q_0(\theta_h + \alpha(-q_0) - 1))^2}{q_1 - q_0}\right) , \tag{34}$$

and (32) into (30) to obtain:

$$\frac{1}{4}\left(2(\lambda - 1)(2p_1 - 2\theta_h q_1 + q_1) + \frac{\lambda(p_1 - \theta_h q_1 + q_1 + q_0(\theta_h + \alpha(-q_0) - 1))^2}{q_1 - q_0}\right) . \tag{35}$$

Since (34) is higher than (35) if and only if:

$$p_1 \geq \frac{q_1(2\theta_h \lambda + \lambda - 1) + \lambda q_0(-2\theta_h + 2\alpha q_0 - 1) + q_0}{2\lambda} = \hat{p}_1^1(q_0, q_1) \ , \tag{36}$$

the best response of firm 0 is:[25]

$$\underline{p}_0^*(p_1, q_0, q_1) = \begin{cases} \underline{p}_0^{L^*}(p_1, q_0, q_1) & \text{if } p_1 > \hat{p}_1^1(q_0, q_1) \\ \underline{p}_0^{H^*}(p_1, q_0, q_1) & \text{if } p_1 < \hat{p}_1^1(q_0, q_1) \\ \left\{\underline{p}_0^{L^*}(p_1, q_0, q_1), \underline{p}_0^{H^*}(p_1, q_0, q_1)\right\} & \text{if } p_1 = \hat{p}_1^1(q_0, q_1). \end{cases} \tag{37}$$

It is interesting to note that:

$$\begin{aligned} \frac{\partial \hat{p}_1^1(q_0, q_1)}{\partial \lambda} &> 0 \\ \frac{\partial \hat{p}_1^1(q_0, q_1)}{\partial \theta_h} &> 0 \\ \frac{\partial \hat{p}_1^1(q_0, q_1)}{\partial \alpha} &> 0. \end{aligned} \tag{38}$$

- $q_0 > q_1$. I adopt the same approach as the previous case. Now, the threshold is:

$$\frac{(2\theta_h - 3)\lambda q_1 + q_1 + q_0(\lambda(-2\theta_h + 2\alpha q_0 + 3) - 1)}{2\lambda} = \hat{p}_1^2(q_0, q_1). \tag{39}$$

The best response of firm 0 is:[26]

$$\overline{p}_0^*(p_1, q_0, q_1) = \begin{cases} \frac{\lambda(p_1 + q_0(\theta_h + \alpha q_0 + 1)) - ((\theta_h + 1)\lambda q_1) + q_1 - q_0}{2\lambda} = \overline{p}_0^{L^*}(p_1, q_0, q_1) & \text{if } p_1 > \hat{p}_1^2(q_0, q_1) \\ \frac{1}{2}(p_1 - \theta_h q_1 + q_0(\theta_h + \alpha q_0)) = \overline{p}_0^{H^*}(p_1, q_0, q_1) & \text{if } p_1 < \hat{p}_1^2(q_0, q_1) \\ \left\{\overline{p}_0^{L^*}(p_1, q_0, q_1), \overline{p}_0^{H^*}(p_1, q_0, q_1)\right\} & \text{if } p_1 = \hat{p}_1^2(q_0, q_1). \end{cases} \tag{40}$$

[25] The best response in equation (37) is well-defined. That is, condition (36) implies condition (10). Moreover, the function (37) is the best response of firm 0 only for those p_1 such that firm 0 does not want to push firm 1 out of the market and become a monopolist. There exists some $\tilde{p}_1(q_0, q_1)$ such that for all $p_1 \geq \tilde{p}_1(q_0, q_1)$, firm 0 wants to become a monopolist. The best response in this case is $p_0 = \underline{p}_0(p_1, q_1, q_0)$, where $\underline{p}_0(p_1, q_1, q_0)$ is the highest p_0 such that all consumers buy good 0. Necessarily, $\tilde{p}_1(q_0, q_1) > \hat{p}_1^1(q_0, q_1)$. If $\hat{p}_1(q_0, q_1) < 0$, then firm 0 wants to serve the median voter for all possible p_1. Note also that there exists an analogous threshold $(\tilde{p}_0(q_0, q_1))$ for firm 1's best response.

[26] See Footnote 25.

The comparative statics of $\hat{p}_1^2\left(q_0, q_1\right)$ is as follows.

$$\begin{aligned}
\frac{\partial \hat{p}_1^2\left(q_0, q_1\right)}{\lambda} &> 0 \\
\frac{\partial \hat{p}_1^2\left(q_0, q_1\right)}{\alpha} &> 0 \qquad (41) \\
\frac{\partial \hat{p}_1^2\left(q_0, q_1\right)}{\theta_h} &< 0
\end{aligned}$$

Therefore, $\hat{p}\left(q_0, q_1\right)$ is as follows.

$$\hat{p}_1\left(q_0, q_1\right) = \begin{cases} \max\left\{\hat{p}_1\left(q_0, q_1\right), 0\right\} & \text{if } q_0 < q_1 \\ \max\left\{\hat{p}_2\left(q_0, q_1\right), 0\right\} & \text{if } q_0 > q_1 \end{cases} \qquad (42)$$

A.2.2 Proof of (ii)

The proof follows from Reny [1999]. The game satisfies the following properties. First, the game satisfies reciprocal upper semi-continuity. The discontinuity set is such that when the payoff of one firm "jumps up," the payoff of the other firm "jumps down." In addition, the sum of the payoffs is a continuous function on Δ. Second, the game satisfies payoff security. Given any pair of prices (p_i, p_j), firm i always has a strategy $p_i^{'}$ that secures a payoff of at least $v_i(p_i, p_j, q_i, q_j)$ even if its opponent slightly deviates from p_j to some $p_j^{'}$.[27] By Corollary 5.2 of Theorem 3.1 in Reny [1999], there exists a (mixed strategy) NE. The NE is not (necessarily) in pure strategies because firm 0's payoff is not quasi-concave.

A.2.3 Proof of (iii)

The function $\pi_1(p_1, p_0, q_1, q_0)$ is strictly concave in p_1 on $D_1(p_0, q_1, q_0)$. However, $\pi_1(p_1, p_0, q_1, q_0)$ is not strictly concave in p_1 on $[0, \infty)$, albeit being quasi-concave. Therefore, for all p_0 such that some $p_1 \in D_1(p_0, q_1, q_0)$ is a best response for firm 1, the best response of firm 1 to p_0 is unique and pure. Then, in any duopoly NE, firm 1 plays a pure strategy.[28]

Let us now consider firm 0. Firm 0's payoff is neither strictly concave in p_0 on $D_0(p_1, q_0, q_1)$, nor quasi-concave in p_0 on $[0, \infty)$. Since firm 1 plays a pure strategy, there are two cases: $p_1 \neq \hat{p}_1(q_0, q_1)$ and $p_1 = \hat{p}_1(q_0, q_1)$.

- $p_1 \neq \hat{p}_1(q_0, q_1)$. By concavity of (29) and (30), if some $p_0 \in D_0(p_1, q_0, q_1)$ is optimal, the best response of firm 0 is a pure strategy. Therefore, there is a unique intersection of best responses, and the NE is unique.
- $p_1 = \hat{p}_1(q_0, q_1)$. By (37), firm 0 has two best responses. As I show later (Appendix A.2.5, equations (50) and (56)), there exists a unique randomization of firm 0 such that $p_1 = p_1 \neq \hat{p}_1(q_0, q_1)$ is optimal for firm 1.

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[27] Edwards and Routledge [2023] shows that any Bertrand game is payoff-secure when marginal costs (with respect to quantity) are constant.

[28] Extending this argument to a possible randomization of firm 0 is straightforward. Let η_0 be a mixed strategy of firm 0, with support S_{η_0}. The expected payoff of firm 1 is $\mathbb{E}_{\eta_0}[\pi_1(p_1, p_0, q_1, q_0)]$. Suppose, toward a contradiction, that firm 1's best response to η_0 is some mixed strategy η_1, with support S_{η_1}. By definition of NE, any $p_1 \in S_{\eta_1}$ must yield firm 1 the same payoff. Then, $S_{\eta_1} \subseteq D_1(p_0, q_1, q_0)$. Now, consider two cases: either firm 0 randomizes over finitely many actions, or it randomizes over infinitely many actions. If $| S_{\eta_0} | < \infty$, $\mathbb{E}_{\eta_0}[\pi_1(p_1, p_0, q_1, q_0)]$ is strictly concave in p_1 on $D_1(p_0, q_1, q_0)$ because it is the sum of finitely many strictly concave functions. If $| S_{\eta_0} | = \infty$, concavity is preserved under the expectation sign as $\pi_1(p_1, p_0, q_1, q_0)$ is bounded above. In both cases, strict concavity guarantees single peakedeness.

A.2.4 Proof of (iv)

If $q_0 = q_1$, the model reproduces the standard Bertrand price competition with homogenous goods. By the standard Bertrand's undercutting argument, if $\lambda \geq \frac{1}{2}$, the unique duopoly NE is such that prices are equal to marginal costs. However, if $\lambda < \frac{1}{2}$, firm 0 still has an incentive to undercut its price below marginal costs to increase the median voter's utility.[29]

[29]There cannot be a private monopoly NE because firm 0 would always have an incentive to undercut the monopoly price of firm 1 and make the median voter better off.

A.2.5 Proof of (v)

I can fully characterize the NE when $\lambda \geq \frac{1}{2}$. I distinguish between two types of NE: duopoly NE (Appendix A.2.5.1) and public monopoly NE (Appendix A.2.5.3). To characterize duopoly NE, I proceed as follows. First, I assume that firms are in a duopoly, and I intersect firms' (duopoly) best response functions. Then, I check that the intersection of best responses implies that firms are indeed in a duopoly. Finally, I check that firms do not want to deviate towards their optimal monopoly price (Appendix A.2.5.2). To characterize monopoly price NE, I solve firm 0's optimal monopoly problem, and I check that neither firm 1 wants to enter nor firm 0 wants to increase its price to give it a positive market share.

When $\lambda < \frac{1}{2}$, it seems complicated to characterize the NE. In particular, after I obtain a duopoly candidate NE, it is difficult to determine when firm 0 has an incentive to "push" firm 1 out of the market.[30] However, by (iii), if it exists a duopoly price NE, the NE is described by the same expressions that characterize the duopoly NE for $\lambda \geq \frac{1}{2}$.

[30]If $\lambda < \frac{1}{2}$, the optimal price of firm 0 conditional on being in a monopoly is $p_0 = 0$. Given some candidate NE prices, it does not seem easy to determine when deviating to $p_0 = 0$ is optimal for firm 0. See also Appendix A.2.5.2.

A.2.5.1 Duopoly NE In this Section, I obtain candidate duopoly NE. There are three possible cases: $q_0 < q_1$, $q_1 > q_0$, and $q_0 = q_1$.

- $q_0 < q_1$. Firm 0's (duopoly) best response function is (37). I now obtain firm 1's (duopoly) best response. In Appendix A.2.5.2, I show that, given candidate NE prices, neither firm wants to deviate and become a monopolist. At the end of this Section, I show under what conditions the candidate NE prices imply that both firms have positive market shares. The payoff of firm 1 is:

$$\pi_1 (p_0, p_1, q_0, q_1) = \left(p_1 - \alpha q_1^2\right) \left(\theta_h - \frac{p_1 - p_0}{q_1 - q_0}\right) . \tag{43}$$

The FOC of (43) (that is necessary for an interior maximizer) requires:

$$\begin{aligned} &\theta_h + \frac{-2p_1 + p_0 + \alpha q_1^2}{q_1 - q_0} = 0 \Rightarrow \\ &p_0 = \frac{1}{2}\left(p_0 + \alpha q_1^2 + \theta_h q_1 - \theta_h q_0\right) = \overline{p}_1^* (p_0, q_0, q_1) \ . \end{aligned} \tag{44}$$

Provided that firm 1 does not want to deviate towards monopoly, $\overline{p}_1^* (p_0, q_0, q_1)$ is the best response of firm 1.

There are two intersections of best responses.

$$\begin{aligned} p_0 &= \frac{1}{3}\left(\alpha q_1^2 - (\theta_h - 2)q_1 + q_0(\theta_h + 2\alpha q_0 - 2)\right) = \underline{p}_0^L (q_0, q_1) \\ p_1 &= \frac{1}{3}\left(2\alpha q_1^2 + (\theta_h + 1)q_1 + q_0(-\theta_h + \alpha q_0 - 1)\right) = \overline{p}_1^L (q_0, q_1) \end{aligned} \tag{45}$$

$$\begin{aligned} p_0 &= \frac{1}{3}\left(\alpha q_1^2 - (\theta_h - 2)q_1 + q_0(\theta_h + 2\alpha q_0 - 2)\right) = \underline{p}_0^H (q_0, q_1) \\ p_1 &= \frac{1}{3}\left(2\alpha q_1^2 + (\theta_h + 1)q_1 + q_0(-\theta_h + \alpha q_0 - 1)\right) = \overline{p}_1^H (q_0, q_1) \end{aligned} \tag{46}$$

The pair (45) is a NE only if (10) holds and $\overline{p}_1^L (q_0, q_1) \geq \hat{p}_1^1 (q_0, q_1)$:

$$\begin{aligned} &\left(0 \leq \lambda \leq \frac{1}{4\theta_h - 1}\right) \text{ or} \\ &\left(\lambda(q_1 + q_0)(-4\theta_h\lambda + \lambda + 4\alpha\lambda(q_1 + q_0) + 1) \leq 0 \text{ and } \frac{1}{4\theta_h - 1} < \lambda < 1\right) \end{aligned} \tag{47}$$

The pair (46) is a NE only if (10) does not hold and $\overline{p}_1^H < \hat{p}_1^1(q_0, q_1)$:

$$\frac{3}{4\theta_h + 1} \leq \lambda \leq 1 \ \text{ and } \ \alpha \leq \frac{4\theta_h\lambda + \lambda - 3}{4\lambda q_1 + 4\lambda q_0}\,. \tag{48}$$

Note that (47) and (48) are never jointly satisfied. However, there are some quality pairs (q_0, q_1) such that neither (47) nor (48) are satisfied. For such qualities, I now show a mixed strategy NE exists.

Let us assume that firm 1 plays $p_1 = \hat{p}_1^1(q_0, q_1)$. Firm 0 is indifferent between $\overline{p}_1^{H^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)$ and $\overline{p}_1^{L^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)$. Assume then that firm 0 randomizes between $\overline{p}_1^{H^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)$ and $\overline{p}_1^{L^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)$ with probabilities h_0 and $1 - h_0$, for some $h_0 \in (0, 1)$. Let σ_0 be this mixed strategy. For all h_0, σ_0 is optimal for firm 0. The expected value of σ_0 is:

$$\begin{aligned}
&\mathbb{E}[\sigma_0] = \\
&h_0\left[\overline{p}_0^{L^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)\right] + (1 - h_0)\left[\underline{p}_0^{H^*}\left(\hat{p}_1^1(q_0, q_1), q_0, q_1\right)\right] = \\
&h_0\left(\frac{(5\lambda - 3)q_1 + q_0(\lambda(4\alpha q_0 - 5) + 3)}{4\lambda}\right) + \\
&(1 - h_0)\left(\frac{(3\lambda - 1)q_1 + \lambda q_0(4\alpha q_0 - 3) + q_0}{4\lambda}\right) \\
&= -\frac{(2h_0 + 1)(q_1 - q_0)}{4\lambda} + \frac{1}{4}(2h_0 + 3)(q_1 - q_0) + \alpha q_0^2\,.
\end{aligned} \tag{49}$$

For $\hat{p}_1^1(q_0, q_1)$ to be a best response of firm 1:

$$h_0 = \frac{4\theta_h\lambda + \lambda - 4\alpha\lambda(q_1 + q_0) - 3}{2(\lambda - 1)} = h_0^*\,. \tag{50}$$

Let σ_0^* denote the NE mixed strategy. Note that:

$$\begin{aligned}
&h_0^* \in (0, 1) \Leftrightarrow \\
&\left(\frac{1}{4\theta_h - 1} < \lambda \leq \frac{3}{4\theta_h + 1} \text{ and } 0 < \alpha < \frac{4\theta_h\lambda - \lambda - 1}{4\lambda q_1 + 4\lambda q_0}\right) \\
&\text{or}\left(\frac{3}{4\theta_h + 1} < \lambda < 1 \text{ and } \frac{4\theta_h\lambda + \lambda - 3}{4\lambda q_1 + 4\lambda q_0} < \alpha < \frac{4\theta_h\lambda - \lambda - 1}{4\lambda q_1 + 4\lambda q_0}\right).
\end{aligned} \tag{51}$$

In the mixed NE:

$$\mathbb{E}\left[\sigma_0^*\right] = \frac{-\alpha\lambda q_1^2 + q_1(\theta_h\lambda + \lambda - 1) + \lambda q_0(-\theta_h + 2\alpha q_0 - 1) + q_0}{\lambda} = \underline{p}_0^M(q_0, q_1)$$
$$p_1 = \hat{p}_1^1(q_0, q_1) = \overline{p}_1^M(q_0, q_1) \ . \tag{52}$$

By comparing (47), (48) and (51), one can see that for any (q_1, q_0) it is impossible for two or more of these conditions to be simultaneously fulfilled. However, at least one of them is always true.

At the end of this Section, I discuss the conditions on (q_0, q_1) such that both firms have a positive market share in the different price NE. In Appendix A.2.5.2, I show that when that is the case, neither firm wants to deviate and become a monopolist. In Appendix A.2.5.3, I show that when that is not the case, there exists a public monopoly NE, and I discuss its property.

- $q_0 > q_1$. I adopt the same approach of the latter case. Some computations are omitted in the interest of brevity. The (duopoly) best response of firm 0 is (40). The (duopoly) best response of firm 1 is:

$$\underline{p}_1^*(p_0, q_0, q_1) = \frac{1}{2}\left(p_0 + \alpha q_1^2 + (\theta_h - 1)q_1 - \theta_h q_0 + q_0\right) \ . \tag{53}$$

There are two intersections of best responses.

$$p_0 = \frac{\alpha\lambda q_1^2 + q_1(2 - (\theta_h + 3)\lambda) + q_0(\lambda(\theta_h + 2\alpha q_0 + 3) - 2)}{3\lambda} = \overline{p}_0^L(q_0, q_1)$$
$$p_1 = \frac{2\alpha\lambda q_1^2 + (\theta_h - 3)\lambda q_1 + q_1 + q_0(\lambda(-\theta_h + \alpha q_0 + 3) - 1)}{3\lambda} = \underline{p}_1^L(q_0, q_1) \tag{54}$$

$$\begin{aligned} p_0 &= \frac{1}{3}\left(\alpha q_1^2 - (\theta_h + 1)q_1 + q_0(\theta_h + 2\alpha q_0 + 1)\right) = \overline{p}_0^H(q_0, q_1) \\ p_1 &= \frac{1}{3}\left(2\alpha q_1^2 + (\theta_h - 2)q_1 + q_0(-\theta_h + \alpha q_0 + 2)\right) = \underline{p}_1^H(q_0, q_1) \end{aligned} \tag{55}$$

There also exists a mixed strategy NE, where firm 1 plays $\hat{p}_1^2(q_0, q_1)$ and firm 0 randomizes between its two best responses with probabilities

$$h_0 = \frac{\lambda(-4\theta_h + 4\alpha(q_1 + q_0) + 5) - 3}{2(\lambda - 1)} = h_0' \tag{56}$$

and $1 - h_0'$. Let σ_S^{**} be the NE mixed strategy of firm 0. Expected prices are as follows.

$$\mathbb{E}\left[\sigma_S^{**}\right] = \frac{-\alpha\lambda q_1^2 + (\theta_h - 2)\lambda q_1 + q_1 + q_0(\lambda(-\theta_h + 2\alpha q_0 + 2) - 1)}{\lambda} = \overline{p}_0^M(q_0, q_1)$$
$$p_1 = \hat{p}_1^2(q_0, q_1) = \underline{p}_{-1}^M(q_0, q_1) \tag{57}$$

(54) is a NE only if:

$$\alpha \leq \frac{4\theta_h\lambda - 3\lambda + 1}{4\lambda q_1 + 4\lambda q_0}. \tag{58}$$

(55) is a NE if and only if:

$$\alpha \geq \frac{4\theta_h\lambda - 5\lambda + 3}{4\lambda q_1 + 4\lambda q_0}. \tag{59}$$

(57) is a NE if and only if:

$$\frac{4\theta_h\lambda - 3\lambda + 1}{4\lambda q_1 + 4\lambda q_0} < \alpha < \frac{4\theta_h\lambda - 5\lambda + 3}{4\lambda q_1 + 4\lambda q_0}. \tag{60}$$

Thus, for any $q_0 < q_1$, it exists at most one price NE.

For further reference, define:

$$p^L(q_0, q_1) = \begin{cases} \underline{p}_0^L(q_0, q_1), \overline{p}_1^L(q_0, q_1) & \text{if } q_0 < q_1 \\ \overline{p}_0^L(q_0, q_1), \underline{p}_{-1}^L(q_0, q_1) & \text{if } q_0 > q_1 \end{cases} \tag{61}$$

$$p^H(q_0, q_1) = \begin{cases} \underline{p}_0^H(q_0, q_1), \overline{p}_1^H(q_0, q_1) & \text{if } q_0 < q_1 \\ \overline{p}_0^H(q_0, q_1), \underline{p}_{-1}^H(q_0, q_1) & \text{if } q_0 > q_1 \end{cases} \tag{62}$$

$$p^M(q_0, q_1) = \begin{cases} \underline{p}_0^M(q_0, q_1), \overline{p}_1^M(q_0, q_1) & \text{if } q_0 < q_1 \\ \overline{p}_0^M(q_0, q_1), \underline{p}_{-1}^M(q_0, q_1) & \text{if } q_0 > q_1 \end{cases} \tag{63}$$

- $q_0 = q_1$. See Appendix A.2.4.

In the remainder of the paper, I refer to $p^k(q_0, q_1)$ as the region of existence of the $k = L, M, H$ price NE. With a slight abuse of notation, I often use the notation p^k, omitting the arguments (q_0, q_1).

Finally, it is important to note that in the p^L region, the median voter always buys good 0. In the p^H region, the median voter always buys good 1. In the p^M

region, firm 0 randomizes between two well-defined best responses. [31]

Note that firm 0 has a positive market share in all the different NE. However, this is not necessarily the case for firm 1. In the p^L NE, firm 1 is out of the market whenever:

$$\begin{aligned} &q_0 < q_1 \text{ and } q_1 < \frac{\theta\lambda + 2\lambda + \alpha\lambda(-q_0) - 1}{\alpha\lambda} = q_1^a \text{ or} \\ &q_0 > q_1 \text{ and } q_1 > \frac{\theta\lambda - 3\lambda + \alpha\lambda(-q_0) + 1}{\alpha\lambda} = q_1^b . \end{aligned} \tag{65}$$

Note that q_1^a is increasing in λ, while q_1^b decreases in λ. Both firms are always active in the p^H and p^M regions. Whenever (65) is satisfied, the NE market structure is a public monopoly.

A.2.5.2 Un-Profitability of Monopolistic Deviations In this Section, I show that given a candidate duopoly NE, firms do not want to "push" their competitor out of the market.

Take a generic pair of candidate NE prices (p_0^*, p_1^*) that, for given a pair of qualities (q_0, q_1), imply

$$\begin{aligned} x_1\left(p_0^*, p_1^*, q_0, q_1\right) > 0 \,, \\ x_0\left(p_0^*, p_1^*, q_0, q_1\right) > 0 \,. \end{aligned} \tag{66}$$

Let us consider the maximization problem of firm i, given p_j^*. By assumption, firm i strictly prefers p_i^* to any $p_i^{'}$ such that $p_i^{'} \neq p_i^*$ and $p_i^{'} \in D_i\left(p_j^*, q_i, q_j\right)$. The payoff function of firm i is increasing in p_i for all $p_i \in \underline{D}_i\left(p_j^*, q_i, q_j\right)$, and decreasing in p_i for all $p_i \in \overline{D}_i\left(p_j^*, q_i, q_j\right)$. Hence, if $\underline{p}_i\left(p_j^*, q_i, q_j\right)$ and $\overline{p}_i\left(p_j^*, q_i, q_j\right)$ are not profitable, any other monopoly price is not profitable as well. [32]

[31] To see this, let us consider the following example. Let $q_0 < q_1$. Then,

$$\begin{aligned} x_0\left(\underline{p}_0^{L^*}\left(\hat{p}_1\left(q_0, q_1\right), q_0, q_1\right), \hat{p}_1\left(q_0, q_1\right), q_0, q_1\right) \geq \frac{1}{2} \\ x_0\left(\underline{p}_0^{H^*}\left(\hat{p}_2\left(q_0, q_1\right), q_0, q_1\right), \hat{p}_2\left(q_0, q_1\right), q_0, q_1\right) \leq \frac{1}{2} \,. \end{aligned} \tag{64}$$

An analogous result applies if $q_0 > q_1$.

[32] The best response functions in Appendix (ii) are well-defined at $p_i = \underline{p}_i\left(p_j, q_i, q_j\right)$ and $p_i = \overline{p}_i\left(p_j, q_i, q_j\right)$. If $\lambda < \frac{1}{2}$, firm 0's payoff is strictly decreasing in p_0 for all $p_0 \leq \underline{p}_0\left(p_1^*, q_0, q_1\right)$, and I cannot exclude that deviations to $p_0 = 0$ are un-profitable.

A.2.5.3 Public Monopoly NE

In this Section, I characterize the public monopoly NE.

Consider a generic pair of candidate NE prices from the p_L region, that is, (61). Call these $\left(p_0^*, p_1^*\right)$. When (65) holds, firm 1 is out of the market. In this case, the NE is characterized as follows. Firm 1 still plays $p_1 = p_1^*$ and firm 0 plays $p_0 = \underline{p}_0\left(p_1^*, q_1, q_0\right)$.

To see this, let us consider the following argument. Let us start from firm 0. It is straightforward that firm 0 does not want to deviate to any $p_0 \in D_0\left(p_1^*, q_0, q_1\right)$. In particular, any $p_0 \in D_0\left(p_1^*, q_0, q_1\right)$ such that the median voter buys from from 0 is dominated by the corner solution $\underline{p}_0\left(p_1^*, q_1, q_0\right)$. Any $p_0 \in D_0\left(p_1^*, q_0, q_1\right)$ such that the median voter buys from from 1 cannot be optimal since $p_1^* > \hat{p}_1\left(q_0, q_1\right)$. In the same way, all $p_0 \in \overline{D}_0\left(p_1^*, q_0, q_1\right)$ are strictly dominated.

What about firm 1? Given any pair of prices in the p^L region, $\pi_1\left(p_0^*, p_1^*, q_1, q_0\right) > 0$. So, if $x_1\left(p_0^*, p_1^*, q_1, q_0\right) < 0$, then $p_1^* < \alpha q_1^2$. Since firm 1 could retrieve market shares only by decreasing its price (any deviation to some $p_1 > p_1^*$ cannot be profitable because it does not increase the market share), but this is not profitable because it would imply negative profits, p_1^* is optimal.

A.2.5.3.1 A Property of the Public Monopoly NE I now prove that in any public monopoly NE,

$$\pi_0 \left(\underline{p}_0 \left(p_1^*, q_1, q_0 \right), p_1^*, q_0, q_1 \right) > 0 \,. \tag{67}$$

Let us define an auxiliary object. $V_0^U(p_0, p_1, q_0, q_1)$ is firm 0's payoff "pretending" that its demand function is unbounded. The unbounded demand of firm 0 is:

$$x_0^U(\hat{\theta}) = \begin{cases} \hat{\theta} - (\theta_h - 1) & \text{if } q_0 \leq q_1 \\ \theta_h - \hat{\theta} & \text{if } q_0 > q_1 \end{cases} \,. \tag{68}$$

The "unbounded" profit and payoff functions are obtained by substituting (68) inside of (2) and (3). The "true" demand function of firm 0 is bounded above by 1, and below by 0.

Fix some p_1^*. Then,

$$p_0^* = \arg \max_{p_0 \in [0,P]} V_0^U(p_0, p_1^*, q_0, q_1) \,, \tag{69}$$

and

$$p_0^* < \underline{p}_0 \left(p_1^*, q_1, q_0 \right) \,. \tag{70}$$

Suppose by contradiction that:

$$p_0^* < \alpha q_0^2 \,. \tag{71}$$

The assumption (71) implies that:

$$\begin{aligned} & V_0^U \left(p_0^*, p_1^*, q_0, q_1 \right) < V_0 \left(p_0^*, p_1^*, q_0, q_1 \right) < \\ & V_0 \left(\underline{p}_0 \left(p_1^*, q_0, q_1 \right), p_1^*, q_0, q_1 \right) = V_0^U \left(\underline{p}_0 \left(p_1^*, q_0, q_1 \right), p_1^*, q_0, q_1 \right) \,, \end{aligned} \tag{72}$$

which contradicts (69) and implies also $\underline{p}_0 \left(p_1^*, q_1, q_0 \right) > \alpha q_0^2$.

The latter result has an important corollary. In any public monopoly NE:

$$V_0^U(p_0^*, p_1^*, q_0, q_1) \geq V_0 \left(\underline{p}_0 \left(p_1^*, q_0, q_1 \right), p_1^*, q_0, q_1 \right) \geq V_0(p_0^*, p_1^*, q_0, q_1) \,. \tag{73}$$

Because of (73), in *Stage* (2), I can "pretend" that firm 0's payoff is $V^U(p_0^*, p_1^*, q_0, q_1)$. If some deviations towards this region are proven to be unprofitable, the same deviations cannot be profitable when considering the real (bounded) NE payoff

$V_0\left(\underline{p}_0\left(p_1^*, q_0, q_1\right), p_1^*, q_0, q_1\right)$. Nonetheless, given SPNE qualities (Proposition 2), firms cannot reach the region of existence of the public monopoly NE. That is, given any candidate SPNE quality of firm j (q_j^*), firm i can only deviate towards some q_i such that the pair $\left(q_i, q_j^*\right)$ sustains a (unique) duopoly price NE. Then, this property, albeit useful, is not used in the following proofs.[33]

[33]When (q_0, q_1) are such that there is a public monopoly NE, the NE is potentially not unique. In particular, firm 1's best response to $\underline{p}_0\left(p_1^*, q_0, q_1\right)$ is potentially not unique. The properties described in this section apply to any monopoly price NE.

A.3 Proof of Proposition 2

In all the following proofs, I assume that firms enter the market in *Stage* (1). Then, I find a NE of *Stage* (2). Finally, I check that, by fixing this NE, firms have indeed an incentive to enter the market in *Stage* (1).

A.3.1 Proof of (i)

Given *Stage* (3) NE prices, each firm's payoff function in *Stage* (2) is continuous in its own action and in the action of its opponent. To see this, consider the discontinuity in firms' demand functions described in Section 5.1. In *Stage* (3), firms' demand functions are discontinuous if $q_i = q_j$ and $p_i \to p_j$. In *Stage* (2), given NE prices $p^* \left(q_i, q_j\right)$:

$$\begin{aligned}
&\lim_{q_i \to q_{j-}} \pi_i \left(q_i, q_j, p^* \left(q_i, q_j\right)\right) = \\
&\lim_{q_i \to q_{j+}} \pi_i \left(q_i, q_j, p^* \left(q_i, q_j\right)\right) = \\
&\pi_i \left(q_j, q_j, p^* \left(q_j, q_j\right)\right) = 0 \; .
\end{aligned} \tag{74}$$

Hence, in *Stage* (2), there exists at least one (mixed strategy) NE (Glicksberg [1952]).[34] Anticipating the NE payoffs of *Stage* (2), in *Stage* (1), firms only choose whether to enter or not, depending on whether their expected payoff is greater than zero.

[34]As in Appendix A.2.2, the NE does not need to be in pure strategies because of the lack of quasi-concavity of payoffs.

A.3.2 Proof of (ii)

Suppose, towards a contradiction, $a_1^* = a_0^* = 1$. I now solve explicitly for the NE of *Stage* (3). Firm 0's payoff is:

$$V_0(p_0, p_1, q_0, q_1) = \max\{\overline{\theta} q_0 - p_0, \overline{\theta} q_1 - p_1\} . \quad (75)$$

Firm 0 has a dominant strategy: $p_0^* = 0$. The best response of firm 1 is:

$$p_1^*(p_0^* = 0) = \begin{cases} \frac{1}{2}\left(\alpha q_1^2 + \theta_h q_1 - \theta_h q_0\right) & \text{if } q_0 < q_1 \\ \frac{1}{2}\left(\alpha q_1^2 + (\theta_h - 1) q_1 - \theta_h q_0 + q_0\right) & \text{if } q_0 \geq q_1 \end{cases} \quad (76)$$

Given subgame NE prices, both firms have positive market shares if and only if:

$$q_1 > q_0 \text{ and } x_1\left(0, p_1^*(0), q_0, q_1\right) \geq \theta_h \Leftrightarrow \frac{\theta_h q_1 - \alpha q_1^2}{\theta_h} < q_0 < q_1 . \quad (77)$$

Let us go back to *Stage* (2). Firm 0's payoff is:

$$V_0(p_0, p_1, q_0, q_1) = \max\{\overline{\theta} q_0, \overline{\theta} q_1 - p_1^*(q_0, q_1)\} . \quad (78)$$

Note that:

$$\frac{\partial p_1^*(q_0, q_1)}{\partial q_0} < 0 . \quad (79)$$

Then, the dominant strategy of firm 0 is $q_0^* = Q$, which yields the median voter the highest possible payoff. By Assumption 1, for any q_1, firm 1 makes negative profits, which contradicts the definition of SPNE (by choosing $a_1 = 1$, firm 1 can secure a zero payoff).[35]

Since *Stage* (3) and *Stage* (2) NE are in dominant strategies, the SPNE is unique. In particular, given firm 0's dominant strategy, firm 1 cannot make non-negative profits. Then, it cannot exist a SPNE where $a_1^* = 1$.

[35] Assumption 1 is crucial for this proof, even if $Q \leq \frac{\theta_h}{4\alpha}$ would be sufficient. The stronger version of this requirement in Assumption 1 is useful to make Q not binding also in the duopoly SPNE.

A.3.3 Proof of (iii)

I proceed in two steps.

1. In any public monopoly SPNE:

$$q_0 = q_0^M \in \arg \max_{q \in [0,Q]} V_0 \left(q, p_0(q) \mid a_1 = 0\right) , \tag{80}$$

   where $p_0(q)$ is firm 0's optimal monopoly price as a function of q. Now, let us define firm 1's optimal entry quality:

$$q_1^E \in \arg \max_{q \in [0,Q]} \pi_1 \left(q, q_0^M, p_1^* \left(q, q_0^M\right), p_0^* \left(q, q_0^M\right)\right) , \tag{81}$$

   where $p_i^* \left(q, q_0^M\right)$ is a mapping from the pair of qualities $\left(q, q_0^M\right)$ to some NE price of firm i. By Lemma 1, such mapping exists. It is unique whenever $\pi_1 \left(q, q_0^M, p_1^* \left(q, q_0^M\right), p_0^* \left(q, q_0^M\right)\right) > 0$.[36] By assumption, if $\lambda = \overline{\lambda}$, then

$$\pi_1 \left(q_1^E, q_0^M, p_1^* \left(q_1^E, q_0^M\right), p_0^* \left(q_1^E, q_0^M\right)\right) > 0 . \tag{82}$$

   Since the function $\pi_1 \left(q, q_0^M, p_1^* \left(q, q_0^M\right), p_0^* \left(q, q_0^M\right)\right)$ is continuous and non-decreasing in λ, then, for all $\lambda > \overline{\lambda}$:

$$\pi_1 \left(q_1^E, q_0^M, p_1^* \left(q_1^E, q_0^M\right), p_0^* \left(q_1^E, q_0^M\right)\right) > 0 . \tag{83}$$

   This is guaranteed by subgame NE prices being continuous and non-decreasing in λ.[37]

2. By (ii), $\overline{\lambda} > 0$. If $\lambda = \frac{1}{2}$, $\pi_1 \left(q_1^E, q_0^M, p_1^* \left(q_1^E, q_0^M\right), p_0^* \left(q_1^E, q_0^M\right)\right) > 0$, then $\overline{\lambda} \leq \frac{1}{2}$.

[36] When the mapping is possibly not unique because the pair of qualities induces a public monopoly NE, by assumption of Proposition 2, I am fixing the NE described in Lemma 1. That is, the monopoly NE prices described in Appendix A.2.5.3. This choice comes with no loss of generality as all possible public monopoly NE are payoff-equivalent for firm 1. Note also that q_0^M and q_1^M exist by the continuity of the payoffs and compactness of $[0, Q]$.

[37] See subgame NE prices obtained in Lemma 1. The function $\pi_1 \left(q, q_0^M, p_1^* \left(q, q_0^M\right), p_0^* \left(q, q_0^M\right)\right)$ has the following properties. First, it is not quasi-concave because by moving q, firm 1 can end up in different price regions. Second, at the "borders" of these regions, and everywhere else, is continuous (see Appendix A.3.1). Third, "inside" of any region, it is non-decreasing in λ, which guarantees the required result.

A.3.4 Proof of (iv)

For the sake of the readability of expressions, I adopt the following normalization in presenting the proof. Let $\alpha = \frac{1}{2}$ and $\theta_h = 2$, so that $\overline{\theta} = \frac{3}{2}$. The normalization, while simplifying the algebra, does not affect any results. The proof for generic (α, θ_h) is available upon request. In this stage, α and θ_h are only multiplicative factors of firms' payoff functions, so that the relation between SPNE qualities and both α and θ_h is linear (see Motta [1993], Lambertini [2006] for analogous results in PD models of private duopoly).

Subgame prices simplify as follows.

$$\begin{aligned}
\underline{p}_0^L(q_0, q_1) &= \frac{\lambda(q_1+4)q_1 - 4q_1 + 2\lambda(q_0-2)q_0 + 4q_0}{6\lambda} \\
\overline{p}_0^L(q_0, q_1) &= \frac{\lambda(q_1-10)q_1 + 4q_1 + 2\lambda q_0(q_0+5) - 4q_0}{6\lambda} \\
\underline{p}_1^L(q_0, q_1) &= \frac{2\lambda(q_1-1)q_1 + 2q_1 + \lambda q_0(q_0+2) - 2q_0}{6\lambda} \\
\overline{p}_1^L(q_0, q_1) &= \frac{2\lambda(q_1+4)q_1 - 2q_1 + \lambda(q_0-8)q_0 + 2q_0}{6\lambda}
\end{aligned} \tag{84}$$

$$\begin{aligned}
\underline{p}_0^H(q_0, q_1) &= \frac{1}{6}\left(q_1^2 + 2q_0^2\right) \\
\overline{p}_0^H(q_0, q_1) &= \frac{1}{6}((q_1-6)q_1 + 2q_0(q_0+3)) \\
\underline{p}_1^H(q_0, q_1) &= \frac{1}{6}\left(2q_1^2 + q_0^2\right) \\
\overline{p}_1^H(q_0, q_1) &= \frac{q_1^2}{3} + q_1 + \frac{1}{6}(q_0-6)q_0
\end{aligned} \tag{85}$$

$$\begin{aligned}
\underline{p}_0^M(q_0, q_1) &= \frac{q_0 - q_1}{\lambda} - \frac{1}{2}(q_1-6)q_1 + (q_0-3)q_0 \\
\overline{p}_0^M(q_0, q_1) &= q_0^2 + \frac{q_1 - q_0}{\lambda} - \frac{q_1^2}{2} \\
\overline{p}_1^M(q_0, q_1) = \hat{p}_1^1(q_0, q_1) &= \frac{5\lambda q_1 - q_1 + \lambda q_0^2 - 5\lambda q_0 + q_0}{2\lambda} \\
\underline{p}_1^M(q_0, q_1) = \hat{p}_1^2(q_0, q_1) &= \frac{\lambda q_0^2 - \lambda q_0 - q_0 + \lambda q_1 + q_1}{2\lambda}
\end{aligned} \tag{86}$$

For further reference, let me now define the following thresholds. Note that these values are not functions of θ_h and α.

λ_i	Value
λ_{m_1}	$\frac{7}{13}$
λ_{m_2}	$\frac{5}{8}$
λ_{p_1}	≈ 0.847
λ_{p_2}	≈ 0.924
λ_{p_3}	≈ 0.93

In some SPNE, the median voter buys good 0. In others, they buy good 1. Moreover, in some SPNE, both firms play a pure strategy in *Stage* (2). In others, one firm plays a mixed strategy. For each type of SPNE, for the sake of brevity, I show the proof only for the first example. The other proofs are analogous and available upon request.

A.3.4.1 Pure Strategies SPNE where the Median Voter Buys good 0

Let us assume that firms believe that in *Stage* (3) they will play the NE prices: $p^L(q_0, q_1)$.

I am looking for a NE in qualities. Any NE quality pair must be such that the following conditions are satisfied. NE qualities, q_i^* and q_j^*, must be such that:

1. the upstream price NE $p^L\left(q_i^*, q_j^*\right)$ exists, or, equivalently, $\left(q_i^*, q_j^*\right)$ lies in the p^L region;
2. given q_j^*,
   (a) q_i^* must maximize the payoff of firm i inside the $p^L\left(q_i, q_j^*\right)$;
   (b) firm i has no incentive to deviate to any other price region.

Given prices $p^L(q_0, q_1)$, I now write down firms' payoffs. $\underline{V}_0^L(q_0, q_1, p^L(q_0, q_1))$ (respectively, $\overline{V}_0^L(q_0, q_1, p^L(q_0, q_1))$) be the *Stage* (2) payoff function of firm 0 when $q_0 \leq q_1$ (respectively, $q_0 > q_1$). Analogously, I define $\underline{\pi}_1^L(q_0, q_1, p^L(q_0, q_1))$ and $\overline{\pi}_1^L(q_0, q_1, p^L(q_0, q_1))$.

$$V_0\left(q_0, q_1, p^L(q_0, q_1)\right) = \begin{cases} \underline{V}_0^L(q_0, q_1, p^L(q_0, q_1)) & \text{if } q_0 \leq q_1 \\ \overline{V}_0^L(q_0, q_1, p^L(q_0, q_1)) & \text{if } q_0 > q_1 \end{cases} =$$
$$= \begin{cases} \frac{\left(\lambda^2\left(\left(q_1^2-70\right)q_0-(q_1-10)q_0^2+q_1(q_1+4)^2-q_0^3\right)-2\lambda(4q_1(q_1+4)+q_0(5q_0-43))+16(q_1-q_0)\right)}{36\lambda} & \text{if } q_0 \leq q_1 \\ \frac{q_0\left(16-\lambda\left(\lambda\left(q_1^2-46\right)+26\right)\right)-q_1(\lambda(q_1-10)+4)^2+\lambda q_0^2(\lambda(q_1-2)-10)+\lambda^2 q_0^3}{36\lambda} & \text{if } q_0 > q_1 \end{cases} \tag{87}$$

$$\pi_1\left(q_0, q_1, p^L(q_0, q_1)\right) =$$
$$\begin{cases} \overline{\pi}_1^L(q_0, q_1, p^L(q_0, q_1)) = \frac{(q_1-q_0)(\lambda(q_1+q_0-8)+2)^2}{36\lambda^2} & \text{if } q_0 \leq q_1 \\ \overline{\pi}_1^L(q_0, q_1, p^L(q_0, q_1)) = \frac{(q_0-q_1)(\lambda(q_1+q_0+2)-2)^2}{36\lambda^2} & \text{if } q_0 > q_1 \end{cases} \tag{88}$$

Let $\underline{q}_i^L(q_j)$ (respectively, $\overline{q}_i^L(q_j)$) denote the (interior) optimal quality of firm i when $q_i \leq q_j$ (respectively, $q_i > q_j$). To characterize these strategies, I combine FOCs and SOCs. FOCs read as follows.

$$\frac{\partial \underline{V}_0^L(q_0, q_1, p^L(q_0, q_1))}{\partial q_0} = \frac{\lambda^2((q_1+q_0)(q_1-3q_0)+20q_0-70)+\lambda(86-20q_0)-16}{36\lambda} = 0 \tag{89}$$

$$\begin{aligned}\frac{\partial \overline{V}_0^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_0} &= \\ \frac{\lambda\left(\lambda\left(3q_0^2 + 2q_0(q_1-2) - q_1^2 + 46\right) - 20q_0 - 26\right) + 16}{36\lambda} &= 0\end{aligned} \tag{90}$$

$$\frac{\partial \underline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} = \frac{(\lambda(q_0 - 3q_1 - 2) + 2)(\lambda(q_0 + q_1 + 2) - 2)}{36\lambda^2} = 0 \tag{91}$$

$$\frac{\partial \overline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} = \frac{(\lambda(3q_1 - q_0 - 8) + 2)(\lambda(q_1 + q_0 - 8) + 2)}{36\lambda^2} = 0 \tag{92}$$

Using FOCs and SOCs, I obtain:

$$\begin{aligned}&\frac{\partial \underline{V}_0^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_0} = 0 \text{ and} \\ &\frac{\partial^2 \underline{V}_0^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_0^2} < 0 \Leftrightarrow \\ &q_0 = \frac{-(\lambda(q_1 - 10)) + \sqrt{2\lambda(-55\lambda + 2q_1(\lambda(q_1 - 5) + 5) + 29) + 52} - 10}{3\lambda} = \underline{q}_0^L(q_1)\end{aligned} \tag{93}$$

$$\begin{aligned}\frac{\partial \overline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} &= 0 \text{ and} \\ \frac{\partial^2 \overline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1^2} &< 0 \Leftrightarrow \\ q_1 = \frac{\lambda(q_0 + 8) - 2}{3\lambda} &= \overline{q}_1^L(q_0)\end{aligned} \tag{94}$$

$$\begin{aligned}&\frac{\partial \overline{V}_0^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_0} = 0 \text{ and} \\ &\frac{\partial^2 \overline{V}_0\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_0^2} < 0 \Leftrightarrow \\ &q_0 = -\frac{\lambda(q_1 - 2) + \sqrt{2\lambda(-67\lambda + 2q_1(\lambda(q_1 - 1) - 5) + 59) + 52} - 10}{3\lambda} = \overline{q}_0^L(q_1)\end{aligned} \tag{95}$$

$$\begin{aligned}\frac{\partial \underline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} &= 0 \text{ and} \\ \frac{\partial^2 \underline{\pi}_1\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1^2} &< 0 \Leftrightarrow \\ q_1 = \frac{\lambda(q_0 - 2) + 2}{3\lambda} &= \underline{q}_1^L(q_0)\end{aligned} \tag{96}$$

I omit the SOCs for the sake of brevity. Each FOC has two solutions, but only one is a maximum. Note that both firms play the same strategy when $\lambda \to 1$.[38] The intersection of $\underline{q}_0^L(q_1)$ and $\overline{q}_1^L(q_0)$ is:

$$q_0^* = \frac{37\lambda + 9\sqrt{\lambda(34-39\lambda)+9} - 43}{16\lambda} < \frac{55\lambda + 3\sqrt{\lambda(34-39\lambda)+9} - 25}{16\lambda} = q_1^* \,. \tag{101}$$

The intersection of $\underline{q}_1^L(q_0)$ and $\overline{q}_0^L(q_1)$ is:

$$q_1^{**} = \frac{-7\lambda - 3\sqrt{\lambda(34-39\lambda)+9} + 25}{16\lambda} < \frac{11\lambda - 9\sqrt{\lambda(34-39\lambda)+9} + 43}{16\lambda} = q_0^{**} \,. \tag{102}$$

Define the vectors: $q^* = (q_0^*, q_1^*)$ and $q^{**} = (q_0^{**}, q_1^{**})$. For q^* or q^{**} to be a SPNE quality pair, it must be that they are not subject to profitable unilateral deviations.

First of all, note that:

$$\begin{aligned} q_0^* &< q_1^* \\ q_0^{**} &> q_1^{**} \,. \end{aligned} \tag{103}$$

Let us start with q^* (the proof for q^{**} is analogous). I need to check that given q_j^*, firm $i \neq j$ does not want to deviate to some $q_i \neq q_i^*$. By Lemma 1, there exists a bi-jective mapping from a pair (q_0, q_1) to some duopoly price NE. Thus, I distinguish between different types of deviations, depending on whether the pair (q_j^*, q_i) is such that the downstream price NE is $p^L\left(q_j^*, q_i\right)$, $p^H\left(q_j^*, q_i\right)$ or

[38]Moreover,

$$\overline{q}_1^L(q_0) > q_0 \Leftrightarrow q_0 \in [0,3] \text{ and } \frac{1}{q_0 - 4} + \lambda > 0 \,. \tag{97}$$

If the condition in (97) is not satisfied, then:

$$\frac{\partial \overline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} > 0 \,. \tag{98}$$

Analogously,

$$\underline{q}_1^L(q_0) \in (0, q_0) \Leftrightarrow \lambda(q_0 + 1) > 1 \,. \tag{99}$$

If the condition in (99) does not hold, then

$$\frac{\partial \underline{\pi}_1^L\left(q_0, q_1, p^L\left(q_0, q_1\right)\right)}{\partial q_1} < 0 \,. \tag{100}$$

However, for any candidate NE, q_0, FOCs are sufficient to find a solution for firm 1's problem.

$p^M\left(q_j^*, q_i\right)$. Payoffs along (101) and (102) are symmetric and they are given by:

$$
\begin{aligned}
&V_0\left(q^*, p^L\left(q^*\right)\right) = \\
&\frac{1}{256\lambda^2}\left[93\sqrt{\lambda(34-39\lambda)+9}+\lambda\left(-108\sqrt{\lambda(34-39\lambda)+9}\right.\right. \\
&\left.\left.+\lambda\left(323\lambda+39\sqrt{\lambda(34-39\lambda)+9}-561\right)+693\right)-407\right]
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
&\pi_1\left(q^*, p^L\left(q^*\right)\right) = \\
&-\frac{3\left(-3\lambda+\sqrt{\lambda(34-39\lambda)+9}-3\right)^3}{512\lambda^3}.
\end{aligned}
\tag{105}
$$

Type (1) Given q_j^*, firm i deviates to some q_i such that the price NE $p^L\left(q_i, q_j^*\right)$ exists.

Consider firm $i=0$. Any deviation $q_0 < q_0^*$ is not profitable by FOC (89).[39] Then, consider some deviation $q_0 > q_1^*$ (Motta [1993] and Lambertini [2006], among others, call these "leapfrogging deviations"). Among these deviations, $\overline{q}_0^L\left(q_1^L\right)$ yields the highest possible payoff. Note that:

$$
\begin{aligned}
&\underline{V}_0^L\left(\underline{q}_0^L(q_1), q_1, p^L\left(\underline{q}_0^L(q_1), q_1\right)\right) \geq \\
&\overline{V}_0^L\left(\overline{q}_0^L(q_1), q_1, p^L\left(\overline{q}_0^L(q_1), q_1\right)\right) \Leftrightarrow \\
&q_1 \geq \frac{3}{2};
\end{aligned}
\tag{108}
$$

[39] In particular, if $q_0 \leq q_1$, (87) is minimized at:

$$
q_0^{min_1} = -\frac{\lambda(q_1-10)+\sqrt{2\lambda(-55\lambda+2q_1(\lambda(q_1-5)+5)+29)+52}+10}{3\lambda}.
\tag{106}
$$

Since $q_0^{min_1}(q_1) < 0$, firm 0's payoff is increasing in the interval $\left[0, \underline{q}_0^L(q_1)\right]$ and decreasing afterwards. Note that $q_0^{min_1}(q_1) < 0$ is the other solution of the FOC of firm 0. If $q_0 > q_1$, (87) is minimized at:

$$
q_0^{min_2}(q_1) = \frac{-(\lambda(q_1-2))+\sqrt{2\lambda(-67\lambda+2q_1(\lambda(q_1-1)-5)+59)+52}+10}{3\lambda}.
\tag{107}
$$

In this case, $q_0^{min_2}(q_1)$ does never lie in the region of existence of the price NE $p^L(q_0, q_1)$. In particular, the highest q_0 such that the NE $p^L(q_0, q_1)$ exists is: $\frac{7\lambda-2\lambda q_1-1}{2\lambda}$ which is lower than $q_0^{min_2}(q_1)$. These conditions ensure that the corners of the regions of existence of the price NE $p^L(q_0, q_1)$ are never attractive for firm 0. In other words, the payoff of 0 is strictly concave inside this region.

$$\begin{aligned}
&\underline{\pi}_1^L\left(\underline{q}_1^L(q_0), q_0, p^L\left(\underline{q}_1^L(q_0), q_0\right)\right) \geq \\
&\overline{\pi}_1^L\left(\overline{q}_1^L(q_0), q_0, p^L\left(\overline{q}_1^L(q_0), q_0\right)\right) \Leftrightarrow \\
&q_0 \geq \frac{3}{2}\,.
\end{aligned} \tag{109}$$

That is, firm i wants to produce a higher quality than q_j if and only if q_j is low enough. Since $q_1^* > \frac{3}{2}$ for $\lambda \geq \frac{1}{2}$, firm 0 does never want to leapfrog the quality of its opponent.

Let us now consider firm 1. Any $q_1 > q_0^*$ is not optimal by FOC.[40] Since $q_0^* > \frac{3}{2}$ for $\lambda \in \left[\frac{7}{13}, \frac{5}{8}\right]$, firm 1 has an incentive to deviate to $\underline{q}_1^L\left(q_0^*\right) < q_0^*$ in this range (and the deviation is also feasible). Let us define $\frac{7}{13} = \lambda_{m_1}$ and $\frac{5}{8} = \lambda_{m_2}$. I conclude that the pair (101) is safe from profitable unilateral deviation of *Type* (1) when $\lambda \in \left[\frac{1}{2}, \lambda_{m_1}\right] \bigcup \left[\lambda_{m_2}, 1\right]$, where $\lambda_{m_2} > \lambda_{m_1}$.

Type (2) Given q_j^*, firm i deviates to some q_i such that the price NE $p^H\left(q_i, q_j^*\right)$ exists.

As in the previous point, I adopt the following approach. I compare (104) and (105) to the payoff that firm i obtains by playing its optimal quality, given q_j^*, in the region where subgame prices are $p^H\left(q_i, q_j^*\right)$. I define the following payoffs.

$$\begin{aligned}
&V_0^H\left(q_0, q_1, p^H\left(q_0, q_1\right)\right) = \\
&= \frac{1}{36}\Bigg[\lambda\Bigg(q_1^3 + q_1^2(q_0+12) - q_1\left(q_0^2+18\right) \\
&- q_0((q_0-6)q_0+36)\Bigg) + 6q_1(3-2q_1) - 6(q_0-6)q_0\Bigg] \\
&\text{if } q_0 \leq q_1; \\
&\text{and} \\
&\frac{1}{36}\left(-\lambda\left(q_1^2-36\right)q_0 - q_1(90\lambda + q_1(\lambda(q_1-24)+12) - 54) + \right. \\
&\left. q_0^2(\lambda(q_1-6)-6) + \lambda q_0^3\right) \\
&\text{if } q_0 > q_1
\end{aligned} \tag{110}$$

[40] An analogous condition of Footnote (39) guarantees that that corner solutions in the p^L region are not attractive.

$$\pi_1^H\left(q_0, q_1, p^H\left(q_0, q_1\right)\right) = \begin{cases} \frac{1}{36}(q_1 - q_0)(q_1 + q_0 - 6)^2 & \text{if } q_0 \leq q_1 \\ \frac{1}{36}(q_0 - q_1)(q_1 + q_0)^2 & \text{if } q_0 > q_1 \end{cases} \tag{111}$$

FOCs and SOCs produce the following (interior) optimal strategies:

$$\begin{aligned} \underline{q}_0^L(q_1) &= \frac{6\lambda - \lambda q_1 + 2\sqrt{\lambda(q_1+3)(\lambda(q_1-6)+3)+9} - 6}{3\lambda} \quad \text{if } q_0 \leq q_1; \\ \overline{q}_0^L(q_1) &= -\frac{\lambda(q_1-6) + 2\sqrt{\lambda(q_1-6)(\lambda(q_1+3)-3)+9} - 6}{3\lambda} \quad \text{if } q_0 > q_1; \end{aligned} \tag{112}$$

$$\begin{aligned} \underline{q}_1^H(q_0) &= \frac{q_0}{3} \quad \text{if } q_0 \leq q_1; \\ \overline{q}_1^H(q_0) &= \frac{q_0+6}{3} \quad \text{if } q_0 > q_1; \end{aligned} \tag{113}$$

Given q_j^*, deviations towards $\underline{q}_i^H\left(q_j^*\right)$ and $\overline{q}_i^H\left(q_j^*\right)$ are never jointly profitable and feasible for firm i in the region of existence of $p^H\left(q_j^*, \underline{q}_i^H\left(q_j^*\right)\right)$, and $p^H\left(q_j^*, \overline{q}_i^H\left(q_j^*\right)\right)$, respectively. However, firms might still want to reach the $p^H\left(q_0, q_1\right)$ region, even if their (interior) optimal strategy in the region is not achievable. In other words, without an interior maximizer, I need to consider corner solutions (see Figure 5). I distinguish between different cases.

- $q_1 = q_0$. This deviation implies zero profits for both firms, which is never optimal for firm 1. Let us check for firm 0. Given q_1^*, if firm 0 deviates to $q_0 = q_1^*$, its payoff is given by (110):

$$\frac{(\lambda-1)\left(7\lambda + 3\sqrt{\lambda(34-39\lambda)+9} - 25\right)\left(55\lambda + 3\sqrt{\lambda(34-39\lambda)+9} - 25\right)}{512\lambda^2}, \tag{114}$$

  which is lower than (104).

- Given q_j^*, firm $i \neq j$ deviates to some q_i such that $\overline{p}_1^H\left(q_i, q_j^*\right) = \hat{p}_1^1\left(q_i, q_j^*\right)$ or $\underline{p}_1^H\left(q_i, q_j^*\right) = \hat{p}_1^2\left(q_i, q_j^*\right)$. It can be shown that these

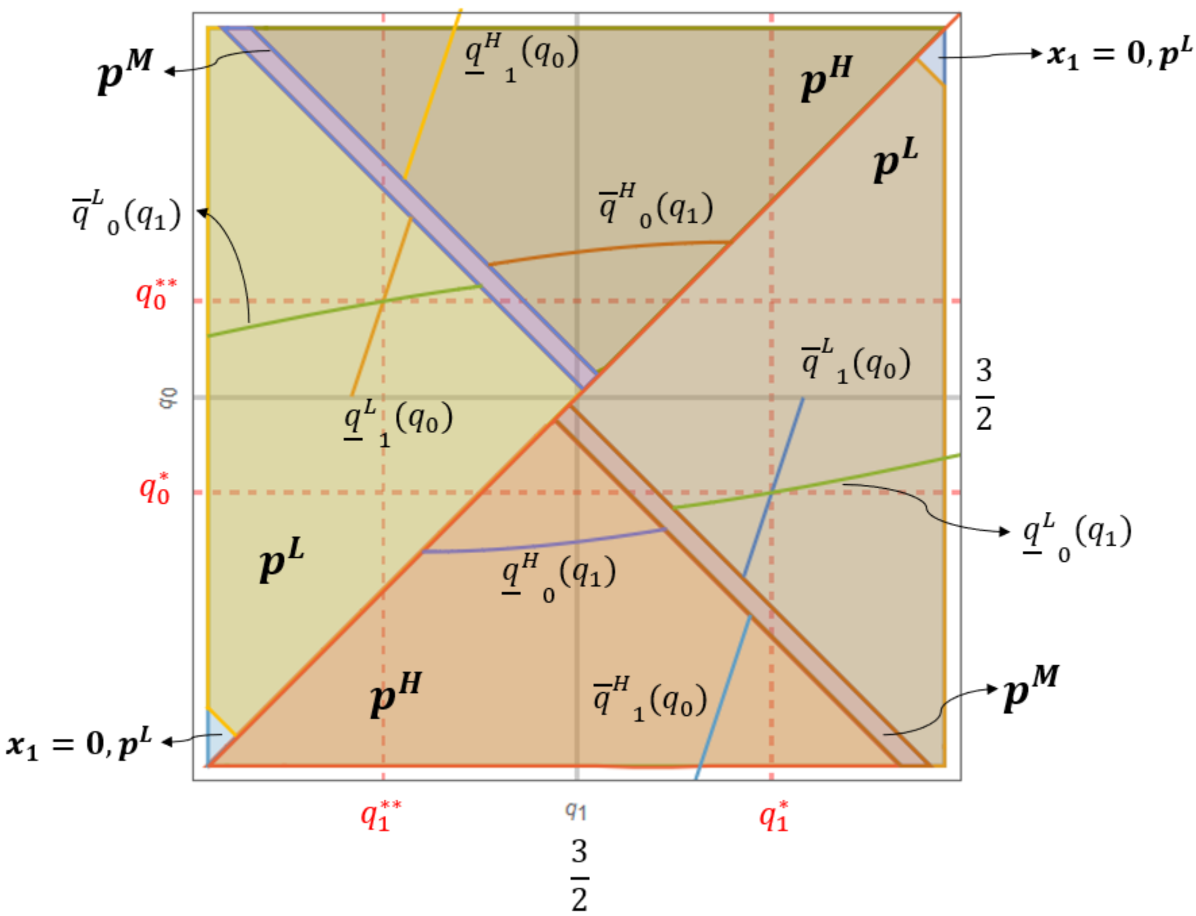

Figure 5: **Optimal Qualities** when $\lambda \geq \lambda_{m_2}$. In the p^L region, I only plot the strategy yielding higher payoffs.

deviations solve the equation:

$$\overline{p}_1^H(q_0,q_1)=\hat{p}_1^1(q_0,q_1)\Rightarrow\frac{(q_1-q_0)(\lambda(2q_1+2q_0-9)+3)}{6\lambda}=0\text{ if }q_0\leq q_1$$
$$\underline{p}_1^H(q_0,q_1)=\hat{p}_1^2(q_0,q_1)\Rightarrow\frac{(q_1-q_0)(\lambda(2q_1+2q_0-3)-3)}{6\lambda}=0\text{ if }q_0>q_1 \tag{115}$$

Take $i=0$ and then $q_1=q_1^*$. Equation (115) reduces to :

$$\frac{\left(3\sqrt{\lambda(34-39\lambda)+9}+\lambda(55-16q_0)-25\right)\left(3\sqrt{\lambda(34-39\lambda)+9}+\lambda(16q_0-17)-1\right)}{768\lambda^2}=0$$

if $q_0<q_1$;

and

$$\frac{1}{384\lambda^2}\Big[-111\sqrt{\lambda(34-39\lambda)+9}$$
$$+\lambda\left(677\lambda+129\sqrt{\lambda(34-39\lambda)+9}-64q_0(\lambda(2q_0-3)-3)-1582\right)+653\Big]=0$$

if $q_0>q_1$.

(116)

There are two solutions.

$$q_0=\frac{17\lambda-3\sqrt{\lambda(34-39\lambda)+9}+1}{16\lambda}=q_0^{d_1}<q_1^*$$
$$q_0=\frac{1}{16\lambda^2}\Bigg[\sqrt{2}\Bigg[\lambda^2\Bigg(-111\sqrt{\lambda(34-39\lambda)+9}+$$
$$\lambda\left(749\lambda+129\sqrt{\lambda(34-39\lambda)+9}-1438\right)+725\Bigg)\Bigg]^{\frac{1}{2}}+12\lambda(\lambda+1)\Bigg]=q_0^{d_2}>q_1^*$$
$$\Leftrightarrow\lambda<\frac{161}{275} \tag{117}$$

Plugging the above solutions into (110) reveals that deviating to $q_0^{d_2}$

is never profitable. However, if firm 0 deviates to $q_0^{d_2}$, its payoff is:

$$\frac{1}{256\lambda^2}\Bigg(57\sqrt{\lambda(34-39\lambda)+9}$$
$$+\lambda\left(-204\sqrt{\lambda(34-39\lambda)+9}+\lambda\left(511\lambda+171\sqrt{\lambda(34-39\lambda)+9}-1205\right)+1017\right)-275\Bigg). \tag{118}$$

I obtain (118) by substituting $q_0^{d_2}$ and q_1^* into (110). This reveals that (118) is higher than (104) for $\lambda > 0.953$. Note that this deviation is no more attractive if $\lambda = 1$ because $q_0^{d_2} = q_0^*$.

Let us now consider firm $i = 1$. (115) has a unique feasible solution. For instance, $\underline{p}_1^H(q_1, q_0^*) = \hat{p}_1^2(q_1, q_0^*)$ implies some $q_1^{d_2} > q_0^*$.

$$\overline{p}_1^H(q_1, q_0^*) = \hat{p}_1^1(q_1, q_0^*) \Rightarrow q_1 = \frac{-9\sqrt{-39\lambda^2+34\lambda+9}+35\lambda+19}{16\lambda} = q_1^{d_1} > q_0^*. \tag{119}$$

Substituting (119) and q_0^* into (111), I obtain the deviation payoff:

$$-\frac{(\lambda+1)^2\left(\lambda+9\sqrt{\lambda(34-39\lambda)+9}-31\right)}{128\lambda^3}, \tag{120}$$

which is higher than (105) if $\lambda > 0.924 = \lambda_{p_2}$.

- Given q_j^*, firm $i \neq j$ deviates to some q_i such that $q_i \to Q$ or $q_i = 0$. With an analogous procedure of the points above, it is relatively easy to show that these "corner" deviations are not profitable.

Type (3) Given q_j^*, firm i deviates to some q_i such that the price NE $p^M\left(q_i, q_j^*\right)$ exists.

I now write down firms' payoff functions along this subgame price NE.

$$V_0^M\left(q_0, q_1, p^M(q_0, q_1)\right) = \begin{cases} \frac{(3-5\lambda)^2 q_1 + q_0(\lambda(-49\lambda+8(\lambda-1)q_0+54)-9)}{16\lambda} & \text{if } q_0 < q_1 \\ \frac{q_0(\lambda(\lambda+8(\lambda-1)q_0-6)+9)-(3-5\lambda)^2 q_1}{16\lambda} & \text{if } q_0 > q_1 \end{cases} \tag{121}$$

$$\pi_1^M\left(q_0, q_1, p^M(q_0, q_1)\right) = \begin{cases} \frac{(q_1-q_0)(\lambda(q_1+q_0-5)+1)^2}{4\lambda^2} & \text{if } q_0 < q_1 \\ \frac{(q_0-q_1)(\lambda(q_1+q_0-1)-1)^2}{4\lambda^2} & \text{if } q_0 > q_1 \end{cases} \tag{122}$$

The mixed strategies price NE exists if and only if:

$$
\begin{aligned}
& q_0 \leq q_1 \text{ and } \overline{p}_1^H(q_0, q_1) < \hat{p}_1^1 < \overline{p}_1^L(q_0, q_1) \Rightarrow \\
& \frac{9\lambda - 2\lambda q_0 - 3}{2\lambda} < q_1 < \frac{7\lambda - 2\lambda q_0 - 1}{2\lambda} \\
& q_0 > q_1 \text{ and } \underline{p}_{-1}^H < \hat{p}_1^2(q_0, q_1) < \underline{p}_{-1}^L(q_0, q_1) \Rightarrow \\
& \frac{5\lambda - 2\lambda q_0 + 1}{2\lambda} < q_1 < \frac{3\lambda - 2\lambda q_0 + 3}{2\lambda}
\end{aligned}
\tag{123}
$$

I now show that by fixing q_j, the payoff function of firm i is continuous. Let us start with firm 1. Suppose $q_0 \leq q_1$. Then,

$$
\begin{aligned}
& \lim_{q_1 \to \frac{9\lambda - 2\lambda q_0 - 3}{2\lambda}} \left\{ \pi_1^H \left(q_0, q_1, p^H(q_0, q_1) \right) \right\} = \\
& \lim_{q_1 \to \frac{9\lambda - 2\lambda q_0 - 3}{2\lambda}} \left\{ \frac{1}{36}(q_1 - q_0)(q_1 + q_0 - 6)^2 \right\} = \\
& \lim_{q_1 \to \frac{9\lambda - 2\lambda q_0 - 3}{2\lambda}} \left\{ \pi_1^M \left(q_0, q_1, p^M(q_0, q_1) \right) \right\} = \\
& \lim_{q_1 \to \frac{9\lambda - 2\lambda q_0 - 3}{2\lambda}} \left\{ \frac{(q_1 - q_0)(\lambda(q_1 + q_0 - 5) + 1)^2}{4\lambda^2} \right\} = \\
& - \frac{(\lambda + 1)^2(\lambda(4q_0 - 9) + 3)}{32\lambda^3} .
\end{aligned}
\tag{124}
$$

$$
\begin{aligned}
& \lim_{q_1 \to \frac{7\lambda - 2\lambda q_0 - 1}{2\lambda}} \left\{ \pi_1^L \left(q_0, q_1, p^L(q_0, q_1) \right) \right\} = \\
& \lim_{q_1 \to \frac{7\lambda - 2\lambda q_0 - 1}{2\lambda}} \left\{ \frac{(q_1 - q_0)(\lambda(q_1 + q_0 - 8) + 2)^2}{36\lambda^2} \right\} = \\
& \lim_{q_1 \to \frac{7\lambda - 2\lambda q_0 - 1}{2\lambda}} \left\{ \pi_1^M \left(q_0, q_1, p^M(q_0, q_1) \right) \right\} = \\
& \lim_{q_1 \to \frac{7\lambda - 2\lambda q_0 - 1}{2\lambda}} \left\{ \frac{(q_1 - q_0)(\lambda(q_1 + q_0 - 5) + 1)^2}{4\lambda^2} \right\} = \\
& - \frac{(1 - 3\lambda)^2(\lambda(4q_0 - 7) + 1)}{32\lambda^3} .
\end{aligned}
\tag{125}
$$

An analogous statement holds for $q_1 < q_0$.

Let us consider firm 0 and $q_0 \leq q_1$. Let us also note that $q_0 = \frac{9\lambda - 2\lambda q_1 - 3}{2\lambda}$ and

$q_0 = \frac{7\lambda - 2\lambda q_1 - 1}{2\lambda}$ bound the existence of the mixed strategies price NE. Then,

$$\lim_{q_0 \to \frac{9\lambda - 2\lambda q_1 - 3}{2\lambda}} \left\{ V_0^M \left(q_0, q_1, p^M(q_0, q_1 \right) \right\} =$$
$$\lim_{q_0 \to \frac{9\lambda - 2\lambda q_1 - 3}{2\lambda}} \left\{ \frac{(3-5\lambda)^2 q_1 + q_0(\lambda(-49\lambda + 8(\lambda - 1)q_0 + 54) - 9)}{16\lambda} \right\} =$$
$$\lim_{q_0 \to \frac{9\lambda - 2\lambda q_1 - 3}{2\lambda}} \left\{ V_0^H \left(q_0, q_1, p^H(q_0, q_1 \right) \right\} =$$
$$\lim_{q_0 \to \frac{9\lambda - 2\lambda q_1 - 3}{2\lambda}} \Bigg\{$$
$$\frac{1}{36}\lambda \left(q_1^3 + q_1^2(q_0 + 12) - q_1 \left(q_0^2 + 18 \right) - q_0((q_0 - 6)q_0 + 36) \right) + 6q_1(3 - 2q_1) - 6(q_0 - 6)q_0 \Bigg\}$$
$$= \frac{\lambda \left(3\lambda(31 - 39\lambda) + 16(\lambda - 1)\lambda q_1^2 + 4(\lambda(\lambda + 6) - 3)q_1 + 9 \right) - 9}{32\lambda^2} \tag{126}$$

and

$$\lim_{q_0 \to \frac{7\lambda - 2\lambda q_1 - 1}{2\lambda}} \left\{ V_0^M \left(q_0, q_1, p^M(q_0, q_1 \right) \right\} =$$
$$\lim_{q_0 \to \frac{7\lambda - 2\lambda q_1 - 1}{2\lambda}} \left\{ \frac{(3-5\lambda)^2 q_1 + q_0(\lambda(-49\lambda + 8(\lambda - 1)q_0 + 54) - 9)}{16\lambda} \right\} =$$
$$\lim_{q_0 \to \frac{7\lambda - 2\lambda q_1 - 1}{2\lambda}} \left\{ V_0^L \left(q_0, q_1, p^L(q_0, q_1 \right) \right\} =$$
$$\lim_{q_0 \to \frac{7\lambda - 2\lambda q_1 - 1}{2\lambda}} \Bigg\{ \tag{127}$$
$$\frac{1}{36\lambda} \Bigg[\lambda^2 \left(\left(q_1^2 - 70 \right) q_0 - (q_1 - 10)q_0^2 + q_1(q_1 + 4)^2 - q_0^3 \right)$$
$$- 2\lambda(4q_1(q_1 + 4) + q_0(5q_0 - 43)) + 16(q_1 - q_0) \Bigg] \Bigg\}$$
$$\frac{\lambda \left(7\lambda(25 - 21\lambda) + 16(\lambda - 1)\lambda q_1^2 + 4(\lambda(9\lambda - 10) + 5)q_1 - 57 \right) + 5}{32\lambda^2}$$

Note again that an analogous condition holds for the case of $q_1 < q_0$. Hence, payoff functions are continuous, and I do not need to deal with corner solutions in this region. So I can focus on interior deviations.

Take firm 1. Suppose that firm 1 deviates to some q_1 such that $q_1 > q_0^*$

and $\left(q_1, q_0^*\right)$ are such that the mixed strategies price NE exists. Then, if it exists an optimal deviation inside this region, it must be characterized by the following equation.

$$\begin{aligned} &\frac{\partial \pi_1^M\left(q_0, q_1, p^M\left(q_0, q_1\right)\right)}{\partial q_1} = \frac{\partial}{\partial q_1}\left[\frac{(\lambda(3q_1 - q_0 - 5) + 1)(\lambda(q_1 + q_0 - 5) + 1)}{4\lambda^2}\right] \\ &\Rightarrow q_1 = \frac{5\lambda + \lambda q_0 - 1}{3\lambda}. \end{aligned} \tag{128}$$

Since $q_0 = q_0^*$, (128) simplifies to:

$$q_1 = \frac{117\lambda + 9\sqrt{\lambda(34 - 39\lambda) + 9} - 59}{48\lambda} = q_1^{d_{m1}}. \tag{129}$$

The pair $\left(q_1^{d_{m1}}, q_0^*\right)$ satisfies (123) if and only if $\frac{119}{141} \leq \lambda \leq \frac{1}{132}\left(3\sqrt{345} + 61\right)$. The payoff from deviation is:

$$\pi_1^M\left(q_0^*, q_1^{d_{m1}}, p^M(q_0^*, q_1^{d_{m1}})\right) = \frac{\left(3\lambda - 9\sqrt{\lambda(34 - 39\lambda) + 9} + 35\right)^3}{13824\lambda^3}, \tag{130}$$

which is lower than (105) for any $\frac{119}{141} \leq \lambda \leq \frac{1}{132}\left(3\sqrt{345} + 61\right)$. Suppose that firm 1 deviates to some q_1 such that $q_1 < q_0^*$ and $\left(q_1, q_0^*\right)$ are such that the mixed strategies price NE exists. Then,

$$\begin{aligned} &\frac{\partial \pi_1^M\left(q_0, q_1, p^M\left(q_0, q_1\right)\right)}{\partial q_1} = \frac{\partial}{\partial q_1}\left[\frac{(q_0 - q_1)(\lambda(q_1 + q_0 - 1) - 1)^2}{4\lambda^2}\right] \\ &\Rightarrow q_1 = \frac{\lambda + \lambda q_0 + 1}{3\lambda}. \end{aligned} \tag{131}$$

However, the intersection of q_0^* and $q_1 = \frac{\lambda + \lambda q_0 + 1}{3\lambda}$ does not satisfy (123).[41]

Let us now consider firm 0. Suppose that firm 0 deviates to some q_0 such that $q_0 < q_1^*$ and (q^*) are such that the mixed strategies price NE exists. Then, if it exists an optimal interior deviation, it is:

$$q_0^{d_{m1}} = \frac{49\lambda^2 - 54\lambda + 9}{16\lambda^2 - 16\lambda}. \tag{132}$$

[41] In particular, by decreasing or increasing q_1, firm 1 can never reach the region of existence of the mixed strategies subgame price NE where $q_1 < q_0$. See Figure 5.

I obtain (132) by taking the FOC of (121). The first deviation (132) satisfies (123) if and only if $0.859 \leq \lambda \leq 0.870$, but it is not profitable. Suppose that firm 0 deviates to some q_0 such that $q_0 > q_1^*$ and the mixed strategies price NE exists. In this case, the FOC of (121) and condition (123) are never jointly satisfied given $q_1 = q_1^*$. Therefore, neither of the two firms is attracted by the region of existence of subgame NE prices in mixed strategies.

Since the pair q_0^*, q_1^* is not subject to profitable unilateral deviations and firms get positive NE payoffs, the following profile of strategies:

$$\begin{aligned}
&\textit{Stage } (1)\text{: } a_i^* = 1 \text{ for } i \in \{0,1\} \\
&\textit{Stage } (2)\text{: } q_0^*, q_1^* \\
&\textit{Stage } (3)\text{: } \underline{p}_0^L\left(q_0^*, q_1^*\right), \overline{p}_1^L\left(q_0^*, q_1^*\right) \\
&\textit{Stage } (4)\text{: All consumers with } \theta \leq \hat{\theta}(\cdot) \text{ buy good 0 and} \\
&\text{all consumers with } \theta > \hat{\theta}(\cdot) \text{ buy good 1}
\end{aligned} \tag{133}$$

is a SPNE for any $\lambda \in \left\{\left[\frac{1}{2}, \lambda_{m_1}\right] \cup \left[\lambda_{m_2}, \lambda_{p_2}\right], 1\right\}$. Note that an analogous proof applies to the following profile of strategies.

$$\begin{aligned}
&\textit{Stage } (1)\text{: } a_i^* = 1 \text{ for } i \in \{0,1\} \\
&\textit{Stage } (2)\text{: } q_0^{L_{**}}, q_1^{L_{**}} \\
&\textit{Stage } (3)\text{: } \overline{p}_0^L\left(q_0^{L_{**}}, q_1^*\right), \underline{p}_1^L\left(q_0^{L_{**}}, q_1^{L_{**}*}\right) \\
&\textit{Stage } (4)\text{: All consumers with } \theta \geq \hat{\theta}(\cdot) \text{ buy good 0 and} \\
&\text{all consumers with } \theta < \hat{\theta}(\cdot) \text{ buy good 1}
\end{aligned} \tag{134}$$

A.3.4.2 Mixed Strategies SPNE where Firm 1 Randomizes and the Median Voter Buys good 0

In this Section, I show that if $\lambda \in (\lambda_{m_1}, \lambda_{m_2})$, there exist a mixed strategy NE where firm 0 plays $q_0 = \frac{3}{2}$ and firm 1 mixes between its two best responses.

Suppose $q_0 = \frac{3}{2}$. Firm 1's best responses are:

$$\underline{q}_1^L\left(\frac{3}{2}\right) = -\frac{\lambda - 4}{6\lambda} \tag{135}$$

and

$$\overline{q}_1^L\left(\frac{3}{2}\right) = \frac{19}{6} - \frac{2}{3\lambda} \tag{136}$$

Given $q_0 = \frac{3}{2}$, the strategies (135) and (136) are such that the price NE p^L exists. These strategies yield firm 1 the following payoff:

$$\frac{(5\lambda - 2)^3}{243\lambda^3}\,. \tag{137}$$

It is relatively easy to show that, given $q_0 = \frac{3}{2}$, firm 1 never wants to deviate to some q_1 such that another price NE exists.

Therefore, let us suppose that firm 1 plays (136) with probability $k_1 \in (0,1)$ and (135) with probability $1 - k_1$. Let us denote with ω_1 this mixed strategy. For any k_1, ω_1 is a best response of firm 1 to $q_0 = \frac{3}{2}$.

For this to be a NE, it must be that firm 0's best response against ω_1 is $q_0 = \frac{3}{2}$. Let us define $S(\omega_1)$ as the support of ω_1. Firm 0's expected payoff given some q_0 such that $p^L(q_0, q_1)$ exists is:

$$\mathbb{E}[V_0(\omega_1, q_0)] =$$

$$\begin{cases} k_1\underline{V}_0^L\left(q_0, \underline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\underline{q}_1^L\left(\frac{3}{2}\right)\right)\right) + (1-k_1)\overline{V}_0^L\left(q_0, \overline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\overline{q}_1^L\left(\frac{3}{2}\right)\right)\right) & \text{if } \underline{q}_1^L\left(\frac{3}{2}\right) \le q_0 \le \overline{q}_1^L\left(\frac{3}{2}\right) \\ k_1\underline{V}_0^L\left(q_0, \underline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\underline{q}_1^L\left(\frac{3}{2}\right)\right)\right) + (1-k_1)\underline{V}_0^L\left(q_0, \overline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\overline{q}_1^L\left(\frac{3}{2}\right)\right)\right) & \text{if } q_0 < \underline{q}_1^L\left(\frac{3}{2}\right) \le \overline{q}_1^L\left(\frac{3}{2}\right) \\ k_1\overline{V}_0^L\left(q_0, \underline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\underline{q}_1^L\left(\frac{3}{2}\right)\right)\right) + (1-k_1)\overline{V}_0^L\left(q_0, \overline{q}_1^L\left(\frac{3}{2}\right), p^L\left(\overline{q}_1^L\left(\frac{3}{2}\right)\right)\right) & \text{if } q_0 > \overline{q}_1^L\left(\frac{3}{2}\right) > \underline{q}_1^L\left(\frac{3}{2}\right) \end{cases} \tag{138}$$

which, if $\underline{q}_1^L\left(\frac{3}{2}\right) < q_0 < \overline{q}_1^L\left(\frac{3}{2}\right)$, reduces to:

$$\frac{1}{7776\lambda^2}\Bigg[\lambda\Bigg(-6k_1(2q_0-3)(\lambda(\lambda(36(q_0-3)q_0+1745)-1936)+560)+$$
$$\lambda^2(6q_0(6q_0(6q_0-13)+1655)+3721)-12\lambda(8q_0(21q_0+58)+1525)+336(10q_0+43)\Bigg)-3136\Bigg] \tag{139}$$

Note that if $\lambda_{m_1} \le \lambda \le \lambda_{m_2}$ and $k_1 = \frac{1}{2}$, (139) is maximized at $q_0 = \frac{3}{2}$. Moreover, any q_0 such that

$$\begin{aligned} &q_0 \le \underline{q}_1^L\left(\frac{3}{2}\right) < \overline{q}_1^L\left(\frac{3}{2}\right) \text{ or} \\ &\text{if } q_0 \ge \overline{q}_1^L\left(\frac{3}{2}\right) > \underline{q}_1^L\left(\frac{3}{2}\right) \end{aligned} \tag{140}$$

it is strictly dominated. Firm 0's expected payoff is then:

$$\begin{aligned} &\underline{V}_0^L\left(q_0, \overline{q}_1^L\left(\frac{3}{2}\right), p^L\left(q_0, \overline{q}_1^L\left(\frac{3}{2}\right)\right)\right) = \\ &\overline{V}_0^L\left(q_0, \underline{q}_1^L\left(\frac{3}{2}\right), p^L\left(q_0, \underline{q}_1^L\left(\frac{3}{2}\right)\right)\right) = \\ &\frac{\lambda(\lambda(4573\lambda-7797)+4872)-784}{1944\lambda^2} \end{aligned} \tag{141}$$

It is relatively easy to check that given ω_1, any q_0 "inside" the p^H or p^M regions is strictly dominated. In particular, one can check that given any strategy in the support of ω_1, any q_0 inside the p^M or p^H regions is strictly dominated by $q_0 = \frac{3}{2}$.

Finally, note that expected payoffs are positive so that firms have an incentive to enter the market in *Stage* (1). Thus, let us denote with ω_1^* the SPNE mixed strategy of firm 1. I can state the following. If $\lambda \in (\lambda_{m_1}, \lambda_{m_2})$, the following strategies profile is a SPNE:

$$\begin{aligned}
&\textit{Stage } (1)\text{: } a_i^* = 1 \text{ for } i \in \{0,1\} \\
&\textit{Stage } (2)\text{: } \frac{3}{2}, \omega_1^* \\
&\textit{Stage } (3)\text{: } p^L\left(\frac{3}{2}, \omega_1^*\right) \\
&\textit{Stage } (4)\text{: If } q_1 \geq \frac{3}{2}\text{also be SPNEers with } \theta \leq \hat{\theta}(\cdot) \\
&\text{buy good 0 and all consumers with } \theta > \hat{\theta}(\cdot) \text{ buy good 1.} \\
&\text{If } q_1 < \frac{3}{2}\text{, all consumers with } \theta \geq \hat{\theta}(\cdot) \text{ buy good 0} \\
&\text{and all consumers with } \theta < \hat{\theta}(\cdot) \text{ buy good 1.}
\end{aligned} \tag{142}$$

Note that (142) may be SPNE also for other values of λ. In Figure 6, I show the strategies in the support of ω_1^* when $\lambda = \lambda_{m_1}$. Let E_0 denote the (collection of) SPNE where the median voter buys good 0. Then, E_0 exists when $\lambda \in \left[\frac{1}{2}, \lambda_{p_2}\right]$.

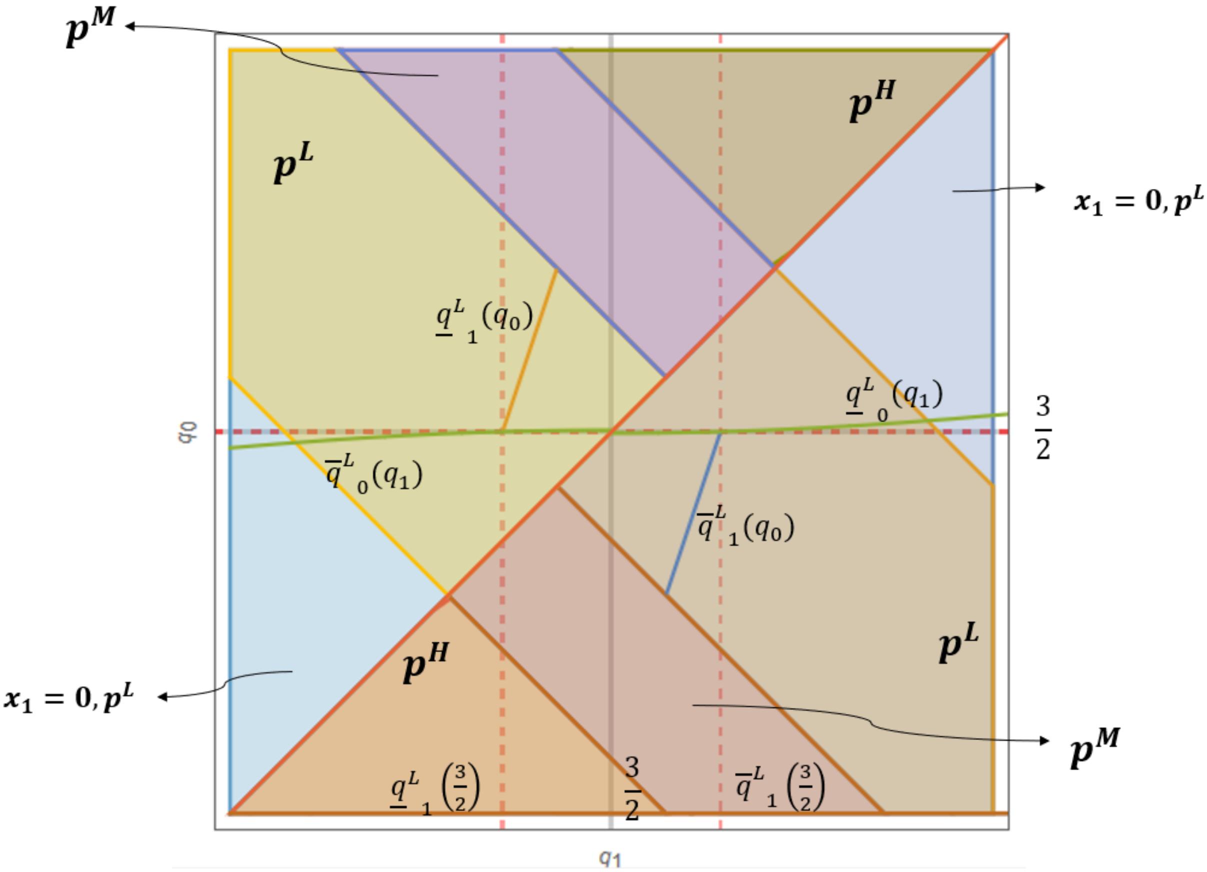

Figure 6: **Mixed Strategy SPNE** when $\lambda = \lambda_{m_1}$. In the p^L region, I only plot the strategy yielding higher payoffs.

A.3.4.3 Pure Strategies SPNE where the Median Voter Buys good 1 Suppose we are in the NE p^H. The intersection of optimal qualities (112) and (113) are:

$$q_1 = \frac{3\left(13\lambda + \sqrt{\lambda(18-23\lambda)+9} - 3\right)}{16\lambda} > q_0 = \frac{3\left(7\lambda + 3\sqrt{\lambda(18-23\lambda)+9} - 9\right)}{16\lambda}; \tag{143}$$

and

$$\begin{aligned} &q_0 = \\ &\frac{1}{16\lambda}\Bigg[29\lambda + \sqrt{\lambda(18-23\lambda)+9} \\ &- 2\sqrt{2}\sqrt{5\sqrt{\lambda(18-23\lambda)+9} + \lambda\left(-287\lambda + 5\sqrt{\lambda(18-23\lambda)+9} + 226\right) + 113} + 29\Bigg] \\ &> q_1 = \frac{9\lambda - 3\sqrt{\lambda(18-23\lambda)+9} + 9}{16\lambda}, \end{aligned} \tag{144}$$

However, given (143) and (144), the conditions such that the price NE $p^H(q_0, q_1)$ exists are not satisfied. In particular, the median voter would buy from firm 0. Hence, I look for corner solutions. Consider the following quality pair.

$$\begin{aligned} &\begin{cases} \overline{p}_1^H(q_0, q_1) = \hat{p}_1^1(q_0, q_1) \\ q_1 = \overline{q}_1^H(q_0) \end{cases} \Rightarrow \\ &q_1 = \frac{3}{8}\left(7 - \frac{1}{\lambda}\right) = q_1^{***} \\ &q_0 = \frac{3}{8}\left(5 - \frac{3}{\lambda}\right) = q_0^{***} \end{aligned} \tag{145}$$

Payoffs along (145) are:

$$\begin{aligned} V_0^H\left(q_0^{***}, q_1^{***}, p^H\left(q_0^{***}, q_1^{***}\right)\right) &= \frac{3(\lambda+1)(\lambda+3)(5\lambda-3)}{128\lambda^2} \\ \pi_1^H\left(q_0^{***}, q_1^{***}, p^H\left(q_0^{***}, q_1^{***}\right)\right) &= \frac{3(\lambda+1)^3}{64\lambda^3}. \end{aligned} \tag{146}$$

First, let us note that the strategy $\underline{q}_0^H\left(q_1^H\right)$ is not feasible for firm 0. Moreover, it can be shown that none of the two firms has a profitable deviation inside the $p^H(q_0, q_1)$ region. Then, to check that (145) is a SPNE, I need to check for deviations toward qualities such that subgame prices are either $p^L(q_0, q_1)$ or $p^M(q_0, q_1)$. Firm 0 can deviate to $\underline{q}_0^L(q_1^{***})$, gaining a payoff of

$$\begin{aligned}
&\frac{1}{31104\lambda^2}\Bigg[721\sqrt{\lambda(1762-2159\lambda)+721}+\\
&\lambda\Bigg(1762\sqrt{\lambda(1762-2159\lambda)+721}+\\
&\lambda\left(-8828\lambda-2159\sqrt{\lambda(1762-2159\lambda)+721}+107214\right)-73248\Bigg)-19306\Bigg],
\end{aligned} \tag{147}$$

which is higher than the public firm's payoff in (146) for $\lambda < 0.93 = \lambda_{p_3}$. If $\lambda \geq 0.93$, firm 0 has no profitable unilateral deviation given q_1^{***}. Analogously, it can be shown that if $\lambda \geq 0.93$, it exists a NE:

$$\begin{aligned}
&\begin{cases} \underline{p}_1^H(q_0, q_1) = \hat{p}_1^2(q_0, q_1) \\ q_1 = \underline{q}_1^H(q_0) \end{cases} \Rightarrow \\
&q_1 = \frac{3(\lambda+1)}{8\lambda} = q_1^{****} \\
&q_0 = \frac{9(\lambda+1)}{8\lambda} = q_0^{****}.
\end{aligned} \tag{148}$$

Payoffs are still (146). If $\lambda < 0.93$, in fact, firm 0 can deviate to $\overline{q}_0^L(q_1^{****})$, gaining again (147). There are two other SPNE that are obtained as corner solutions.

$$\begin{aligned}
&\begin{cases} \overline{p}_1^H(q_0, q_1) = \hat{p}_1^1(q_0, q_1) \\ q_0 = \underline{q}_0^H(q_1) \end{cases} \Rightarrow \\
&q_1 = -\frac{6}{\lambda+1} - \frac{15}{8\lambda} + \frac{57}{8} = q_1^{*****} \\
&q_0 = \frac{3}{8}\left(\frac{16}{\lambda+1} + \frac{1}{\lambda} - 7\right) = q_0^{*****}.
\end{aligned} \tag{149}$$

The NE (149) exists if and only if $\lambda \geq 0.847 = \lambda_{p_1}$ because otherwise firm 1 has an incentive to deviate toward $\overline{q}_1^L(q_0^{*****})$. In the same way, if $\lambda \geq 0.847$, it exists the NE:

$$\begin{cases} \underline{p}_1^H(q_0, q_1) = \hat{p}_1^2(q_0, q_1) \\ q_0 = \overline{q}_0^H(q_1) \end{cases} \Rightarrow$$
$$q_1 = \frac{6}{\lambda+1} + \frac{15}{8\lambda} - \frac{33}{8} = q_1^{******}$$
$$q_0 = -\frac{6}{\lambda+1} - \frac{3}{8\lambda} + \frac{45}{8} = q_0^{******}. \tag{150}$$

Payoffs along (149) and (150) are symmetric and are given by:

$$V_0^H\left(q_0^{*****}, q_1^{*****}, p^H\left(q_0^{*****}, q_1^{*****}\right)\right) =$$
$$V_0^H\left(q_0^{******}, q_1^{******}, p^H\left(q_0^{******}, q_1^{******}\right)\right) =$$
$$\frac{3(\lambda(\lambda(\lambda(\lambda(965\lambda-1237)-150)+558)-15)-57)}{128\lambda^2(\lambda+1)^2};$$
$$\pi_1^H\left(q_0^{*****}, q_1^{*****}, p^H\left(q_0^{*****}, q_1^{*****}\right)\right) =$$
$$\pi_1^H\left(q_0^{******}, q_1^{******}, p^H\left(q_0^{******}, q_1^{******}\right)\right) =$$
$$\frac{3(\lambda+1)(\lambda(13\lambda-6)-3)}{64\lambda^3}. \tag{151}$$

Therefore, it is possible to state the following results. If $\lambda \geq \lambda_{p_3}$, the following profile of strategies are SPNE.

Stage (1): $a_i^* = 1$ for $i \in \{0, 1\}$
Stage (2): q_0^{***}, q_1^{***}
Stage (3): $p^H\left(q_0^{***}, q_1^{***}\right)$ (152)
Stage (4): All consumers with $\theta \leq \hat{\theta}(\cdot)$
buy good 0 and all consumers with $\theta > \hat{\theta}(\cdot)$ buy good 1.

Stage (1): $a_i^* = 1$ for $i \in \{0, 1\}$
Stage (2): q_0^{****}, q_1^{****}
Stage (3): $p^H\left(q_0^{****}, q_1^{****}\right)$ (153)
Stage (4): All consumers with $\theta \geq \hat{\theta}(\cdot)$
buy good 0 and all consumers with $\theta < \hat{\theta}(\cdot)$ buy good 1.

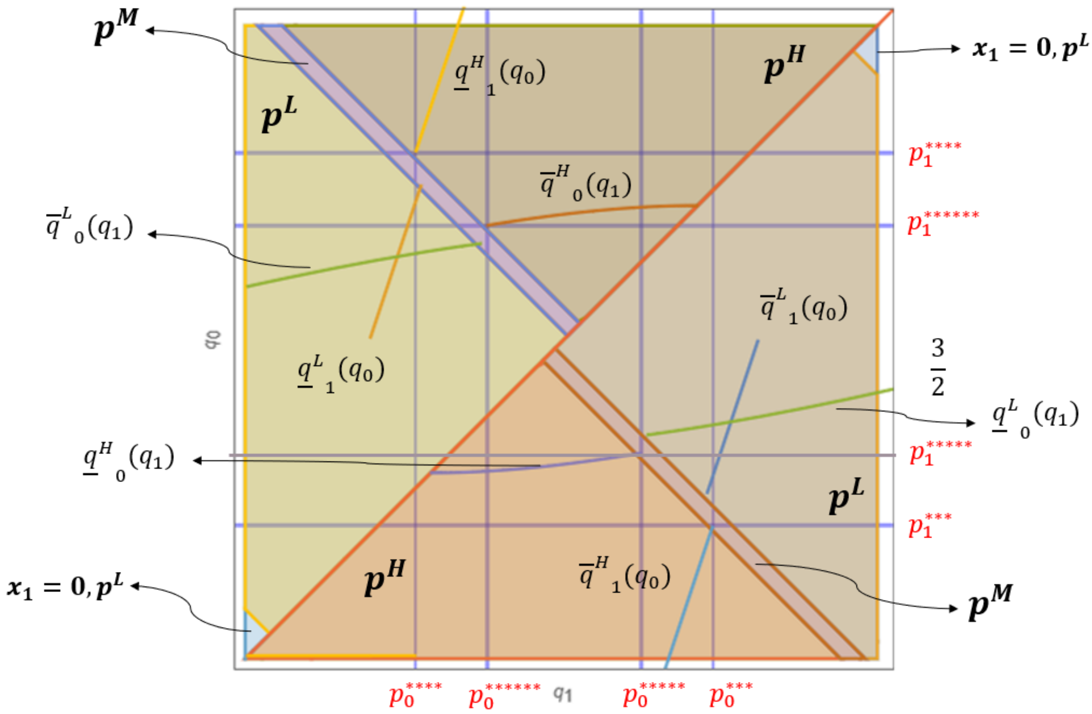

Figure 7: **"Corner" SPNE** when $\lambda \geq \lambda_{m_2}$. In the p^L region, I only plot the strategy yielding higher payoffs.

If $\lambda \geq \lambda_{p_1}$, the following profile of strategies are SPNE.

$$\begin{aligned}
&\textit{Stage (1)}: a_i^* = 1 \text{ for } i \in \{0,1\} \\
&\textit{Stage (2)}: q_0^{*****}, q_1^{*****} \\
&\textit{Stage (3)}: p^H\left(q_0^{*****}, q_1^{*****}\right) \qquad (154) \\
&\textit{Stage (4)}: \text{All consumers with } \theta \leq \hat{\theta}(\cdot) \\
&\text{buy good 0 and all consumers with } \theta > \hat{\theta}(\cdot) \text{ buy good 1.}
\end{aligned}$$

$$\begin{aligned}
&\textit{Stage (1)}: a_i^* = 1 \text{ for } i \in \{0,1\} \\
&\textit{Stage (2)}: q_0^{******}, q_1^{******} \\
&\textit{Stage (3)}: p^H\left(q_0^{******}, q_1^{******}\right) \qquad (155) \\
&\textit{Stage (4)}: \text{All consumers with } \theta \geq \hat{\theta}(\cdot) \\
&\text{buy good 0 and all consumers with } \theta < \hat{\theta}(\cdot) \text{ buy good 1.}
\end{aligned}$$

I refer to E_1 (respectively, $E_1(2)$) to the SPNE where median voter buys good 1, that is, (152), and (153) (respectively, (154), and (155)). In Table 1, I recap the duopoly SPNE and clarify the notation.

The proof that pairs of SPNE are payoffs-equivalent is straightforward. It is sufficient to substitute SPNE qualities and prices inside payoff functions. Welfare-equivalence holds as a direct consequence of payoffs-equivalence.

Table 1: Characterization of Duopoly SPNE for $\lambda \geq \frac{1}{2}$.

/	Median Voter Buys?	Existence	Strategies	Qualities Ordering
(133)	$0 \rightarrow E_0$	$\lambda \in \left[\frac{1}{2}, \lambda_{m_1}\right] \bigcup \left[\lambda_{m_2}, \lambda_{p_2}\right]$	Pure	$q_0 < q_1$
(134)	$0 \rightarrow E_0$	$\lambda \in \left[\frac{1}{2}, \lambda_{m_1}\right] \bigcup \left[\lambda_{m_2}, \lambda_{p_2}\right]$	Pure	$q_0 > q_1$
(142)	$0 \rightarrow E_0$	$\lambda \in (\lambda_{m_1}, \lambda_{m_2})$	Mixed	$\mathbb{E}[q_1] = q_0$
(152)	$1 \rightarrow E_1$	$\lambda \geq \lambda_{p_3}$	Pure	$q_0 < q_1$
(153)	$1 \rightarrow E_1$	$\lambda \geq \lambda_{p_3}$	Pure	$q_0 > q_1$
(154)	$1 \rightarrow E_1(2)$	$\lambda \geq \lambda_{p_1}$	Pure	$q_0 < q_1$
(155)	$1 \rightarrow E_1(2)$	$\lambda \geq \lambda_{p_1}$	Pure	$q_0 > q_1$

A.4 SPNE Discussion

To illustrate the different possible SPNE, it is convenient to define the following thresholds:[42]

$$\frac{1}{2} < \lambda_{m_1} < \lambda_{m_2} < \lambda_{p_1} < \lambda_{p_2} < \lambda_{p_3} < 1\,. \tag{156}$$

When $\lambda \leq \lambda_{p_2}$, Proposition 2 identifies two pairs of (payoffs-equivalent) duopoly SPNE qualities such that the median voter buys good 0. Let these SPNE qualities be $q_0^* < q_1^*$ and $q_1^{**} > q_0^{**}$. When λ is low, firm 0 produces an "intermediate" quality, neither too high nor too low. Combined with a low price in *Stage* (3), this strategy appeals to most consumers. To survive in the market, firm 1 produces a very high (or very low) quality, serving consumers with very high (or very low) WTP. When $\lambda > \lambda_{p_2}$, firm 1 wants to deviate to some q_1 inside the p^H region, and these pairs stop to be SPNE. Along the deviation path to the p^H region, firm 1 faces a trade-off between a higher markup and higher product differentiation, two key drivers of profits. On the one hand, the region p^H would always tempt firm 1 because of the high markups. However, deviation towards that region would imply lower product differentiation, thereby decreasing profits.[43] The degree of product differentiation along this possibly profitable deviation increases in λ. Therefore, the deviation becomes profitable for firm 1 only when λ is high enough. Firm 0 is attracted by the p^H region only if λ is high enough for the increase in public profits to compensate for the loss in the median voter's payoff.

In the region $\lambda \leq \lambda_{p_2}$, there exists an interval $(\lambda_{m_1}, \lambda_{m_2})$ where the SPNE is in mixed strategies. The lack of pure strategies SPNE is due to the discontinuity of best response functions. Inside the p^L region, each firm i wants to produce a higher quality than q_j if and only if q_j is below a certain threshold $\hat{q}$ (Motta

[42]See Table A.3.4 for the expression of these thresholds.

[43]See Figure 6.

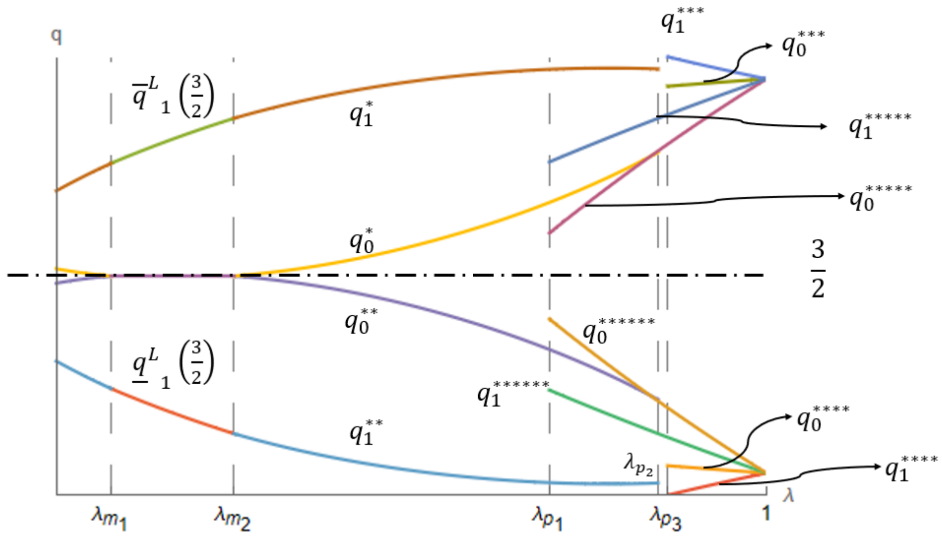

Figure 8: **SPNE Qualities for** $\lambda \geq \frac{1}{2}$. See Table 1 for the classification of SPNE, and Table A.3.4 for the expressions of the different thresholds for λ. The plot is obtained fixing some θ_h, α, which imply $\hat{q} = \frac{3}{2}$.

[1993]). If $\lambda = \lambda_{m_1}$, then $q_0^* = \hat{q}$ and $q_0^{**} = \hat{q}$. So, a marginal increase in λ makes firm 1 want to leapfrog the quality of its opponent. Take, for example, the pair (q_0^*, q_1^*). As soon as λ "hits" the threshold λ_{m_1}, then $q_0^* > \hat{q}$ and firm 1's best response "jumps". This deviation is profitable as long as $q_0^* > \hat{q} \Leftrightarrow \lambda < \lambda_{m_2}$. [44] In the region $(\lambda_{m_1}, \lambda_{m_2})$, I show that there exists a SPNE where firm 0 plays $\hat{q}$ and firm 1 randomizes between its two best responses. Note that the strategies in the mixed strategy SPNE approach those of the pure strategy SPNE when $\lambda \to \lambda_{m_1}$ and $\lambda \to \lambda_{m_2}$ (see Figure 8 in Appendix (iv)).

When λ is high enough ($\lambda \geq \lambda_{p_1}$), there exists at least two SPNE where the median voter buys good 1. In particular, there are two pairs of SPNE qualities if $\lambda \in [\lambda_{p_1}, 1]$, ($q_0^{****}, q_1^{****}$ and $q_0^{******}, q_1^{******}$), and two more ($q_0^{***}, q_1^{***}$ and q_0^{*****}, q_1^{*****}) if $\lambda \in [\lambda_{p_3}, 1]$. To obtain these equilibria, I intersect the optimal quality of firm $i \in \{0, 1\}$ with the highest q_j such that the p^H price equilibrium exists. Therefore, I refer to these as corner SPNE.[45] In Figure 8 A.8, I show equilibrium qualities and their relationship with the different thresholds. See Table 1 for a classification of the different SPNE.

[44] An analogous reasoning applies for the pair (q_0^{**}, q_1^{**})

[45] The intersection of FOCs inside the p^H region is not feasible.

A.5 Proof of Proposition 3

The proof is straightforward and obtained by substituting SPNE qualities and prices in the profits and welfare functions.

A.6 Proof of Proposition 4

Without loss of generality, I adopt the normalization I adopted in the previous Section: $\theta_h = 2$, $\alpha = \frac{1}{2}$.

In *Stage* (4), the game runs as in the previous Section.

In *Stage* (3), firm 0 commits to $p_0 = \frac{1}{2}q_0^2$. When $q_0 < q_1$, the profit function of firm 1 is maximized if

$$p_1 = \frac{1}{4}(q_1(q_1 + 4) + (q_0 - 4)q_0) = \overline{p}_1^{c_*}(q_0, q_1) \ . \tag{157}$$

If $q_1 < q_0$, the optimal price of firm 1 is:

$$p_1 = \frac{1}{4}(q_1(q_1 + 2) + (q_0 - 2)q_0) = \underline{p}_1^{c_*}(q_0, q_1) \ . \tag{158}$$

Let us now consider *Stage* (2). If firm 0 serves the median voter, its payoff is:

$$(1 - \lambda)\left(q_0 - \frac{1}{2}q_0^2\right) \ . \tag{159}$$

(159) is maximized at $q_0 = \frac{3}{2}$. If evaluated at $q_0 = \frac{3}{2}$, (159) reduces to:

$$\frac{1}{8}(-9)(\lambda - 1) \ . \tag{160}$$

If firm 0 does not serve the median voter, its payoff is:

$$\frac{1}{4}(\lambda - 1)((q_1 - 2)q_1 + (q_0 - 4)q_0) \ , \tag{161}$$

which is maximized at $q_0 = 2$, yielding a payoff of:

$$\frac{1}{4}(\lambda - 1)((q_1 - 2)q_1 - 4) \ . \tag{162}$$

The profit of firm 1 is:

$$\frac{1}{16}(q_1 - q_0)(q_1 + q_0 - 4)^2 \ , \tag{163}$$

which is maximized at:

$$\frac{q_0 + 4}{3} = \overline{q}_1^{c_*}(q_0) \ . \tag{164}$$

The intersection of $q_0 = \frac{3}{2}$ and (164) is

$$\begin{aligned} \frac{11}{6} &= q_1^{c_*} \\ \frac{3}{2} &= q_0^{c_*} . \end{aligned} \tag{165}$$

(165) are SPNE qualities because (160) is higher than (162) for $q_1 = \frac{11}{6}$, and because neither of the two firms has an incentive leapfrog the quality of its opponent (since an analogous version of (109) and (108) hold). Analogously, it can be shown that there exists a SPNE where $q_0 = \frac{3}{2} = q_0^{c_{**}}$ and $q_1 = \frac{9}{6} = q_1^{c_{**}}$.

The welfare comparison is straightforward.

A.7 Proof of Proposition 5

I start by briefly summarizing this "modified" model setup. Without loss of generality, suppose $q_i > q_j$. There are now two indifferent consumers: $\hat{\theta}_1 = \frac{p_j}{q_j}$; and $\hat{\theta}_2 = \frac{p_i - p_j}{q_i - q_j}$. Hence, under duopoly, the demand for firm j is: $\hat{\theta}_2 - \hat{\theta}_1$. The demand for firm i is: $\theta_h - \hat{\theta}_2$. Without loss of generality, let $\theta_h = 1$ and $\alpha = \frac{1}{2}$.[46] If consumers do not buy any good, they get a zero payoff.

Let us consider the price stage. As in Appendix A.2.2, the existence of a NE is guaranteed by the results in Reny [1999]. Firm 0's payoff is:

$$V_0(\pi_0, q_1, q_0, p_1, p_0) = \lambda\pi_0 + (1-\lambda)\max\{0, \overline{\theta}q_1 - p_1, \overline{\theta}q_0 - p_0\}\,. \tag{166}$$

The FOC of (166) w.r.t. p_0 is non-decreasing in λ. Then, the equilibrium price of firm 0 is non-decreasing in λ. By strategic complementarity of prices, $\hat{\theta}_1 = \frac{p_j}{q_j}$ is also non-decreasing in λ.

[46]See Appendix A.3.4.

A.8 Additional Plots

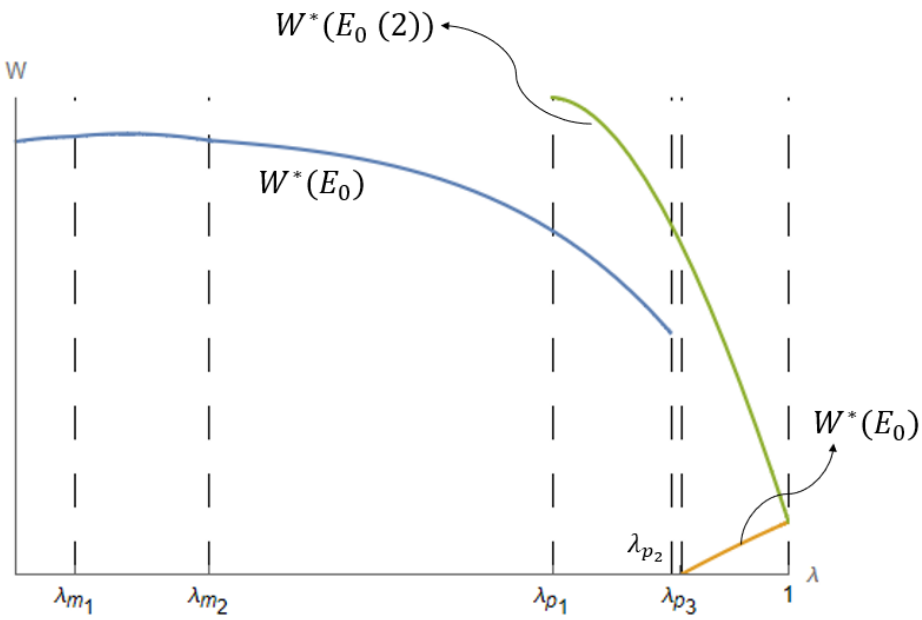

Figure 9: **SPNE Welfare for** $\lambda \geq \frac{1}{2}$. See Table 1 for the classification of SPNE.

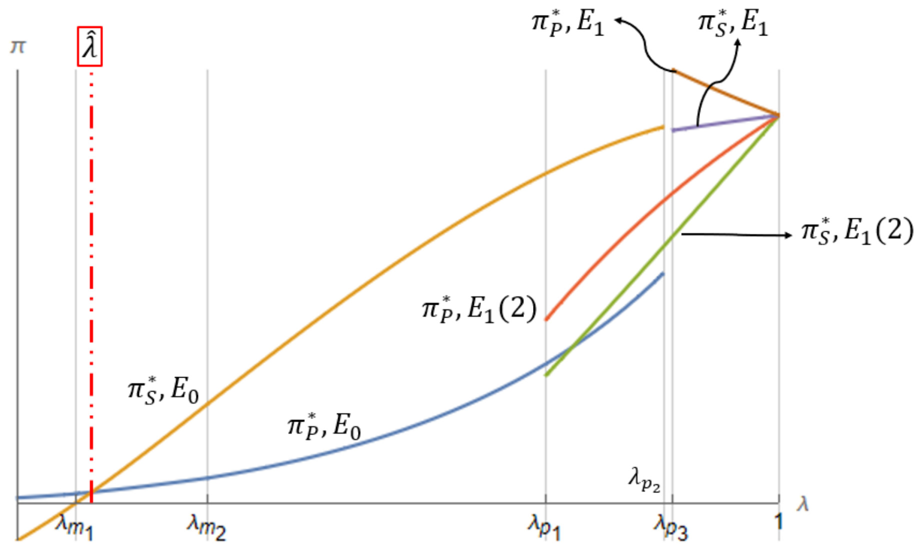

Figure 10: **SPNE Profits for** $\lambda \geq \frac{1}{2}$. See Table 1 for the classification of SPNE.