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Capital Risk, Fiscal Policy, and the Distribution of Wealth*

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Abstract

We develop a continuous-time model of a production economy where households face leverage constraints, uninsurable labour income shocks, and capital depreciation risk. We derive a numerical approximation of the model's competitive equilibrium and compare it with a benchmark economy with no capital risk. Introducing capital risk generates a positive risk premium while fostering aggregate capital accumulation and safe asset demand. At the same time, it exacerbates wealth inequality by making poor households' net worth more volatile than their wealthier peers. In this framework, we investigate the impact of fiscal policy on households' wealth distribution and welfare. Fiscal policy influences the equilibrium wealth distribution by changing the risk premium. This channel unevenly impacts households' consumption and asset allocation decisions, depending on their wage and net worth levels. Tax cuts on risky capital may benefit wealthy or poor households, depending on whether they are financed by raising taxes on safe assets or labour.

JEL classification: C61; E21; E62; G11.

Keywords: Fiscal policy; incomplete market; portfolio choices; wealth distribution

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1 Introduction

A well-known result in macroeconomic theory is that, in a representative-agent model with complete markets and unproductive public expenditure, no agents would choose redistributive capital income taxation, independently of their initial or long-run net worth levels (Judd, 1985; Chamley, 1986; Jones et al., 1993). More recent studies (e.g., Domeij and Heathcote, 2004; Heathcote, 2005; Conesa et al., 2009; Boar and Midrigan, 2022), however, highlight that in the presence of uninsurable idiosyncratic risk capital taxes may be welfare improving and may have very different effects in the short and the long run.

A common assumption of these papers is that investing in firm capital carries no idiosyncratic risk, neglecting the steady decline in small- and medium-sized firms going public over the last 20 years (Gao et al., 2013), and that entrepreneurial equity represents a very concentrated risk for a sizable fraction of households who invest more than two-third of her holdings in a single (private) company (Moskowitz and Vissing-Joergensen, 2002). Consequently, they overlook a crucial effect of capital income taxes, which impact households' net worth distribution by affecting the expected returns on their assets *jointly with* their risk-bearing capacity.[1] With this in mind, this paper advances previous literature by investigating the role of fiscal policies in redistributing net worth and risk in a heterogeneous-agent economy with capital and income risks, incomplete financial markets, and unproductive public debt.

Based on the seminal work of Achdou et al. (2022), we work in continuous time. The model features a representative firm, a unit mass of ex-ante identical households, and the government. The firm uses labour and capital to produce output. Households contribute to both factors of production, providing labour inelastically and allocating their net worth between capital, which depreciates stochastically, and riskless government bonds. As in Bewley (1986) and Aiyagari (1994), households make their choices subject to borrowing constraints and face uninsurable labour-income shocks. The government, as in Aiyagari and McGrattan (1998), collects taxes on financial assets and labour income and issues bonds to finance unproductive public expenditure.

[1] A recent literature highlights that there is a fundamental relationship between public-debt, idiosyncratic capital risk, and taxation (Gersbach et al., 2023) and between limited idiosyncratic risk pooling and the business cycle (Dindo et al., 2022).

The first part of the paper solves the model for its competitive equilibrium and investigates its long-run (stationary) dynamics numerically. In particular, we explore the effect of introducing uninsurable capital risk by using a Domeij and Heathcote (2004) economy as a benchmark.

We find that introducing capital risk in the economy increases wealth inequality, fattening the tails of wealth distribution and increasing the share of financially-constrained households. Coherently, consumption-to-income ratios decrease because of stronger precautionary motifs.

Households' asset allocation decisions differ based on their net worth (and wage) levels. Poor households (i.e., with a higher marginal propensity to consume) invest in riskier assets to seek higher expected returns. Conversely, wealthier households behave in a "financially sophisticated" manner, tilting their asset allocation towards risk-free bonds for hedging purposes. This choice contributes to increase inequality by mitigating wealthy households' worth volatility, allowing them to accumulate larger endowments with a higher probability in the long run.

The second part of the paper investigates the impact of different tax policies. We find that lowering risky capital taxes fosters capital accumulation in the aggregate and increases gross wages. The policy reduces wealth inequality as well by inducing more conservative consumption policies across poor households, decreasing the share of financially constrained individuals and fostering risky investments among the wealthiest. Noticeably, the policy effect is more substantial when the capital tax cuts are compensated by raising additional taxes on labour rather than riskless bond income.

The last part of the paper explores the effect of these two alternatives on households' long-run welfare. Our analysis indicates that when risky capital tax cuts are financed by raising additional taxes on bonds, most households experience welfare gains, except for the highest net-worth individuals. Conversely, financing the policy by raising labour income taxes benefits the wealthiest and the expenses of the poorest. Finally, in the same spirit of Domeij and Heathcote (2004) we highlight that tax policies' short- and long-term effects may differ significantly. In the short run, for instance, a capital tax cut financed by additional taxes on bonds improves welfare for all individuals, with the wealthiest benefiting the most.

The paper proceeds as follows. Section 2 reviews the most closely related studies. Section

3 describes the model and the algorithm employed in its numerical solution. Section 4 compares the model's competitive equilibrium with a benchmark economy without capital risk and analyzes the effects of fiscal policies on macroeconomic aggregates, wealth distribution, and welfare. Section 5 concludes.

2 Related literature

From a broad perspective, we relate to several studies on the effects of fiscal policy in incomplete-market economies (e.g., Domeij and Heathcote, 2004; Heathcote, 2005; Conesa et al., 2009; Boar and Midrigan, 2022, among others). Heathcote (2005) studies taxes in Aiyagari (1994)-type economies and finds that income tax cuts provide a more significant boost to consumption and a smaller investment stimulus when asset markets are incomplete; in a similar framework Domeij and Heathcote (2004) show that reducing capital taxes entail substantial re-distributional effects, whose sign and magnitude have large differences in the short and long run. Conesa et al. (2009) find similar results in an OLG model where households face idiosyncratic, uninsurable income and productivity shocks, showing that capital tax rates are largely positive (about 25 per cent).

More recent contributions show that a uniform flat tax on capital and labour income combined with a lump-sum transfer is nearly optimal when households face income and productivity shocks (Boar and Midrigan, 2022) and that capital taxes can provide redistribution benefits in the short run (Dyrda and Pedroni, 2023), whereas increasing labour taxes in the medium to long run can mitigate the intertemporal distortion. Krueger et al. (2021) explore similar issues in a two-period OLG model, showing that the optimal time-invariant tax on capital increases with income risk. While we do not deal with optimal taxation, our paper differentiates from these works by studying the effects of tax policies in a context where households are subject to idiosyncratic capital risk, i.e., considering their portfolio choice of investing in risky capital or risk-free bonds.

Due to the presence of market incompleteness in our model, portfolio choices are critical determinants of the distribution of wealth across individuals. We thus connect to the literature on capital market risk and wealth inequality (e.g., Benhabib et al., 2016; Gomez, 2016; Campbell et al., 2019). Some recent contributions have analyzed the impact of capital shocks on the wealth distribution.[2] Among others, Benhabib et al. (2016) proves that the stationary wealth distribution in Bewley economies with idiosyncratic capital income risk is heavy-tailed. Gomez (2016) studies the impact of the heterogeneous exposure to aggregate risk on inequality and its relationship to asset prices. Campbell et al. (2019) empirically investigates stock return heterogeneity as the primary driver of wealth inequality, finding that larger accounts are better diversified, having a pivotal role in exacerbating wealth differentials. While our focus is not on the stock market, we highlight a similar mechanism in our paper.

Unlike these studies, we focus on the interaction between capital risk and fiscal policy and how they jointly affect wealth inequality. Moreover, we consider the role of public debt.

Finally, from a technical standpoint, the model's structure and solution build on Achdou et al. (2022) and exhibit several similarities with the mathematical theory of mean-field games introduced by Lasry and Lions (2007). Despite these shared elements, our model, much like other heterogeneous agent economies, deviates from the classical MFG models (see Gomes et al., 2016; Carmona and Delarue, 2018) because the interaction between individual agents' decisions and their overall distribution occurs (indirectly) via market prices rather than (directly) through their utility functions.

3 Model

Time is continuous and indexed as $t \in [0, \infty)$. The economy features three types of agents: a representative firm, a unit mass of ex-ante identical households, and the government. The firm produces output using labour and risky capital. Households supply labour inelastically and allocate their net worth between risk-free bonds and capital. As in Bewley (1986) and Aiyagari (1994), households face uninsurable labour endowment shocks and borrowing constraints, provoking a non-trivial wealth distribution. As in Benhabib et al. (2016), we assume that investing in firm capital is a source of idiosyncratic risk (e.g., private business risk) and that risk sharing is impossible. The government raises taxes on capital and labour

[2]The importance of considering idiosyncratic investment risk was early recognised by Angeletos (2007).

and issues riskless bonds to finance its exogenous public expenditure.

We now review each actor in greater detail.

3.1 Firms

The production sector consists of a continuum of identical and perfectly competitive firms. In the aggregate, there exists a representative firm which uses capital K_t and labour L_t to generate output Y_t utilizing the following Cobb-Douglas technology:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

where A parametrizes Total Factor Productivity (TFP) and α is the economy's capital share.

The firm trades production factors in competitive markets; her rental rate is such that marginal revenues equal marginal costs; that is,

$$R_t = \alpha AK_t^{\alpha-1} L_t^{1-\alpha}, \tag{2}$$

$$w_t = (1-\alpha) AK_t^{\alpha} L_t^{-\alpha}. \tag{3}$$

Accordingly, the firm breaks even and earns no profit in equilibrium.

3.2 Households

There is a unit mass of ex-ante identical households, indexed $i \in [0, 1]$, who have net worth $n_{t,i} \in [0, +\infty)$ and unit labour endowment. Households enjoy utility from instantaneous consumption flows $c_t^{1-\gamma} e^{-\rho t}$, where γ is their relative risk aversion, and ρ their subjective discount rate. They face stochastic labour-income shocks, modelled as a 2-state Markov chain $z_{t,i} \in \{1, z < 1\}$ which changes regime with (constant) Poisson intensity $\lambda_i \in \{\lambda_1, \lambda_z\}$. Labour income is taxed at the constant rate τ_l.

At each instant, households allocate their net worth between risky capital $k_{t,i} \geq 0$, which the firms use in their production process, and riskless bonds $b_{t,i}$ issued by the government. The former asset yields returns at the rate R_t (see Eq. (2)) but depreciates at the stochastic rate

$$-\delta dt + \sigma dZ_{t,i}, \tag{4}$$

where δ is a positive constant and $Z_{t,i}$ is a standard Brownian motion.[3] The latter asset yields the risk-free rate r_t, which is determined in equilibrium. The government taxes capital and bond earnings at the constant rates τ_k and τ_b, respectively.

Formally, households choose the triple $\{c_{t,i}, b_{t,i}, k_{t,i} \geq 0\}_{t\in[0,\infty)}$ solving the following program:

$$V_{0,i} =: \max_{\{c_{t,i},b_{t,i},k_{t,i}\geq 0\}_{t\in[0,\infty)}} \mathbb{E}_0^i \left[\int_0^{\theta_i} e^{-\rho t} \frac{c_{t,i}^{1-\gamma} - 1}{1-\gamma} dt + e^{-\rho\theta_i} V_{\theta_i,-i} \right], \tag{5}$$

subject to $n_{t,i} = k_{t,i} + b_{t,i} \geq \underline{n}$ and

$$dn_{t,i} = \left[b_{t,i}\left(1-\tau_b\right) r_t + w_t z_{t,i}\left(1-\tau_l\right) - c_{t,i}\right] dt + k_{t,i}\left(1-\tau_k\right)\left[\left(R_t - \delta\right) dt + \sigma dZ_{t,i}\right], \tag{6}$$

where θ_i denotes the random time when the labour-income shock z_i changes state and $V_{\theta_i,-i}$ is the value function after the regime change.

By imposing the balance sheet constraint and using standard stochastic dynamic programming arguments (see e.g., Pham, 2009), one can show that the solution to the problem in Eq. (5) associates with the following Hamilton-Jacobi-Bellman Equation (HJBE) (suppressing time-dependence for notational convenience):

$$\rho V_i = \max_{c_i,k_i\geq 0} \left\{ \begin{array}{c} \frac{\partial V_i}{\partial n}\left(n\left(1-\tau_b\right) r + w z_i\left(1-\tau_l\right) - c + k_i\left(\left(1-\tau_k\right)\left(R-\delta\right) - r\left(1-\tau_b\right)\right)\right) + \\ + \frac{\partial V_i}{\partial t} + \frac{c_i^{1-\gamma}-1}{1-\gamma} + \frac{1}{2}\frac{\partial^2 V_i}{\partial n^2} k_i^2 \sigma^2 + \lambda_i\left(V_i - V_{-i}\right) \end{array} \right\}. \tag{7}$$

The associated first-order conditions are

$$c_i^* = \left(\frac{\partial V_i}{\partial n}\right)^{-\frac{1}{\gamma}}, \tag{8}$$

$$k_i^* = \min\left\{ -\frac{\frac{\partial V_i}{\partial n}}{\frac{\partial^2 V_i}{\partial n^2}} \frac{\left(1-\tau_k\right)\left(R-\delta\right) - \left(1-\tau_b\right) r}{\sigma^2}, n - \underline{n} \right\} \tag{9}$$

and $b_i^* = n - k_i^*$.

[3] A detailed discussion of stochastic capital depreciation appears in Waelde (2011).

Similarly to what happens in a representative agent model, households' optimal consumption (Eq. (8)) is such that its marginal utility equals that of their net worth endowment. The capital allocation policy (Eq. (9)) is similar to the classical Merton (1969) portfolio rule, accommodating the constraint $k_{t,i} \geq 0$, the borrowing constraint $n \geq \underline{n}$, and taking into account that labour income is stochastic.

3.3 Government

The government uses tax revenues T_t and raises debt B_t to finance the exogenous constant public spending level G. Therefore, the stock of public debt obeys the following law of motion:

$$dB_t = r_t B_t dt + \underbrace{(G - T_t)\, dt}_{\text{Primary deficit}}. \tag{10}$$

where

$$T_t = K_t(R_t - \delta)\tau_k + L_t w_t \tau_l + B_t r_t \tau_b. \tag{11}$$

In summary, the public debt grows because of the interest paid and increases or decreases depending on the sign of the primary deficit, i.e., tax revenues minus public expenditure.

3.4 Equilibrium

We now define the model's competitive equilibrium and characterize its steady state. For this purpose, let $\pi_i(n)$ denote the joint density function of households' net worth distribution in each state of the Markov chain. To simplify notation, we omit all time subscripts and normalize households' aggregate labour supply to one without loss of generality.

Definition 1. ***(Competitive equilibrium)*** *A competitive equilibrium is a set of aggregates (capital and public debt levels), factor prices, risk-free rate, consumption, asset allocations, and a net worth distribution such that: 1) households' solve the problem in Eq. (5); 2) public debt evolves as in Eq. (10); 3) all markets (capital, bonds, labour) clear.*

At each instant, the equilibrium level of the risk-free rate r_t is such that aggregate households' net worth equals their total capital and bond holdings, i.e.,

$$\sum_i \int_{\bar{n}}^{\infty} n\pi_i(n)dn = \underbrace{\sum_i \int_{\bar{n}}^{\infty} k_i^*(n)\pi_i(n)dn}_{=K} + \underbrace{\sum_i \int_{\bar{n}}^{\infty} b_i^*(n)\pi_i(n)dn}_{=B} . \tag{12}$$

In the steady state, the public debt level B is such that Eq. (10) equals zero; therefore,

$$B = \frac{w\tau_l + (R-\delta)\tau_k K - G}{(1-\tau_b)r}. \tag{13}$$

Households' net worth distribution $\pi_i(n)$ satisfies the following coupled system of Fokker-Plank Equations (FPE):

$$\frac{\partial}{\partial t}\pi_i(n) = -\frac{\partial}{\partial n}\left(\mu_n^* \pi_i(n)\right) + \frac{1}{2}\frac{\partial^2}{\partial n^2}\left(\sigma_n^{*2}\pi_i(n)\right) + \lambda_i \pi_{-i}(n) - \lambda_{-i}\pi_i(n), \tag{14}$$

where $\mu_{n,i}^*$ and $\sigma_{n,i}^*$ are the drift and diffusion terms of Eq. (6) after substituting the optimal policies in Eqs. (8) and (9).

The steady state of the competitive equilibrium can be thus fully characterized by the forward-backwards PDE system, which includes the stationary versions of the HJBE in Eq. (7) and FPE in Eq. (14), such that $\partial \pi_i / \partial t = \partial V_i / \partial t = 0$. Importantly, the HJB and the FP equations "interact" through Eqs. (12) and (13).

Concerning this last point, it is relevant to highlight that the equilibrium's characterization has similarities with those of the so-called Mean-Field Games (MFGs) introduced by Lasry and Lions (2007). In particular, as in MFGs, the equilibrium is the solution to a fixed-point problem in which households' optimal strategies (and the corresponding prices) are such that the distribution of their future individual states matches that of the overall population. An essential difference with the MFG literature is that the *coupling* between HJBE and JP equations does not occur through households' utility (or cost) function but through the market clearing condition.

Another aspect that we would like to stress is that, as explained in Achdou et al. (2022) (see Online Appendix C.5), the existence and uniqueness results developed in the MFGs literature do not apply to the backwards-forward system describing the competitive equi-

Parameter	Interpretation	Value
ρ	Subjective discount	0.04
α	Capital share	0.36
δ	Depreciation	0.025
γ	Relative risk aversion	1.0
A	Total factor productivity	1.0
τ_k	Capital tax	0.23
τ_b	Bond tax	0.23
τ_l	Income tax	0.34
σ	Idiosyncratic volatility	0.2
λ_1	Transition intensity	0.04
λ_z	Transition intensity	0.12
z	Low state income shock	0.65
G	Public expenditure	0.521
$\bar{n}$	Leverage constraint	0

Table 1: Model calibration.

librium in our model. More specifically, this happens because the Hamiltonian operator implicit in Eq. (5) is not additive separable in $\partial V/\partial n$ and π because r is a function of the net worth distribution π. A comprehensive discussion of the existence and uniqueness of mean-field games solutions can be found in Carmona and Delarue (2018).

3.5 Calibration and solution method

Being unable to characterize the equilibrium further analytically, we resort to numerical methods. In this section, we calibrate the model's parameters and present a sketch of the algorithm adopted for its solution.

The baseline parameterization appears in Table 1. The subjective discount rate $\rho = 0.04$, the capital share $\alpha = 0.36$, the capital depreciation rate $\delta = 0.025$, and the relative risk aversion $\gamma = 1$ are set to standard values in the macroeconomic literature. Total factor productivity A is normalized to one. Tax rates $\tau_k = \tau_b = 0.23$ and $\tau_l = 0.34$ take values in line with the averages across OECD countries. Consistently with Kelly et al. (2016), the idiosyncratic volatility of capital depreciation equals $\sigma = 0.2$.

Coherently with OECD (2022) data, we set the labour-income transition rates $\lambda_1 = 0.04$ and $\lambda_z = 0.12$ to match an employment rate of around 0.75 and an auto-correlation of 0.84.

In line with Domeij and Heathcote (2004), we set the low-state income-shock parameter to $z = 0.65$ to generate a variability of the labour income process of 15 per cent. Finally, we set the public expenditure level to match the 2022 US debt-to-GDP ratio in the steady state (about 130%), and fix the leverage constraint parameter to an arbitrary level of $\underline{n} = 0$.

To approximate the solution to the HJBE, we implement an implicit upwind scheme (details appear in Candler, 2001) by imposing the following boundary conditions at $n_{min} = 0$ and $n_{max} \to \infty$ (in the numerical implementation, we approximate the latter state by using some large but finite constant).

When households' net worth is large, their labour income becomes negligible and the value function can be written as $V(n) = v_0 + v_1 \log n_{max}$, for some unknown constants v_0 and v_1. Therefore,

$$\frac{\partial V}{\partial n}(n^{\max}) = v_1 (n^{\max})^{-1}. \tag{15}$$

By substituting Eq. (15) in Eq. (9) and rearranging, one gets that

$$\frac{\partial^2 V}{\partial n^2}(n^{\max}) = -v_1 \gamma (n^{\max}) = -\frac{1}{n^{\max}} \frac{\partial V}{\partial n}(n^{\max}). \tag{16}$$

Imposing that $n \leq n^{max}$ to the drift of Eq. (6) implies that

$$c(n^{\max}) \geq r n^{\max}(1-\tau_b) + n^{\max} \frac{\left[(R-\delta)(1-\tau_k) - (1-\tau_b) r\right]^2}{\sigma^2} + zwl(1-\tau_l). \tag{17}$$

Substituting Eq. (17) in Eq. (8) and rearranging yields the first boundary condition

$$\frac{\partial V}{\partial n}(n^{\max}) = \left(r n^{\max}(1-\tau_b) + n^{\max} \frac{\left[(R-\delta)(1-\tau_k) - (1-\tau_b) r\right]^2}{\sigma^2} \right)^{-1}. \tag{18}$$

We find the second boundary condition by using that, when $n_{min} = 0$, then $k_{min} = 0$ and $c_{min} \leq zw$, which imply

$$\frac{\partial V}{\partial n}(n_{\min}) = \frac{1}{c_{\min}} = \frac{1}{zw}. \tag{19}$$

To approximate the solution of the FPE, we use an initial guess and apply the adjoint operator obtained by transposing the matrix containing the numerical solution of the HJBE (see Achdou et al., 2022, for details). As discussed in the same paper, the finite-difference up-wind scheme satisfies the so-called Barles and Souganidis (1991) conditions under which the numerical approximation of each PDE converges to its (unique) viscosity solution.

In summary, we approximate the competitive equilibrium by implementing the steps summarized in the following algorithm.

Algorithm 1 (Equilibrium approximation)

Guess initial aggregate capital level K_h
Guess initial risk-free rate r_j
for $h = 1 : H$ **do** (outer loop)
    for $j = 1 : J$ **do** (inner loop)
        Compute $\{R_{j,h}, w_{j,h}\}$.
        Approximate HJBE stationary solution; return $\left\{c^*_{h,j}, k^*_{h,j}, b^*_{h,j}\right\}$.
        Approximate FPE stationary solution; return $\pi_{h,j}$
        Compute $B_{h,j} = (w_{h,j}\tau_l + R_{h,j}\tau_k K_h - G)/r_j$.
        Approximate market clearing $\approx \sum_i \sum_n n\pi_{h,j}\Delta n - K_h - B_{h,j}$
        if $|\text{market clearing}| \leq \epsilon_r$ **then**
            $r_{j+1} \leftarrow r_j$
        else $r_{j+1} \leftarrow r_j + \textit{update}$
        end if
    end for
    Approximate $\tilde{K}_h \approx \sum_i \sum_n k^*_{h,j}\pi_{h,j}\Delta n$
    if $|\tilde{K}_h - K_h| \leq \epsilon_K$ **then**
        $K_{h+1} \leftarrow K_h$
    else $K_{h+1} \leftarrow \tilde{K}_h$
    end if
    if $|K_{h+1} - K_h| + |r_{j+1} - r_j| \leq \epsilon_\omega$ **then**
        Break
    end if
end for
Return $\left\{c^*_{h,k}, k^*_{h,k}, b^*_{h,k}, r_j, K_h, \pi_{h,k}\right\}$.

4 Numerical results

In this section, we first compare the numerical solution of the model with that of a benchmark economy in which capital investments entail no uncertainty, as in Domeij and Heathcote (2004).[4] Second, we investigate the effect of changing the tax mix between capital,

[4] In the benchmark economy, households invest their whole net worth in bonds provided by the government, bearing no capital uncertainty. A detailed description of the model appears in Appendix A.1.

bonds and labour on the distribution of wealth and welfare. Third, we discuss the impact of different tax policies in the short and long run.

4.1 Capital risk, asset allocations, and wealth distribution

The top panels of Figure 1 compare the numerical solutions of households' consumption functions, consumption-to-income ratios, and net worth stationary densities of the benchmark (Domeij and Heathcote, 2004, economy, in dotted lines) and in the Capital-Risk (CR) economy described in Section 3.5 (solid lines).[5] The bottom three panels of the figure display the steady-state asset allocations in the latter economy and the aggregate amount of capital and bonds. To complete the picture, Table 2 reports the key macroeconomic aggregates of both economies; Table 3 displays quantiles of households' net worth distribution q_j, the Gini coefficient, and the share of financially constrained households $\Pi(0)_i$.

What first stands out is that introducing CR increases wealth inequality relative to the benchmark economy with no capital risk. This happens both because the tails of the distribution become fatter (see Figure 1, Panel (b)) and there is a stark increase in the share of constrained agents (Table 3, Columns 1-3). These results are evident when looking at the level of the Gini coefficients and comparing higher with lower percentiles of the household's net worth distribution (see Table 3, Columns 4-7).

The second result is that CR decreases consumption-to-income rates but increases consumption levels (see Figure 1, Panels (I), (a) and (b)). The first pattern materializes because capital shocks stimulate households' precautionary motif, thereby fostering capital accumulation.[6] The latter occurs because a higher capital stock increases labour income by fostering wages (see Table 2).

The third main effect of including CR in the model is that households make substantially different asset allocation decisions depending on their net worth levels (see Figure 1, Panels (II) - (a) and (b)). Poorer households hold a significant share (if not all) of their net worth in capital. Conversely, wealthier ones tilt their portfolios towards riskless assets; the more,

[5]We define households' instantaneous income as $\text{income}_i := k_i^*(R_t\text{-}\delta)(1\text{-}\tau_k)+w_t\tau_l z_i(1\text{-}\tau_l)+b_i^* r_t(1\text{-}\tau_b)$.

[6]Notice that this happens as long as the market imperfection introduced by CR is not too large. For instance, a value of $\sigma = 0.3$ with our parameters leads to a lower capital value in the steady-state relative to the benchmark.

Model	Y	B	B/Y	K	r	R	w	$N_1/\sum_i N_i$
Benchmark	2.04	2.65	130.25%	7.23	10.15%	-	1.30	77.63%
High risk ($\sigma = 0.2$)	2.20	2.87	130.25%	8.98	3.12%	8.82%	1.41	77.15%
Low risk ($\sigma = 0.15$)	2.28	3.08	134.90%	9.89	4.54%	8.30%	1.46	77.08%

Table 2: Equilibrium aggregates in the Benchmark and Capital Risk models.

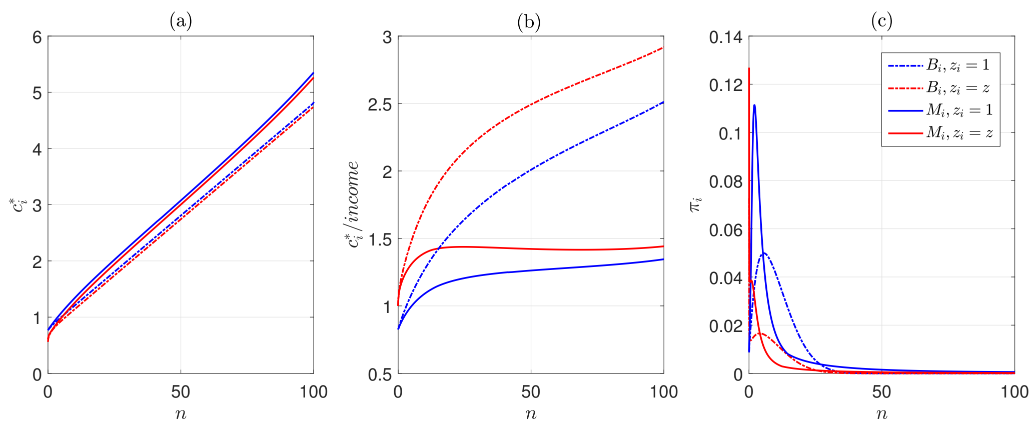
the higher their net worth. The interpretation behind this perhaps counter-intuitive result is the following.

As poor households have a higher propensity to consume ($:= dc_i^*/d(\text{income})$) and capital earns a positive premium over bonds, they are willing to invest their whole net worth into risky assets to earn (in expectation) higher returns on their savings ($k_i^* + b_i^*$). Put differently, the first moment of asset returns drives their decisions. We can rationalize this behaviour as that of small- and medium-sized entrepreneurs investing their whole net worth in their enterprises ("inside equity"). As a result of this decision, low-net-worth households have a very volatile endowment, fostering the share of those whose financial constraint is binding. Instead, when households accumulate net worth, the marginal utility of their consumption decreases, and they find it convenient to hedge their net worth fluctuations by considering both the risk and expected returns on their assets holdings. In other words, wealthy households are "financially sophisticated" because they care about the first two moments of their asset returns. This behaviour allows them to mitigate their net worth volatility and accumulate more in the long run, fattening the distribution's heavy right-hand side tail.

The considerations about the increased inequality in the distribution we drew so far carry over to the wealth distribution within high- or low-income individuals. However, it is worth noticing that the distribution across types is hardly affected by CR. Interestingly, a smaller fraction of the net worth is held by the high-income type of individuals when there is CR relative to the benchmark case (see the last column of Table 2).

To further explore the effect of introducing capital risk, Figure 2 compares the competitive equilibrium with the parametrization discussed above and with a lower level of capital risk, $\sigma = 0.15$. Table 2 reports the corresponding macroeconomic aggregates, while Table 3 displays relevant statistics of the wealth distribution.

(I) Benchmark vs Capital Risk models: consumption and distribution



(II) Benchmark vs Capital Risk models: allocations

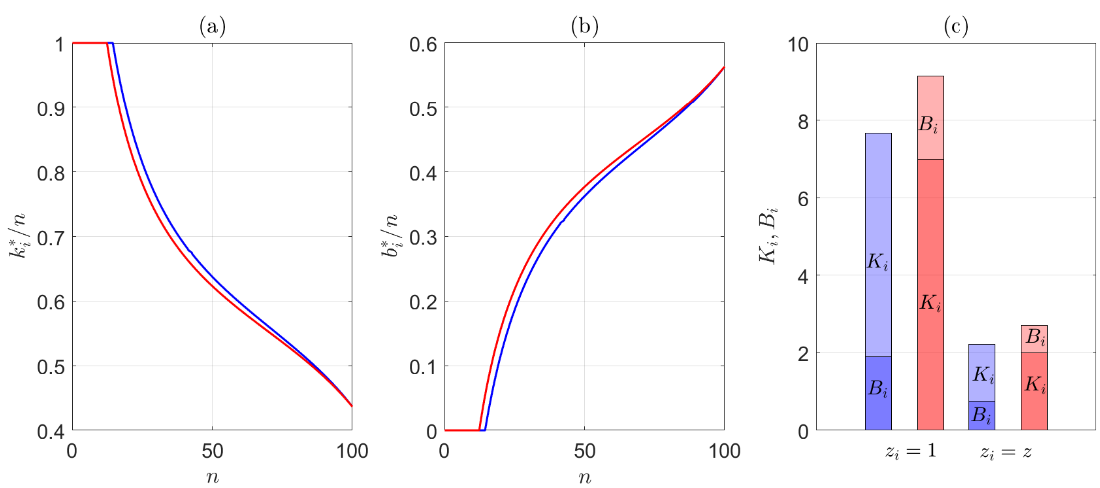
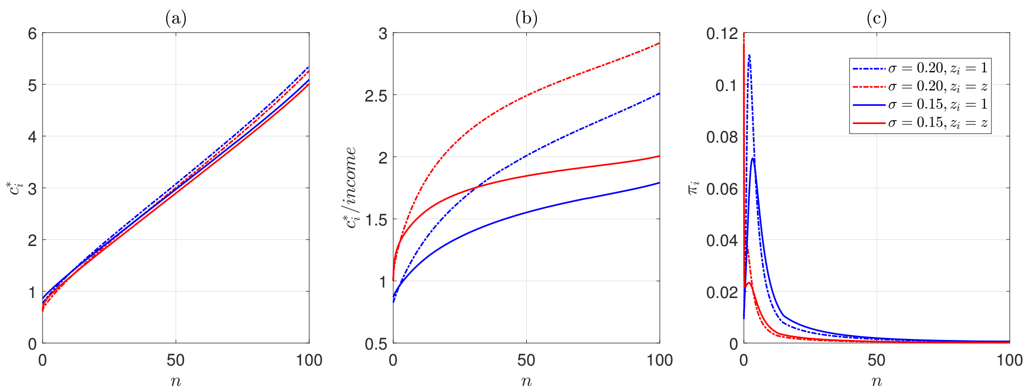


Figure 1: Numerical solutions of the Benchmark (dotted) and the Capital Risk models (solid).

(I) High vs low Capital Risk volatility: consumption and distribution



(II) High vs low Capital Risk volatility: allocations

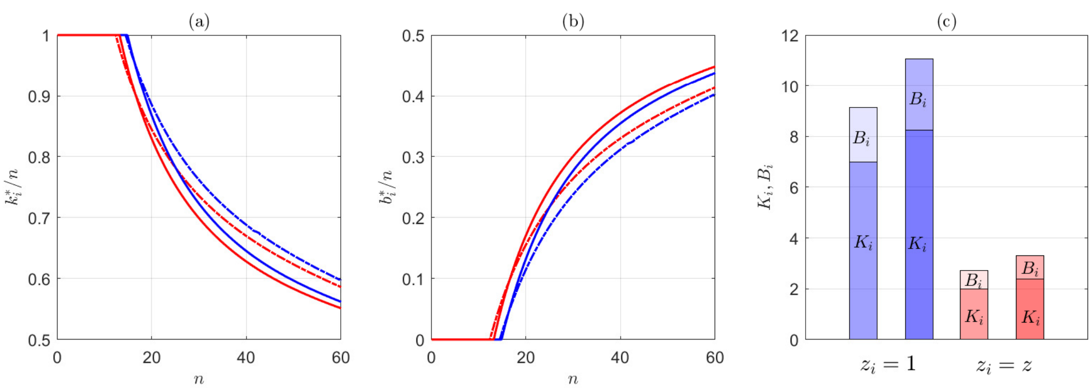


Figure 2: Numerical solutions of the model with high (dotted) and low (solid) capital risk.

	$\Pi(0)$	$\Pi_1(0)$	$\Pi_z(0)$	q_5	q_{10}	q_{50}	q_{95}	Gini	Gini_1	Gini_z
Benchmark	0.81%	0.12%	2.91%	1.3	2.3	8.7	22.6	0.3720	0.3580	0.4124
$\sigma = 0.2$	1.36%	0.12%	5.08%	0.6	1.2	4.6	54.0	0.6371	0.6206	0.6853
$\sigma = 0.15$	1.37%	0.14%	5.08%	0.7	1.4	5.5	56.2	0.6142	0.5981	0.6615

Table 3: Capital risk and the distribution of wealth.

As intuition suggests, a lower level of CR ($\sigma = 0.15$) mitigates wealth inequality while fostering capital accumulation in the aggregate and across all percentiles of its distribution. Indeed, the peaks of the wealth distributions for both the high- and low-income earners shift to the right (see 2 (I) Panel (c)) and the median individual's wealth is 20% higher than with $\sigma = 0.2$. Moreover, capital holdings are less concentrated across different wealth levels. A higher fraction of individuals hold all of their wealth in risky assets, and such share decreases only for high wealth percentiles. Coherently, the risk premium reduces from 5.70 tp 3.75 per cent, a variation explained by both a higher risk-free rate r and a drop in expected returns R.

A lower capital risk also induces poorer individuals to consume more, while wealthier individuals consume less (see Figure 2, Panel (I)). The former result is the combined effect of the reduced need to save to face future uncertainty and the increased wage level due to the larger capital stock. Conversely, wealthier individuals consume less because, faced with lower risk premiums, they tilt their portfolio towards bonds and thus earn lower returns on average. However, they also find it convenient to start diversifying their portfolios at a higher level of net worth (see Figure 2, Panel (II)). As a result of these forces, reducing capital risk boosts output and government debt in the aggregate; however, it leaves the distribution of capital across income-earner types unaffected.

4.2 Fiscal policy

We now explore how fiscal policy affects the competitive equilibrium in the long run. In particular, we evaluate the effects of changing the tax mix between financial assets (i.e., capital and bonds) and labour and financial assets. The first simulation varies the tax rate on risky capital τ_k (plus/minus three percentage points) and keeps the labour tax rate τ_l,

	Y	B	B/Y	K	r	R	w	τ_b/τ_l
				Panel (a) - Risky vs riskless asset taxes				
$\tau_k = 20\%$	2.23	2.60	160.51%	9.33	4.49%	8.61%	1.43	0.32
$\tau_k = 23\%$	2.20	2.87	130.25%	8.98	3.12%	8.82%	1.41	0.23
$\tau_k = 26\%$	2.17	3.11	143.43%	8.64	3.75%	9.05%	1.39	0.15
				Panel (b) - Human vs financial capital taxes				
$\tau_l = 31\%$	2.10	2.51	119.51%	7.84	4.64%	9.63%	1.34	0.33
$\tau_l = 34\%$	2.20	2.87	130.25%	8.98	3.12%	8.82%	1.41	0.23
$\tau_l = 37\%$	2.31	3.14	136.26%	10.17	3.57%	8.16%	1.47	0.13

Table 4: Fiscal policy and macroeconomic aggregates.

adjusting τ_b to hold tax revenues T constant. The second one changes τ_l while adjusting the tax rate across all financial assets $\tau_k = \tau_b$.

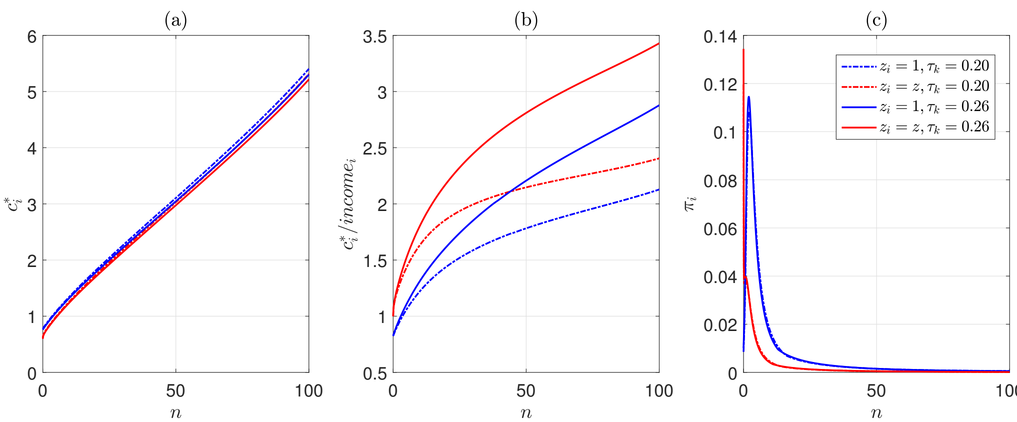
The effect of the former policy on cross-sectional quantities is summarized in Figure 3; aggregate outcomes appear in Table 4, Panel (a).

At first glance, the effect of changing the financial asset tax mix is relatively straightforward. Lowering the tax on risky capital fosters capital accumulation and increases wages. At the same time, a lower bond demand pushes up the risk-free interest rate; accordingly, the risk premium falls by about two percentage points. The distributional effects of the policy are far less intuitive because redistributing taxes from risky to riskless assets reduces wealth inequality. The mechanism behind this result is two-folded. First, there are fewer financially constrained agents because labour wages are relatively higher, and the consumption rate across the whole population is thus lower (see Figure 3, Panel (I), (a) and (b)). Second, higher taxes on bonds (and lower on capital) foster wealthier households' capital investment (i.e., mitigates their hedging motif), thereby increasing their net worth volatility (see Figure 3, Panel (II), (a) and (b)).

The result of the second policy, that is, the cross-sectional and aggregate effect of changing the tax mix between human and financial capital on the equilibrium, are summarized in Figure 4 and Table 4, Panel (b).

Similarly to the outcome of our previous analysis, substituting capital with labour income taxes has a positive effect on net worth and capital accumulation (and thus output; see Figure 4, Panel (II), (c) and Table 4, Panel (b), Columns 2 and 4) because capital investment

(I) Risky vs riskless asset taxes: consumption and distribution



(II) Risky vs riskless asset taxes: allocations

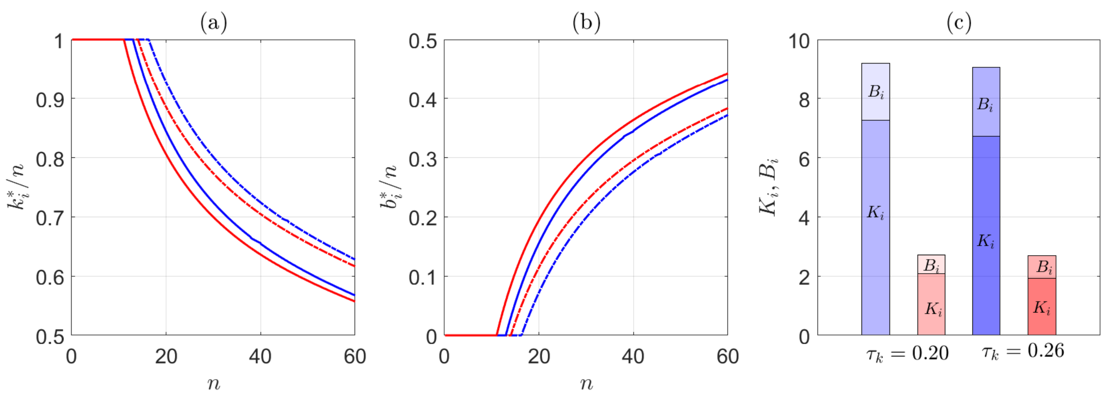


Figure 3: Numerical solutions for different financial asset tax policies.

	$\Pi(0)$	$\Pi_1(0)$	$\Pi_z(0)$	q_5	q_{10}	q_{50}	q_{90}	q_{95}	Gini	Gini_1	Gini_z
Panel (a) - Risky vs riskless asset taxes											
$\tau_k = 20\%$	1.28%	0.11%	4.79%	0.7	1.2	4.7	34.1	54.4	0.6336	0.6170	0.6821
$\tau_k = 23\%$	1.36%	0.12%	5.08%	0.6	1.2	4.6	34.0	54.0	0.6371	0.6206	0.6853
$\tau_k = 26\%$	1.45%	0.13%	5.38%	0.6	1.1	4.5	33.8	53.5	0.6400	0.6237	0.6879
Panel (b) - Human vs financial capital taxes											
$\tau_l = 31\%$	1.74%	0.17%	6.48%	0.5	1.0	4.0	28.7	47.4	0.6445	0.6260	0.6986
$\tau_l = 34\%$	1.36%	0.12%	5.08%	0.6	1.2	4.6	34.0	54.0	0.6371	0.6206	0.6853
$\tau_l = 37\%$	1.04%	0.09%	3.90%	0.8	1.3	5.2	39.1	59.7	0.6283	0.6136	0.6716

Table 5: Fiscal policy and the distribution of wealth.

yields higher post-taxes returns (Table 4, Panel (b), Columns 5 and 6). As a result, the equilibrium risk premium lowers and gross wages increase, while net ones slightly decrease (Column 7). These forces encourage households to behave more carefully, reducing their consumption levels and rates (see Figure 4, Panel (I), (a) and (b)). This result, combined with the higher net return from both capital investments (see Table 4, Panel (b)), determines a relatively uniform shift of the wealth distribution to the right, with a lower mass of constrained individuals (mainly belonging to the low-income type) and a slightly fatter centre and right-tail of the distribution (see Figure 4, Panel (I), (c) and Table 5, Panel (b)).

All in all, the progressive effect of having lower capital taxes overtakes the regressive one of having lower post-tax wages, leading to an overall decrease in the Gini coefficient of about one percentage point (see Table 5, Panel (b)). Finally, we remark that distributional effects are much more extensive for this tax policy than the one that only concerns taxes on financial assets.

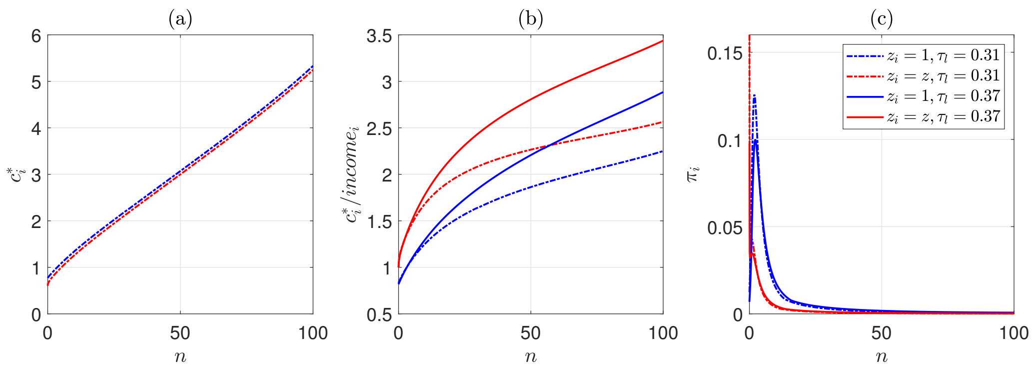
4.3 Welfare

We now investigate the effect of the two tax policies on welfare by comparing households' value functions $V_i(n)$ (see Eq. (5)) in the steady state before and after the tax policy implementation.[7]

We begin the analysis by focusing on the effects of taxing risky vs riskless assets, whose

[7] We interpret the value function's absolute change $\Delta V_i(n)$ relative to the baseline parametrization as a measure of the welfare gain/loss of a household of type i and net worth n.

(I) Human vs financial capital taxes: consumption and distribution



(II) Human vs financial capital taxes: allocations

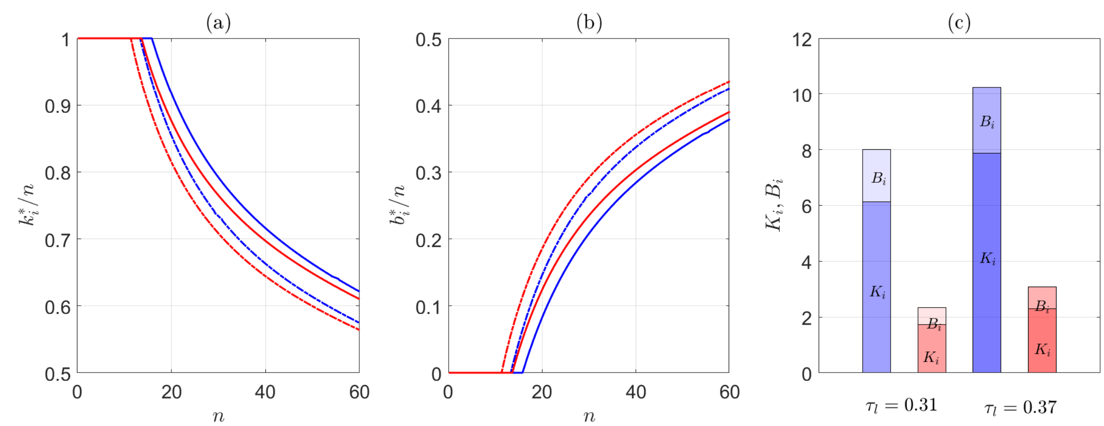


Figure 4: Numerical solutions of the model for different human vs financial capital tax policies.

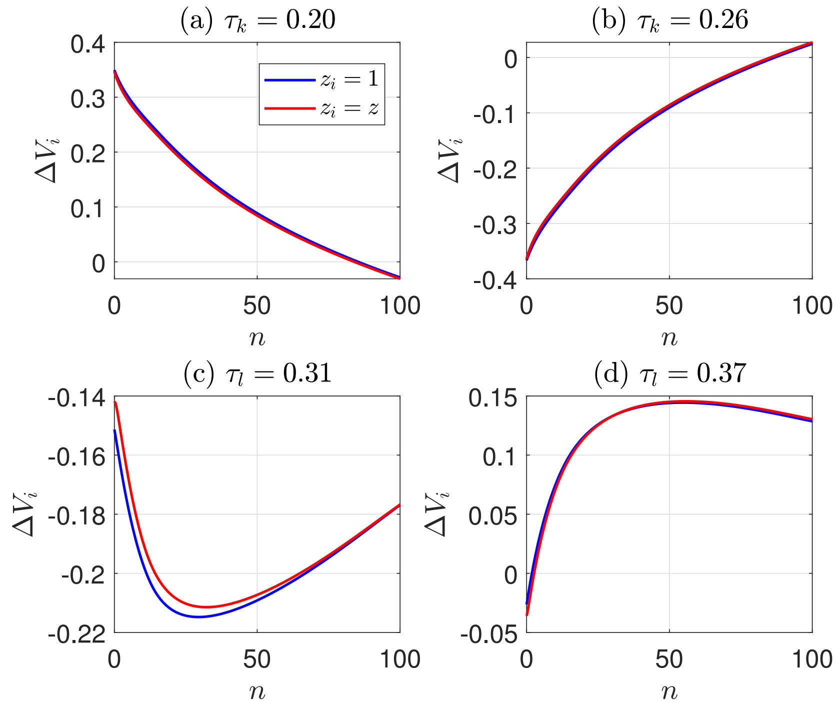


Figure 5: Tax policy and welfare.

equilibrium outcomes are displayed in Figure 5, Panels (a) and (b).

Coherently with the results of Section 4.1, reducing (increasing) risky capital taxes at the expenses of bonds improves (reduces) welfare for most households. However, gains (losses) concentrate among low (high) income households and are decreasing (increasing) in their net worth levels. The reason is that low-net-worth households invest most of their net worth in capital, while wealthy ones (also the most financially sophisticated) allocate an increasingly high share of their holdings in bonds (see Figure 3, Panel (II)).

The remaining panels of Figure 5 display the welfare effect of financing tax cuts on capital (labour) by raising taxes on labour (capital). Unlike the previous policy, increasing labour taxes generates welfare gains across most individuals except for the poorest ones. Conversely, lower labour taxes reduce welfare across the whole population. Interestingly, both policies affect agents non-linearly, benefitting (hurting) the middle net worth levels the most.

The results on welfare gains/losses are coherent with the fact that reallocating taxes from capital to labour fosters aggregate consumption (but only across wealthy households) and

	Y	B	B/Y	K	r	R	w	τ_b/τ_l
	Panel (a) - Risky vs riskless asset taxes							
Short run	2.46	2.25	110.00%	9.39	4.46%	8.58%	1.43	0.32
Baseline	2.20	2.87	130.25%	8.98	3.12%	8.82%	1.41	0.23
Long run	2.23	2.60	160.51%	9.33	4.49%	8.61%	1.43	0.32
	Panel (b) - Human vs financial capital taxes							
Short run	2.25	2.25	100.01%	9.6	3.97%	8.46%	1.44	0.33
Baseline	2.20	2.87	130.25%	8.98	3.12%	8.82%	1.41	0.23
Long run	2.30	3.14	136.26%	10.17	3.57%	8.16%	1.47	0.13

Table 6: Fiscal policy and macroeconomic aggregates in the short and long run

capital accumulation while dampening risk-free asset demand. In fact, in equilibrium, the policy reduces returns on risky capital and (after-tax) wages more than bonds, benefiting wealthier and less income-dependent households the most (see Table 4, Panel (b)).

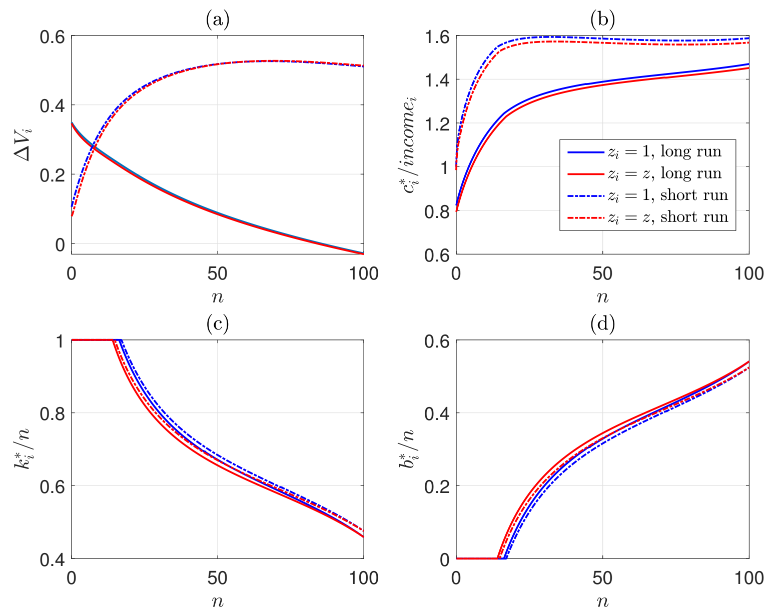
What is also relevant to stress is that even though the policy hurts poor households the most, it also generates a reduction in the share of financially constrained, thereby reducing the probability of being in a low-net-worth state (see Table 5, Panel (b) and Figure 4, Panel (I), (c)). In other words, the policy entails a loss in consumption in adverse states but also a lower probability of experiencing them.

4.4 Short vs long run

So far, our analysis has examined how fiscal policies affect the economy in its steady state, i.e., in the long run. However, as highlighted by Domeij and Heathcote (2004), the conclusions drawn from this approach may be misleading. For this reason, we complement the long-run perspective provided in Section 4.3 by evaluating policies in the short run. In practice, we evaluate the general equilibrium effect of tax changes by holding the wealth distribution constant, computing variations in individual households' strategies, and letting prices adjust accordingly. The idea is that while agents can immediately adjust their behaviour, the wealth distribution takes time before adjusting.

We first focus on the policy that shifts the tax burden from capital to bonds (see Figure 6, Panel (I) and Table 6). Immediately following the tax change, the interest rate and the

(I) Risky vs riskless asset taxes: short vs long run



(II) Human vs financial capital taxes: short vs long run

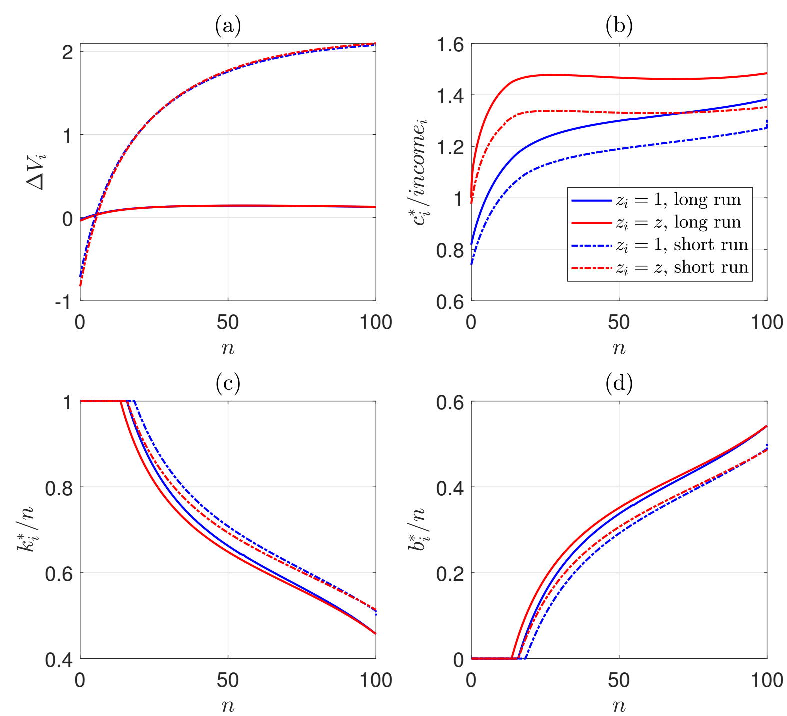


Figure 6: Fiscal policy in the short (dashed) and long (solid) run.

cost of capital adjust (r spikes up to 4.64 per cent, while R drops to 8.58 per cent) above their new steady-state levels. As a result, the risk premium shrinks below its long-run level (see Table 6, Panel (b)). As a result of these changes in relative prices, households reallocate net worth towards capital, which increases its aggregate level while debt decreases (see Table 6, Column 4). Accordingly, output and wages increase.

The top panel of Figure 6 compares the policy's short vs welfare effects, highlighting a stark contrast between the two. Whereas in the long run benefits accrue to poor households, in the short run they affect wealthy ones the most. Moreover, the short-run welfare gain across wealthy households is more prominent than among poorer ones.

To complete the discussion, the second panel of Figure 6 compares the effects of the fiscal policy that decreases taxes on financial assets and increases labour income taxes in the short and long run.

Similarly to the previous policy, macroeconomic aggregates overreact in the short run, as riskless and risky rates of returns rise by 0.4 and 0.3 percentage points above their new steady-state levels. Accordingly, capital stock and output increase, realizing immediately half of their long-run increments, while wages increase only slightly (see Table 6, Columns 4-7). Unlike the outcome of the first policy, however, increments in risk-free returns in the short run overtake those of risky capital returns and aggregate debt drops. Noticeably, this pattern is in stark contrast with the long-run effect on debt, which increases at the steady-state relative to the pre-tax policy level (see Table 6, Column 2).

When looking at the policy effect on welfare in Figure 6, Panel (II), we notice that short-run changes in the value function are much larger than at the steady state and are monotonically increasing in the level of wealth. In other words, as in the long-run, the poorest experience a welfare reduction, while the wealthiest are positively affected by the income tax change in the short-run. However, the magnitude of the welfare change is much more prominent in the short than in the long run.

5 Conclusions

We have developed and solved numerically an equilibrium continuous-time model of a production economy where households face leverage constraints, uninsurable labour income shocks, and capital depreciation risk. Within this framework, we show that capital risk fosters aggregate capital accumulation and safe asset demand, while increasing wealth inequality. The reason is that poor households hedge a lower share of their idiosyncratic capital risk in equilibrium, having a more volatile net worth than their wealthier peers. As a result, low-net-worth households are more likely to be financially constrained, while wealthy ones accumulate large endowments on average. We analyze the effect of fiscal policy on households' wealth inequality and welfare in the presence of capital risk. We show that redistributing tax revenues from risky capital to risk-free bonds or labour income may have remarkably heterogeneous effects across households' wealth distribution due to their interaction with the economy's risk premiums.

While the joint presence of uninsurable capital risk (and thus portfolio choice) and government debt is novel in the literature, our study has some limitations, which we acknowledge. First, we do not allow households to choose their labour supply endogenously (as, for example, in Marcet et al., 2007), which may have significant policy implications. Second, our welfare analysis focuses on a "static" comparison between its short- and long-run effects. Since our results hint at non-monotone adjustments between steady states, an interesting extension would be to characterize the full transition dynamics, taking the whole path while evaluating welfare gains and losses. We leave these extensions to future research.

A Appendix

A.1 Benchmark economy (Domeij and Heathcote, 2004)

The benchmark economy is a continuous-time version of the baseline model in Domeij and Heathcote (2004). As explained in the main text, in this model there is no difference between risky capital and risk-free bonds, and households solve the following problem:

$$V_{0,i}^b =: \max_{\{c_{t,i}, n_{t,i} \geq 0\}_{t \in [0,\infty)}} \mathbb{E}_0^i \left[\int_0^{\tau_i} e^{-\rho t} u(c_t) dt + e^{-\rho \tau_i} V_{-i}^b \right], \tag{20}$$

subject to

$$\dot{n}_{t,i} = n_{t,i} r_t \left(1 - \tau_k\right) + w_t z_{t,i} \left(1 - \tau_l\right) - c_{t,i}, \tag{21}$$

and $n_{t,i} \geq 0$. The HJBE associated to this problem is

$$\rho V_i = \max_{c_i} \left\{ u(c_i) + \frac{\partial V_i}{\partial n} \left(nr \left(1 - \tau_k\right) + w z_i \left(1 - \tau_l\right) - c_i \right) + \lambda_i \left(V_i - V_{-i} \right) \right\},$$

whose first-order condition yields

$$c_i^* = u^{'-1} \left(\frac{\partial V_i}{\partial n} \right). \tag{22}$$

In the steady-state competitive equilibrium, the market clearing condition is

$$N := \sum_i \int n \pi_i(n) dn = B + K,$$

where the public debt equals

$$B = \frac{w \tau_l + r \tau_k N - G}{r}. \tag{23}$$

As for our our specification, we solve the model numerically by applying the algorithm described in Section 3.5.

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