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Rational Learning and Term Structures*

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Abstract

We study the impact of information processing and rational learning on the term structures of equity risk, risk premia, and bond yields. In contrast with the full information economy, learning gives rise to an upward-sloping term structure of bond yields and downward-sloping term structures of equity risk and risk premia. Moreover, learning generates lower interest rates and a larger risk premium than in an otherwise identical economy with full information. Therefore, information processing and learning help explain jointly the observed level and timing of both equity return moments and interest rates. Economic growth forecast data lends support to the model.

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1 Introduction

To build forecasts about future economic conditions, investors need to gather, analyze, and filter a tremendous amount of information. By efficiently and rationally processing this flow of information over time, investors update their forecasts and use them to dynamically invest in financial markets. Therefore, information processing impacts asset prices through investors' trades.

We study the role of information processing and learning in an asset-pricing model with time-varying economic growth.[1] Investors, unable to directly observe economic fundamentals, use a Bayesian model to update their beliefs about future growth prospects. We show that rational learning changes the agents' perception of economic growth risk across different horizons, which in equilibrium translates into patterns of the term structures of equity risk premia, equity return volatility, and interest rates that are in line with the empirical evidence (van Binsbergen, Brandt, and Koijen, 2012; van Binsbergen, Hueskes, Koijen, and Vrugt, 2013). Information processing and rational learning help to overcome the empirically inconsistent predictions of the full information framework.

We analyze an economy in which aggregate output is driven by two sources of risk: a long-lasting component and a contemporaneous shock. The long-lasting component of output growth must be small to generate a realistic model-implied volatility of output growth. Such a component is difficult to detect statistically, even in large samples. Therefore, we consider a model of learning in which the representative agent observes changes in output but cannot observe the long-lasting component driving the change.

Following Marfè (2017), we assume that the long-lasting component of output growth depends on two latent factors. The first is the usual long-run component considered in the long-run risk literature. The second is a mean-reverting, transitory component that captures business cycle fluctuations. The long-run component implies that output growth

[1] The importance of time variation in economic growth for asset pricing and portfolio choice is highlighted by Veronesi (1999, 2000), Brennan and Xia (2001), Xia (2001), and Bansal and Yaron (2004) among others.

risk tends to increase with the time horizon, whereas the transitory component has the exact opposite impact. Since the impact of the transitory component dampens that of the long-run component, the term structure of output growth risk is about flat, as in the data (Marfè, 2017; Dew-Becker, 2017).

In line with the empirical findings of van Binsbergen et al. (2012), we show that information processing and rational learning imply a downward-sloping term structure of equity risk premia, whereas the full information model yields an upward-sloping term structure. Moreover, the slope of the term structure of interest rates is positive under information processing, whereas it is negative under full information. The equity risk premium under information processing is sizable and larger than under full information. Also, the short-term real bond yield is smaller under the information processing than under full information. These results provide evidence that the predictions of the model with information processing and learning are consistent with data.

The economic mechanism is novel and works as follows. Under full information, each priced source of risk has a specific effect on the slope of risk premia. Specifically, the long-run component, the transitory component, and the contemporaneous shock to output have respectively an upward-sloping, a downward-sloping, and a neutral effect on the term structure of equity risk premia. The upward-sloping effect dominates even if output growth risk is flat across different horizons. Under information processing, the investor needs to infer the latent components using only one source of information: the realized output growth rate. As a result, learning yields an endogenous, perfect correlation between shocks to realized output growth, shocks to the long-run component, and shocks to the transitory component. This implies that the economy, as perceived by the investor, is driven by a unique priced source of risk. In such a case, the slope of the term structure of equity risk premia depends on the price sensitivities to the filtered long-run component and the filtered transitory component. Because prices are more sensitive to the transitory component than to the long-run component, the impact of the transitory component dominates and yields a downward-sloping term structure of equity risk premia. Moreover, since bonds are used to hedge equity risk, short-term bonds are more expensive and therefore feature lower yields than long-term bonds.

To provide empirical support for the economic mechanism of the model, we show that the theoretical relation between beliefs about expected growth and equilibrium asset prices can be recovered in actual data. Using actual prices, we extract model-implied beliefs about expected economic growth and document that they align well with observed forecasts of economic growth obtained from the Survey of Professional Forecasters.

Our paper is closely related to the literature providing theoretical foundations for the observed shape of the term structures of equity risk premia and equity return volatility. Belo, Collin-Dufresne, and Goldstein (2015) shows that, when dividend dynamics are such that leverage ratios are stationary, the term structures of dividend growth volatility, equity risk premia, and equity return volatility are downward sloping as observed empirically. Croce, Lettau, and Ludvigson (2015) show that when investors have limited information and bounded rationality, the observed timing of equity risk premia and the high risk premium can be explained simultaneously. Instead, we show that under a realistic description of the timing of economic fundamentals rational learning is sufficient to explain in equilibrium the empirical evidence about the timing of equity risk and risk premium. Importantly, we do not need to evoke bounded rationality. Hasler and Marfè (2016) show that the observed term structures of equity risk premia and volatility can be explained by the existence of recoveries following rare economic disasters. Marfè (2017) shows that, when labor impacts dividend payout, the equilibrium term structures of equity risk premia and equity returns volatility are consistent with the data. Ai, Croce, Diercks, and Li (2018) show that, when investment responds positively and negatively to respectively short-term and long-term productivity shocks, the term structure of equity risk premia over short maturities is downward-sloping. Hasler and Khapko (2018) derive conditions under which a simple model featuring standard preferences and realistic output dynamics can produce the observed shapes of the terms structures of equity and interest rates.

Our results contribute to the existing literature by showing that the latent nature of expected growth and rational learning alone help to rationalize the empirical evidence about the timing of equity risk and risk premia. This economic mechanism is simple, arises naturally, and complements the alternative economic channels mentioned above. Moreover, this mechanism can explain the pro-cyclical variation in the slope of the term structure of equity risk premia (van Binsbergen et al., 2013; Bansal, Miller, Song, and Yaron, 2019). Incomplete information and the need for learning become more pronounced in bad times (Bloom, 2009, 2014; Jurado, Ludvigson, and Ng, 2015). In good times uncertainty is low and investors are better informed. According to our results, this leads to an upward-sloping term structure of risk premia. In bad times uncertainty is high and investors are forced to actively learn. In this case, learning generates a downward-sloping term structure of risk premia.

The paper is also related to the long-run risk literature, which was launched by Bansal and Yaron (2004). In this literature, the long-run component is highly persistent, which yields highly volatile consumption growth rates over long horizons. However, empirical evidence documents that consumption growth risk is about the same at short and long horizons (Marfè, 2017; Dew-Becker, 2017). This suggests that the high empirical level of the equity premium is unlikely to be rationalized by highly volatile long-horizon cash-flows (Beeler and Campbell, 2012). In addition, long-run risk models imply an upward-sloping term structure of equity risk premia, which is inconsistent with the empirical findings of van Binsbergen et al. (2012), van Binsbergen et al. (2013), van Binsbergen and Koijen (2017), and Weber (2018), as well as a downward-sloping term structure of interest rates. In our model, rational learning, in the presence of a transitory component, dampens the upward-sloping impact of the long-run component. Therefore, our model helps explain jointly the flat term structure of consumption growth risk, the downward-sloping term structure of equity risk premia, the upward-sloping term structure of interest rates, and the high equity risk premium.

The remainder of the paper is organized as follows. Section 2 describes the economic fundamentals and learning problem; Section 3 solves for equilibrium asset prices; Section 4 presents the results and Section 5 concludes. Derivations and supplementary material are provided in the Appendix.

2 Economic Fundamentals

In this section we describe the economy and discuss the implications of learning on the term structure of output growth risk. The only information available to the representative agent is the one generated by the aggregate output process (which is equal to the aggregate consumption in equilibrium). Importantly, the underlying factors driving the output growth dynamics are not directly observable. This introduces learning into the decision problem of the agent and has implications for the agent's perception of risk at different horizons.

2.1 Output Dynamics

The aggregate output, C, has the following dynamics

$$d \log C_t = dy_t + dz_t, \tag{1}$$

where y is an integrated process with time-varying expected growth, x. The dynamics of y and x are

$$dy_t = (\mu + x_t)dt + \sigma_y dB_{y,t}, \tag{2}$$

$$dx_t = -\lambda_x x_t dt + \sigma_x dB_{x,t}, \tag{3}$$

and the mean-reverting process z satisfies

$$dz_t = -\lambda_z z_t dt + \sigma_z dB_{z,t}. \tag{4}$$

The three Brownian motions, B_y, B_x, and B_z are independent. The agent observes only the level of output C and, through Bayesian updating, filters out the unobservable factors $\theta = (x, z)'$. The full filtration generated by observing all three Brownian shocks is denoted by $\mathbf{F}$.

The aim of considering both components, y and z, is to introduce some flexibility in modeling the timing of risk (Marfè, 2017). The first component, y, is an integrated process which depends on the time integral of x. That is, shocks in x accumulate and therefore permanently affect future output levels. For this reason, we call x the stochastic drift of the permanent component. The second component, z, depends on the current value of z. Since the process z is mean-reverting, shocks in z dissipate as time passes and therefore have a transitory impact on output. For this reason, we call z the transitory component. Without the z component, the aggregate output follows the standard dynamics considered in the literature on incomplete information and learning (e.g. Gennotte, 1986; Detemple, 1986) as well as in the long-run risk literature pioneered by Bansal and Yaron (2004). In this case, as we will show, the term structure of output growth risk is upward-sloping because of the accumulation of x shocks in y. Adding the transitory component z generates risk in the short term that dissipates in the longer term. Consequently, the transitory component induces a downward-sloping effect on the term structure of output growth risk. The existence of both x and z therefore provides flexibility in the modeling of the timing of output growth risk.

Equations (1), (2), and (4) imply the following dynamics for the logarithm of output

$$d \log C_t = (\mu + x_t - \lambda_z z_t)dt + \sqrt{v} dB_t, \tag{5}$$

where $v \equiv \sigma_y^2 + \sigma_z^2$ is the instantaneous variance and $dB_t \equiv (\sigma_y dB_{y,t} + \sigma_z dB_{z,t})/\sqrt{v}$ is an increment of a standard Brownian motion.

2.2 Bayesian Learning

The expected growth rate of output varies over time due to shocks that come from two sources: the drift of the permanent component x_t defined in (3) and the transitory component z_t defined in (4). The agent only has access to information generated by the observation of the realized aggregate output path in (5), and thus does not have access to the full information contained in the filtration $\mathbf{F}$. Therefore, all her actions must be adapted to her observation filtration $\mathbf{F}^o = \{\mathcal{F}_t^o\}_{t\geq 0}$, defined as the flow of information generated by the path of output. In other words, the agent needs to filter out through Bayesian updating the unobservable components $\theta = (x, z)'$ by observing the history of output only. Proposition 1 provides the dynamics of the filtered state variables.

Proposition 1. *With respect to the agent's observation filtration, the dynamics of output C, and the filtered state variables $\widehat{x}_t$, and $\widehat{z}_t$ satisfy*

$$d\log C_t = (\mu + \widehat{x}_t - \lambda_z \widehat{z}_t)dt + \sqrt{v}d\widehat{B}_t, , \tag{6}$$

$$d\widehat{x}_t = -\lambda_x \widehat{x}_t dt + \widehat{\sigma}_{x,t} d\widehat{B}_t, \tag{7}$$

$$d\widehat{z}_t = -\lambda_z \widehat{z}_t dt + \widehat{\sigma}_{z,t} d\widehat{B}_t. \tag{8}$$

where $\widehat{x}_t \equiv \mathbb{E}\left[x_t \,|\, \mathcal{F}_t^o\right]$, $\widehat{z}_t \equiv \mathbb{E}\left[z_t \,|\, \mathcal{F}_t^o\right]$, $\widehat{B}_t$ is an $\mathcal{F}_t^o$-Brownian motion, and

$$\widehat{\sigma}_{x,t} = \frac{\gamma_{x,t} - \lambda_z \gamma_{xz,t}}{\sqrt{v}}, \qquad \widehat{\sigma}_{z,t} = \frac{\sigma_z^2 + \gamma_{xz,t} - \lambda_z \gamma_{z,t}}{\sqrt{v}}.$$

The posterior variance-covariance matrix Γ_t is defined as follows:

$$\Gamma_t \equiv \begin{pmatrix} \gamma_{x,t} & \gamma_{xz,t} \\ \gamma_{xz,t} & \gamma_{z,t} \end{pmatrix} = \begin{pmatrix} \mathrm{Var}\left[x_t \,|\, \mathcal{F}_t^o\right] & \mathrm{Cov}\left[x_t, z_t \,|\, \mathcal{F}_t^o\right] \\ \mathrm{Cov}\left[x_t, z_t \,|\, \mathcal{F}_t^o\right] & \mathrm{Var}\left[z_t \,|\, \mathcal{F}_t^o\right] \end{pmatrix} \tag{9}$$

and its elements satisfy

$$\frac{d\gamma_{x,t}}{dt} = \sigma_x^2 - 2\lambda_x\gamma_{x,t} - v^{-1}\left(\gamma_{x,t} - \lambda_z\gamma_{xz,t}\right)^2, \tag{10}$$

$$\frac{d\gamma_{z,t}}{dt} = \sigma_z^2 - 2\lambda_z\gamma_{z,t} - v^{-1}\left(\sigma_z^2 - \lambda_z\gamma_{z,t} + \gamma_{xz,t}\right)^2, \tag{11}$$

$$\frac{d\gamma_{xz,t}}{dt} = -\left(\lambda_x + \lambda_z\right)\gamma_{xz,t} - v^{-1}\left(\gamma_{x,t} - \lambda_z\gamma_{xz,t}\right)\left(\sigma_z^2 - \lambda_z\gamma_{z,t} + \gamma_{xz,t}\right). \tag{12}$$

Proof. See Appendix B.1.

Equation (6) gives the dynamics of log-output, $\log C_t$, projected on the observable filtration, while Equations (8) and (7) describe the agent's updating rule of the expectation of the latent state variables x_t, and z_t. We refer to $\widehat{x}_t$ and $\widehat{z}_t$ as the filter estimates. Equations (10), (11), and (12) provide the dynamics of the posterior variance-covariance matrix (9) and hence capture the evolution of uncertainty associated with the estimation of the unobserved components.

Note that the posterior variance-covariance matrix is a deterministic function of time. In accord with the literature (e.g., Scheinkman and Xiong, 2003; Dumas, Kurshev, and Uppal, 2009), we replace Γ_t with its steady-state (i.e., $\Gamma \equiv \lim_{t\to\infty}\Gamma_t$). That is, we assume that the agent has already observed a long enough history of output paths to reach the most precise variance estimate of the unobserved components. We will use $\widehat{\sigma}_x$ and $\widehat{\sigma}_z$ to denote the corresponding steady-state volatilities of the two filter estimates. The steady-state volatilities $\widehat{\sigma}_x$ and $\widehat{\sigma}_z$ are characterized in Appendix B.1.

2.3 Timing of Growth Risk

The goal here is to study how growth risk varies across different horizons. To this end, we follow Belo et al. (2015) and Marfè (2017) and compute an annualized measure of output growth volatility under the full filtration $\mathbf{F}$:

$$\sigma_C(t,\tau) = \sqrt{\frac{1}{\tau}\log\left(\frac{\mathbb{E}_t[C_{t+\tau}^2 \mid \mathcal{F}_t]}{\mathbb{E}_t[C_{t+\tau} \mid \mathcal{F}_t]^2}\right)}, \tag{13}$$

or under the observation filtration $\mathbf{F}^o$:

$$\widehat{\sigma}_C(t,\tau) = \sqrt{\frac{1}{\tau}\log\left(\frac{\mathbb{E}_t[C_{t+\tau}^2 \mid \mathcal{F}_t^o]}{\mathbb{E}_t[C_{t+\tau} \mid \mathcal{F}_t^o]^2}\right)}, \tag{14}$$

where τ denotes the horizon.

To compare risk across horizons, we also look at the corresponding term structures of variance ratios:

$$VR_C(t,\tau) = \frac{\sigma_C^2(t,\tau)}{\sigma_C^2(t,1)} \qquad \text{and} \qquad \widehat{VR}_C(t,\tau) = \frac{\widehat{\sigma}_C^2(t,\tau)}{\widehat{\sigma}_C^2(t,1)},$$

with reference of one year.

We study how learning affects the perception of risk across horizons, and then, how learning alters the shape of the term structures of growth risk. To provide a clear intuition, our analysis focuses first on the simplified models with either a permanent shock only or a transitory shock only, and then considers the general case that accounts for both permanent and transitory shocks.

2.3.1 The Case of Permanent Shocks Only

In this subsection, we assume that output is an integrated process with its drift driven by the process x only. That is, $d\log C_t = dy_t$.

Under the full filtration $\mathbf{F}$, the term structure of risk is monotone increasing. The more volatile (i.e., the larger $\sigma_x > 0$) or the more persistent (i.e., the smaller $\lambda_x > 0$) the drift x, the higher the level of the term structure of growth risk. The same properties are also shared by the term structure of risk under the partial information filtration $\mathbf{F}^o$. The aforementioned results are formalized in Proposition 2 below.

Proposition 2. *The following properties hold for any horizon $\tau > 0$:*

$$
\begin{aligned}
\partial_\tau \sigma_C(t,\tau) &> 0, & \partial_\tau \widehat{\sigma}_C(t,\tau) &> 0, \\
\partial_{\sigma_x} \sigma_C(t,\tau) &> 0, & \partial_{\sigma_x} \widehat{\sigma}_C(t,\tau) &> 0, \\
\partial_{\lambda_x} \sigma_C(t,\tau) &< 0, & \partial_{\lambda_x} \widehat{\sigma}_C(t,\tau) &< 0.
\end{aligned}
$$

Proof. See Appendix B.2.

These results obtain because x is the instantaneous drift of the integrated process y. Fluctuations in x accumulate over time and contribute to the integrated path of y. Thus, the longer the horizon, the larger the accumulated variation of x and so the larger the variance of y.

The same reasoning applies to the term structure of growth risk under the observation filtration. However, the term structures under $\mathbf{F}$ and $\mathbf{F}^o$ are not equal. They are both increasing and share the same short-run and long-run limits but the output growth variance perceived by the agent under the partial information filtration is larger than that under the full information filtration. The difference is a hump-shaped function of the horizon. These results are formalized in Proposition 3 below.

Proposition 3. *The following properties hold:*

$$
\begin{aligned}
\lim_{\tau\to 0} \sigma_C^2(t,\tau) &= \lim_{\tau\to 0} \widehat{\sigma}_C^2(t,\tau) = \sigma_y^2, \\
\lim_{\tau\to \infty} \sigma_C^2(t,\tau) &= \lim_{\tau\to \infty} \widehat{\sigma}_C^2(t,\tau) = \sigma_y^2 + \frac{\sigma_x^2}{\lambda_x^2}.
\end{aligned}
\tag{15}
$$

Moreover, for any finite horizon $\tau > 0$:

$$
\widehat{\sigma}_C^2(t,\tau) - \sigma_C^2(t,\tau) > 0,
$$

and

$$\partial_\tau \left(\widehat{\sigma}_C^2(t,\tau) - \sigma_C^2(t,\tau)\right) \begin{cases} > 0 & \tau < \overline{\tau}_x, \\ < 0 & \tau > \overline{\tau}_x, \end{cases} \tag{16}$$

where

$$\overline{\tau}_x \equiv -\frac{1}{2\lambda_x}\left(1 + 2\mathcal{L}(-1, -\frac{1}{2\sqrt{e}})\right) > 0$$

and $\mathcal{L}(k,z)$ is k-th solution of the Lambert-W (or product logarithm) function.

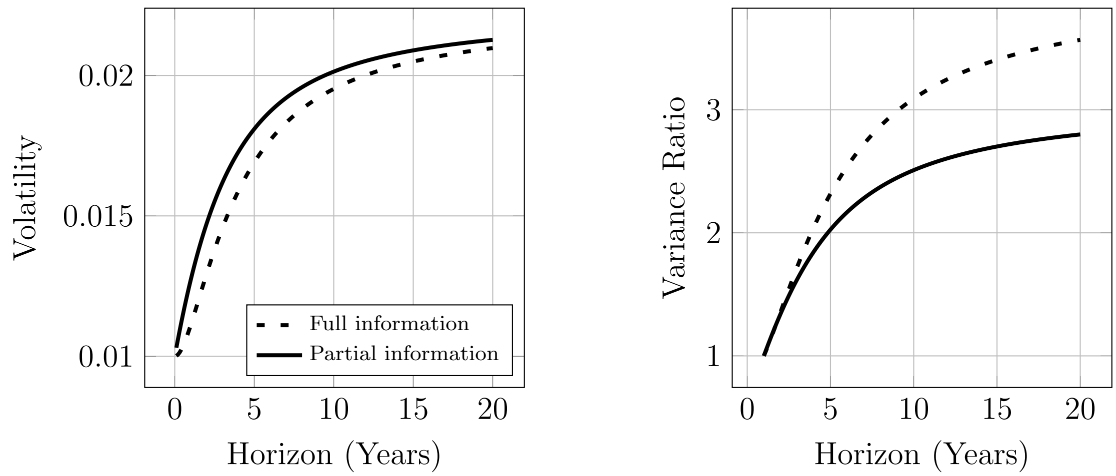
Proof. See Appendix B.3.

Why does the term structure of growth risk under the partial information filtration lie above that obtained under the full information filtration? Because the agent observes only the level of output, uncertainty is generated by a unique Brownian motion under her observation filtration. As a result, the filtered variables are driven by a unique Brownian motion, and hence, are instantaneously perfectly correlated. Positive (negative) shocks to the level y are perceived to come together with positive (negative) shocks to its expected growth x. Such positive correlation increases $\widehat{\sigma}_C^2(t,\tau)$ relative to $\sigma_C^2(t,\tau)$ because y and x are instead uncorrelated under $\mathbf{F}$.

Moreover, the difference between the perceived and true growth risk is a hump shaped function of the horizon: It increases up to a threshold $\overline{\tau}_x$, and decreases afterwards. At the horizon $\overline{\tau}_x$, the divergence between the agent's perception of growth risk under the full and partial information is maximal. We note that the threshold $\overline{\tau}_x$ is decreasing in λ_x. Consequently, the difference between the perceived and the true growth risk increases for a longer horizon when the persistence of x is high, or in other words, when the mean-reversion speed λ_x is low.

Figure 1 illustrates the term structure of growth risk in the model with only permanent shocks.

Figure 1: Term Structure of Growth Risk with Permanent Shocks Only.
Parameter values are $\mu = 0.025$, $\sigma_y = 0.01$, $\lambda_x = 0.5$, $\sigma_x = 0.01$.



2.3.2 The Case of Transitory Shocks Only

In this subsection, we assume that the drift of output is only driven by transitory shocks z. That is, $d \log C_t = dy_t + dz_t$, where $dy_t = \mu dt + \sigma_y dB_{y,t}$.

Both under the full and partial information filtrations, the term structure of growth risk is monotone decreasing. The reason is that shocks to z affect output in a transitory way; as time passes, the impact of these shocks on output weakens. Therefore, risk is higher in the short term than in the long term. The more volatile (i.e., the larger $\sigma_z > 0$) or the more persistent (i.e., the smaller $\lambda_z > 0$) the process z, the higher is the level of the term structure of growth risk. These results are summarized in Proposition 4 below.

Proposition 4. *The following properties hold for any horizon $\tau > 0$:*

$$
\begin{aligned}
&\partial_\tau \sigma_C(t,\tau) < 0, && \partial_\tau \widehat{\sigma}_C(t,\tau) < 0,\\
&\partial_{\sigma_z} \sigma_C(t,\tau) > 0, && \partial_{\sigma_z} \widehat{\sigma}_C(t,\tau) > 0,\\
&\partial_{\lambda_z} \sigma_C(t,\tau) < 0, && \partial_{\lambda_z} \widehat{\sigma}_C(t,\tau) < 0.
\end{aligned}
$$

Proof. See Appendix B.4.

Similar to the case of permanent shocks only, the term structures of growth risk under **F** and $\mathbf{F}^o$ are not equal. At any finite horizon, the risk perceived under the partial information filtration is higher. In addition, the difference between the partial and full information term structures is a hump-shaped function of the horizon. These results are summarized in Proposition 5 below.

Proposition 5. *The following properties hold:*

$$\begin{aligned} \lim_{\tau\to 0} \sigma_C^2(t,\tau) &= \lim_{\tau\to 0} \widehat{\sigma}_C^2(t,\tau) &= \sigma_y^2 + \sigma_z^2, \\ \lim_{\tau\to \infty} \sigma_C^2(t,\tau) &= \lim_{\tau\to \infty} \widehat{\sigma}_C^2(t,\tau) &= \sigma_y^2. \end{aligned} \tag{17}$$

Moreover, for any finite horizon $\tau > 0$:

$$\widehat{\sigma}_C^2(t,\tau) - \sigma_C^2(t,\tau) > 0,$$

and

$$\partial_\tau \left(\widehat{\sigma}_C^2(t,\tau) - \sigma_C^2(t,\tau)\right) \begin{cases} > 0 & \tau < \overline{\tau}_z, \\ < 0 & \tau > \overline{\tau}_z, \end{cases} \tag{18}$$

where

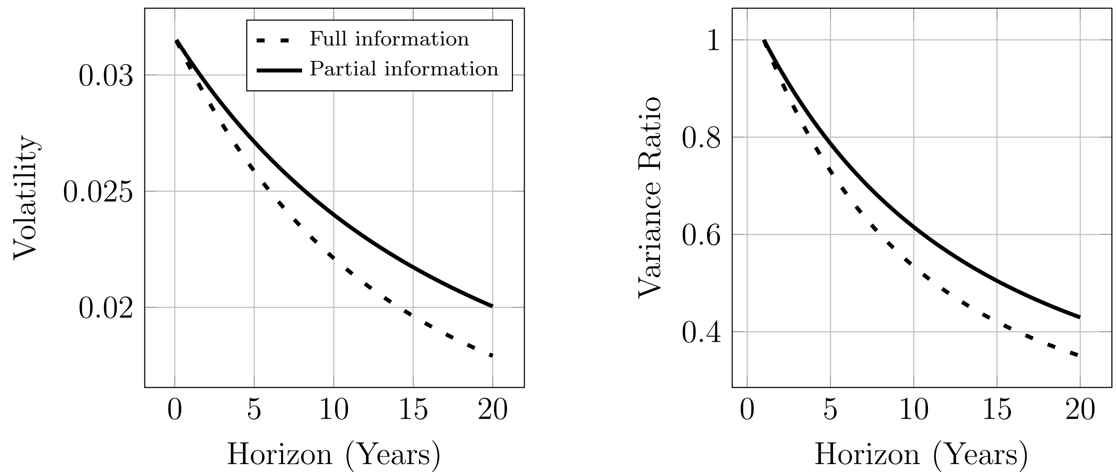
$$\overline{\tau}_z \equiv -\frac{1}{2\lambda_z}\left(1 + 2\mathcal{L}(-1, -\frac{1}{2\sqrt{e}})\right) > 0$$

and $\mathcal{L}(k,z)$ is k-th solution of the Lambert-W function.

Proof. See Appendix B.5.

Figure 2 illustrates the term structure of growth risk in the model with only transitory shocks.

Figure 2: Term Structure of Growth Risk with Transitory Shocks Only.
Parameter values are $\mu = 0.025$, $\sigma_y = 0.01$, $\lambda_z = 0.1$, $\sigma_z = 0.03$.



2.3.3 The Case of Permanent and Transitory Shocks

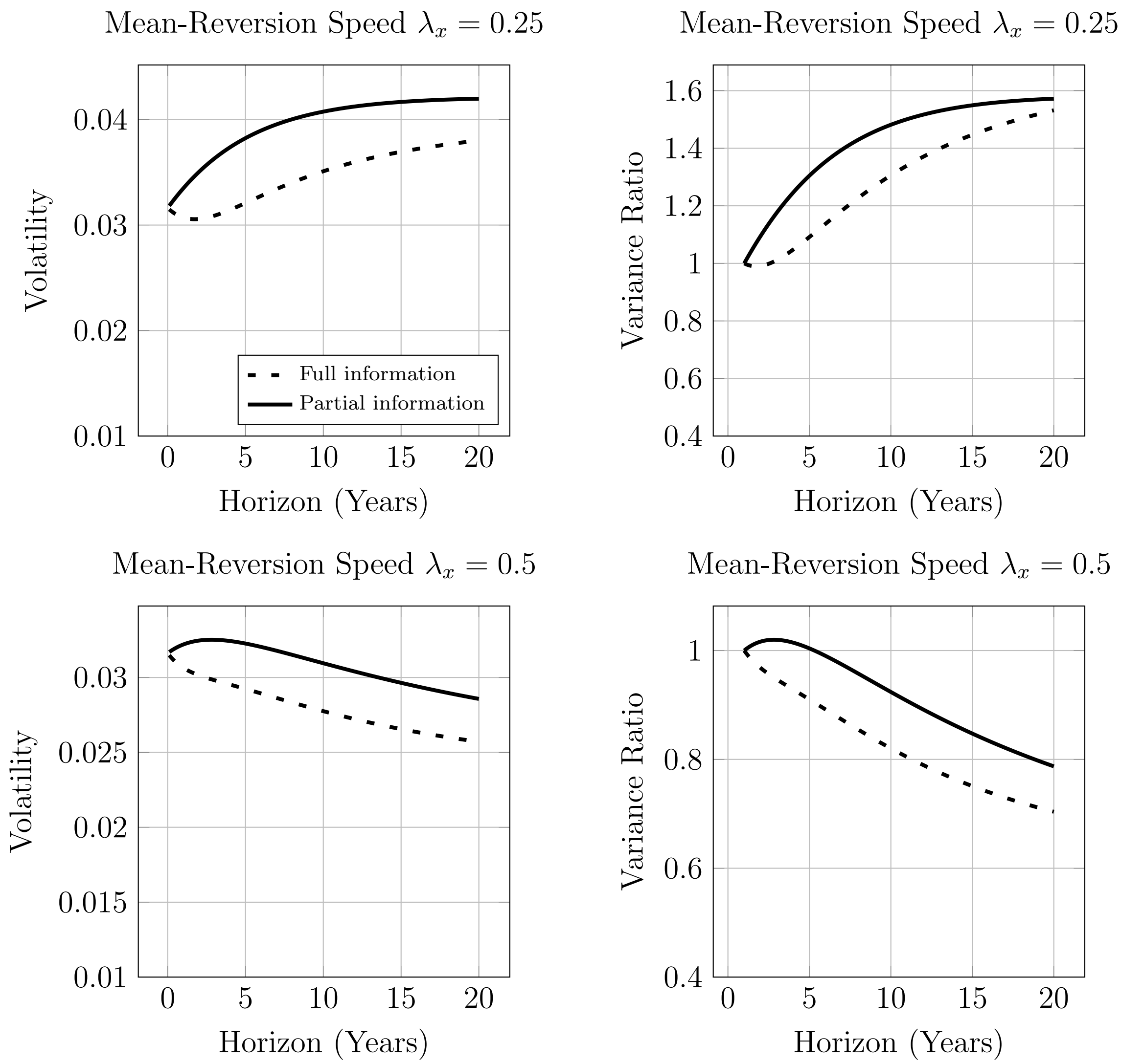
When aggregate output is driven by both permanent and transitory shocks, the term structure of growth risk can be either increasing or decreasing depending on which of the two shocks dominates. The limits of the annualized variance of output growth at short and long horizon are respectively

$$\begin{aligned}
\lim_{\tau\to 0} \sigma_C^2(t,\tau) &= \lim_{\tau\to 0} \widehat{\sigma}_C^2(t,\tau) = \sigma_y^2 + \sigma_z^2, \\
\lim_{\tau\to \infty} \sigma_C^2(t,\tau) &= \lim_{\tau\to \infty} \widehat{\sigma}_C^2(t,\tau) = \sigma_y^2 + \tfrac{\sigma_x^2}{\lambda_x^2}.
\end{aligned}$$

At the short end, growth risk is driven by the volatility of the transitory shock, σ_z. On the contrary, it is the volatility of the permanent shock (scaled by the mean-reversion speed), σ_x, that influences growth risk at the long end. The lower the mean-reversion speed of the permanent shock, λ_x, the higher the output growth volatility at the long end. Figure 3 illustrates the term structures of growth risk in the model with both permanent and transitory shocks for different values of the mean-reversion speed λ_x.

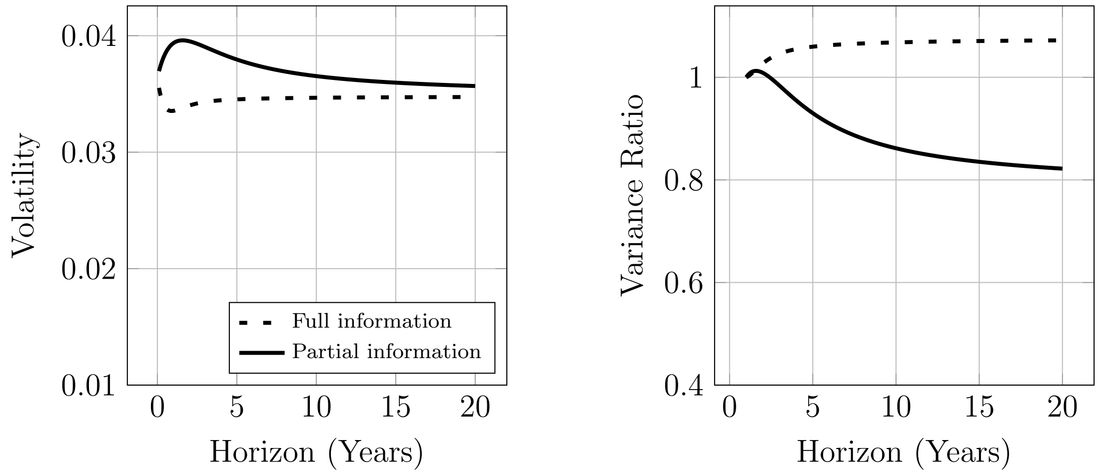
Importantly, in the economy with both permanent and transitory shocks, partial information and learning can alter the shape of the term structure of growth risk perceived by the agent. Figure 4 illustrates the case when the true term structure of growth risk is decreasing

Figure 3: Term Structure of Growth Risk with Permanent and Transitory Shocks.
Parameter values are $\mu = 0.025$, $\sigma_y = 0.01$, $\sigma_x = 0.01$, $\lambda_z = 0.1$, $\sigma_z = 0.03$.



up to one year and then flat. However the agent, unable to directly observe the shocks that drive output, perceives the term structure as increasing for up to one year and decreasing afterwards.

Figure 4: Term Structure of Growth Risk with Permanent and Transitory Shocks.
Parameter values are $\mu = 0.025$, $\sigma_y = 0.01$, $\lambda_x = 1.5$, $\sigma_x = 0.05$, $\lambda_z = 0.6$, $\sigma_z = 0.035$.



3 Asset Pricing

In this section we study the role of learning in the context of a general equilibrium asset pricing model. We focus on the equilibrium term structures of dividend strips and interest rates. We compare the shape of these term structures derived under partial information to those in the full information economy.

We consider an endowment economy (Lucas, 1978) in which the output process follows the dynamics in (1). In equilibrium, the representative agent's aggregate consumption is equal to the output. As we have seen in Section 2, the permanent shock x induces an upward-sloping effect on the term structure of growth risk, whereas the transitory shock z induces a downward-sloping effect. The two shocks jointly allow for a flexible shape of the term structure of growth risk.

The representative agent features recursive preferences in the spirit of Kreps and Porteus (1979), Epstein and Zin (1989), Weil (1989), and Duffie and Epstein (1992). These preferences allow for the separation between the elasticity of intertemporal substitution and the coefficient of relative risk aversion. Given a consumption process C, the utility at time t is defined as

$$U_t \equiv \left[(1-\delta^{dt})C_t^{\frac{1-\gamma}{\theta}} + \delta^{dt}\mathbb{E}_t\left[U_{t+dt}^{1-\gamma}\,|\,\mathcal{F}_t^o\right]^{\frac{1}{\theta}}\right]^{\frac{\theta}{1-\gamma}},$$

where δ is the time discount factor, γ is the coefficient of risk aversion, ψ is the elasticity of intertemporal substitution, and $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$.

Note that expectations are taken under the observation filtration $\mathbf{F}^o$. Thus, the dynamics of aggregate consumption depend on the filter estimates $\widehat{x}$ and $\widehat{z}$, as provided in equations (6)–(8). The only source of uncertainty is the $\mathcal{F}_t^o$–Brownian motion $\widehat{B}_t$.

In order to derive the price of dividend strips and equity, we assume a simple dynamics for dividends. In line with the existing literature (e.g., Abel, 1999; Bansal and Yaron, 2004) we define dividends as levered consumption:

$$D_t = e^{-\beta_d t}C_t^{\phi},$$

where $\phi \geq 1$ is the leverage parameter and β_d is a parameter that determines the growth rate of dividends. Note that we do not need to alter the learning problem of the agent; Observing the dividend process does not bring any additional information compared to observing only the path of consumption. This is because the dividend process is a deterministic function of consumption.

Recursive preferences lead to a non-affine state-price density. Therefore, to solve for prices and preserve analytic tractability, we follow the methodology presented by Eraker and Shaliastovich (2008), which is based on the Campbell and Shiller (1988) log-linearization. The discrete time (continuously compounded) log-return on aggregate wealth W (e.g., the claim on the aggregate consumption stream $\{C_t\}_{t\geq 0}$) can be expressed as

$$\log R_{t+1} = \log\frac{W_{t+1}+C_{t+1}}{W_t} = \log\left(e^{wc_{t+1}}+1\right) - wc_t + \log\frac{C_{t+1}}{C_t},$$

where $wc \equiv \log(W/C)$. A log-linearization of the first summand around the mean log wealth-consumption ratio leads to

$$\log R_{t+1} \approx k_0 + k_1 wc_{t+1} - wc_t + \log \frac{C_{t+1}}{C_t},$$

where the endogenous constants k_0 and k_1 satisfy

$$k_0 = -\log\left((1-k_1)^{1-k_1} k_1^{k_1}\right) \qquad \text{and} \qquad k_1 = e^{\mathbb{E}(wc_t|\mathcal{F}^o)} / \left(1 + e^{\mathbb{E}(wc_t|\mathcal{F}^o)}\right).$$

Campbell, Lo, and MacKinlay (1997) and Bansal, Kiku, and Yaron (2012) provide evidence of the high accuracy of this log-linearization, which we assume exact hereafter. We follow Eraker and Shaliastovich (2008) and consider the continuous time counterpart defined as:

$$d\log R_t = k_0 dt + k_1 d(wc_t) - (1-k_1) wc_t dt + d\log C_t. \tag{19}$$

Recursive preferences lead to the following Euler equation, which enables us to characterize the state-price density, M, that prices any asset in the economy:

$$\mathbb{E}\left[\exp\left(\log \frac{M_{t+\tau}}{M_t} + \int_t^{t+\tau} d\log R_s\right) \mid \mathcal{F}_t^o\right] = 1.$$

The state-price density satisfies

$$d\log M_t = \theta \log \delta dt - \frac{\theta}{\psi} d\log C_t - (1-\theta) d\log R_t.$$

Proposition 6 below characterizes the state-price density, the risk-free rate, and the price of risk in our economy.

Proposition 6. *The equilibrium state-price density has dynamics given by*

$$\frac{dM_t}{M_t} = -r_t dt - \Lambda d\widehat{B}_t,$$

where the risk-free rate satisfies

$$r_t = r_0 + r_{\widehat{x}}\widehat{x}_t + r_{\widehat{z}}\widehat{z}_t,$$

with

$$\begin{aligned}
r_0 &= -\frac{1-\gamma}{1-1/\psi}\log\delta + \frac{1/\psi-\gamma}{1-1/\psi}\log k_1 + \gamma\mu - \frac{1}{2}\Theta\left(\widehat{\sigma}_y, \widehat{\sigma}_x, \widehat{\sigma}_z\right), \\
r_{\widehat{x}} &= \frac{1}{\psi}, \\
r_{\widehat{z}} &= -\frac{\lambda_z}{\psi},
\end{aligned}$$

and the market price of risk equals

$$\Lambda = \gamma\widehat{\sigma}_y + \left(\frac{(\gamma - 1/\psi)}{1/k_1 - (1-\lambda_x)}\right)\widehat{\sigma}_x + \left(\gamma - \frac{\lambda_z(\gamma-1/\psi)}{1/k_1-(1-\lambda_z)}\right)\widehat{\sigma}_z,$$

where $\widehat{\sigma}_y \equiv \sqrt{v} - \widehat{\sigma}_z$, *and* $\widehat{\sigma}_x$, $\widehat{\sigma}_z$ *are defined in Appendix B.1.* $\Theta\left(\widehat{\sigma}_y, \widehat{\sigma}_x, \widehat{\sigma}_z\right)$ *is defined in Appendix B.6.*

Proof. See Appendix B.6.

Proposition 7 below characterizes the zero-coupon bond price and yield.

Proposition 7. *The equilibrium price of the zero-coupon bond with time to maturity* τ *is given by*

$$B(t,\tau) = \mathbb{E}\left[\frac{M_{t+\tau}}{M_t} \mid \mathcal{F}_t^o\right] = e^{q_0(\tau) + q_{\widehat{x}}(\tau)\widehat{x}_t + q_{\widehat{z}}(\tau)\widehat{z}_t},$$

where $q_0(\tau)$ is derived in Appendix B.7

$$
\begin{aligned}
q_{\widehat{x}}(\tau) &= -\frac{1}{\lambda_x \psi}\left(1 - e^{-\lambda_x \tau}\right), \\
q_{\widehat{z}}(\tau) &= \frac{1}{\psi}\left(1 - e^{-\lambda_z \tau}\right).
\end{aligned}
$$

The yield to maturity τ is defined as $YTM(t,\tau) = -(1/\tau)\log B(t,\tau)$.

Proof. See Appendix B.7.

Proposition 8 below characterizes the dividend strip price and its return moments.

Proposition 8. *The equilibrium price of the dividend strip with time to maturity τ is given by*

$$
S(t,\tau) = \mathbb{E}\left[\frac{M_{t+\tau}}{M_t} D_{t+\tau} \,\middle|\, \mathcal{F}_t^o\right] = e^{-\beta_d t + \phi \widehat{y}_t + w_0(\tau) + w_{\widehat{x}}(\tau)\widehat{x}_t + w_{\widehat{z}}(\tau)\widehat{z}_t},
$$

where $w_0(\tau)$ is derived in Appendix B.8 and

$$
\begin{aligned}
w_{\widehat{x}}(\tau) &= \frac{1}{\lambda_x \psi}(1 - e^{-\lambda_x \tau})(\phi\psi - 1) \\
w_{\widehat{z}}(\tau) &= \frac{1}{\psi}\big(1 - e^{-\lambda_z \tau}(1 - \phi\psi)\big).
\end{aligned}
$$

The return premium of the dividend strip with time to maturity τ is given by

$$
RP(t,\tau) = -\frac{1}{dt}\langle \frac{dM_t}{M_t}, \frac{dS(t,\tau)}{S(t,\tau)} \rangle = \left(\phi \widehat{\sigma}_y + w_{\widehat{x}}(\tau)\widehat{\sigma}_x + w_{\widehat{z}}(\tau)\widehat{\sigma}_z\right)\Lambda.
$$

The return volatility of the dividend strip with time to maturity τ is given by

$$
Vol(t,\tau) = \sqrt{\frac{1}{dt}\langle \frac{dS(t,\tau)}{S(t,\tau)} \rangle} = \left|\phi \widehat{\sigma}_y + w_{\widehat{x}}(\tau)\widehat{\sigma}_x + w_{\widehat{z}}(\tau)\widehat{\sigma}_z\right|.
$$

Proof. See Appendix B.8.

The log return on equity, $\log R^e$, is defined in a similar way as the return on aggregate wealth in (19):

$$d \log R_t^e = k_{0,d}dt + k_{1,d}d(pd_t) - (1 - k_{1,d})pd_t dt + d \log D_t,$$

where $pd_t \equiv \log P_t/D_t$ and $k_{0,d}, k_{1,d}$ are endogenous constants. The equity price can be approximated as an exponential affine function of the state variables. Proposition 9 below characterizes the equity price and its return moments.

Proposition 9. *The equilibrium price of equity is given by*

$$P_t = \int_0^{\infty} \mathbb{E}_t \left[\frac{M_{t+\tau}}{M_t} D_{t+\tau} \,\middle|\, \mathcal{F}_t^o \right] d\tau \approx \; D_t e^{A_d + B_{\widehat{x},d}\widehat{x}_t + B_{\widehat{z},d}\widehat{z}_t},$$

where A_d is derived in Appendix B.9 and

$$\begin{aligned} B_{\widehat{x},d} &= \frac{\phi - 1/\psi}{1 - k_{1,d}(1 - \lambda_x)}, \\ B_{\widehat{z},d} &= -\frac{\lambda_z(\phi - 1/\psi)}{1 - k_{1,d}(1 - \lambda_z)}. \end{aligned}$$

The equity risk premium is given by

$$RP(t) = -\frac{1}{dt}\left\langle \frac{dM_t}{M_t}, \frac{dP_t}{P_t} \right\rangle = \left(\phi\widehat{\sigma}_y + B_{\widehat{x},d}\widehat{\sigma}_x + (\phi + B_{\widehat{z},d})\,\widehat{\sigma}_z\right)\Lambda.$$

The equity return volatility is given by

$$Vol(t) = \sqrt{\frac{1}{dt}\left\langle \frac{dP_t}{P_t} \right\rangle} = \left|\phi\widehat{\sigma}_y + B_{\widehat{x},d}\widehat{\sigma}_x + (\phi + B_{\widehat{z},d})\,\widehat{\sigma}_z\right|.$$

Proof. See Appendix B.9.

4 Results

In this section we calibrate the model, provide empirical support for it, and study its asset-pricing predictions on the level and timing of both equity return moments and interest rates.

4.1 Model Calibration and Empirical Support

In order to understand the role of learning on the term structures of both equity return moments and interest rates, it is important to parametrize the output dynamics so that it matches the empirical properties of growth risk across different horizons. We calibrate the output dynamics parameters as follows. First, we match the short term (e.g., one-year) empirical level of volatility. Second, we match the empirical observation that output growth volatility is flat across horizons. Evidence of this empirical observation is provided in Hasler and Marfè (2016) and Marfè (2017), who document that the variance ratios of output growth rates in the U.S. are approximately flat around unity. Also, Dew-Becker (2017) documents that robust estimators of long-run growth volatility are very close to estimates of the one-year volatility.

Therefore, we set the output growth parameter $\mu = 2.5\%$ and obtain the other output dynamics parameters $\Theta = \{\sigma_y, \sigma_x, \lambda_x, \sigma_z, \lambda_z\}$ by minimizing the following objective:

$$\Theta^{\star} = \arg\min \left\{ \left[\sigma_C(t,1) - 3\%\right]^2 + \alpha \int_0^{50} \left[VR_C(t,\tau) - 1\right]^2 d\tau \right\}$$

where α is a weighting constant. This minimization procedure yields: $\sigma_y = 0.0002$, $\sigma_x = 0.040$, $\lambda_x = 1.346$, $\sigma_z = 0.033$, and $\lambda_z = 0.549$.

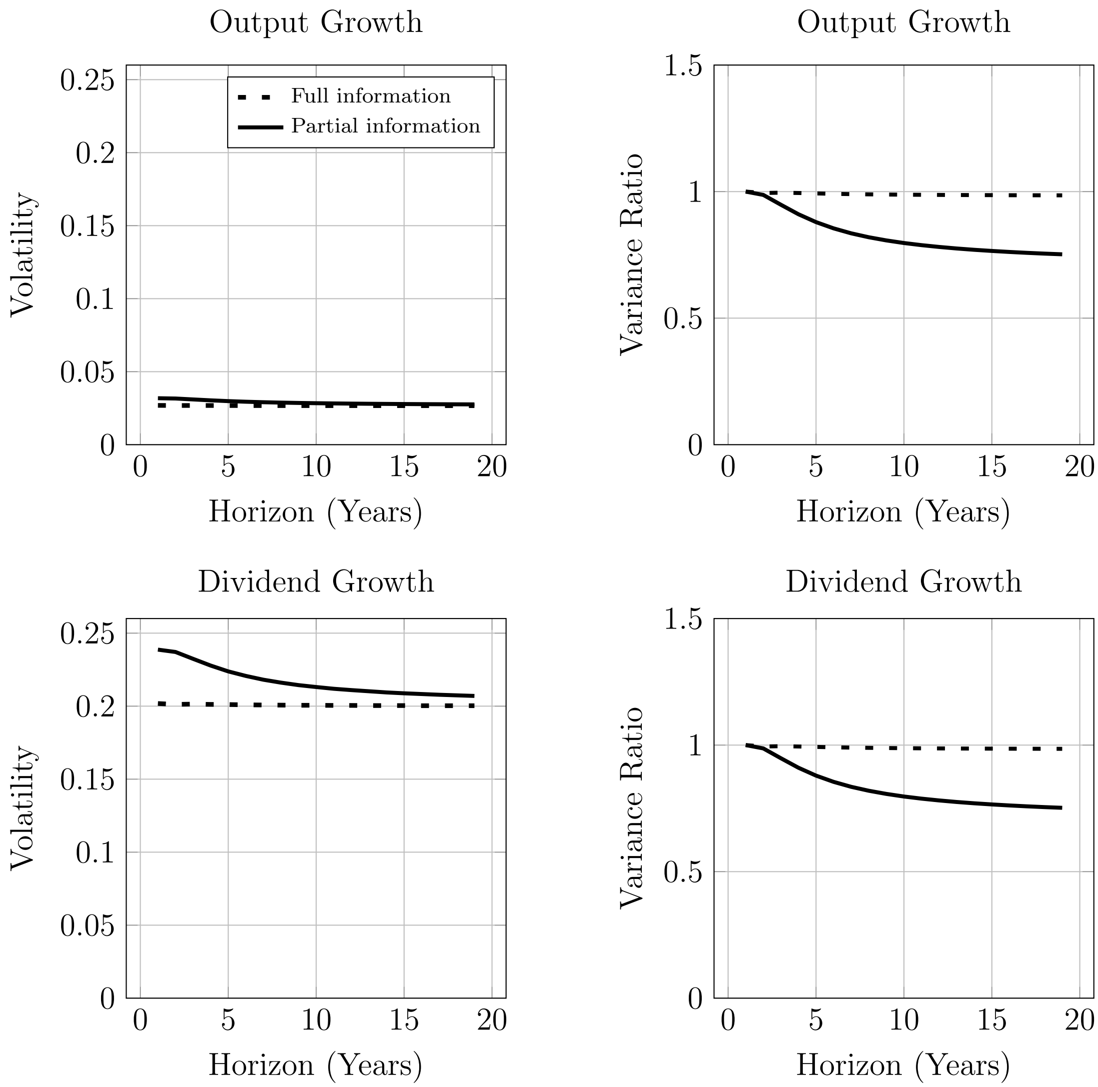
In addition we set $\beta_d = \mu(\phi - 1)$ such that the long-run growth rate of dividends equals that of output. The parameter ϕ captures the excessive volatility of dividends relative to output. We use $\phi = 7.5$ to approximately match the one-year volatility of shareholders' remuneration in the U.S., which is about 20% (see Belo et al. (2015)). Therefore, we use the label dividend with slight abuse of terminology, as we actually consider the more appropriate full shareholders' remuneration consisting of dividends plus net repurchases. Finally, we choose a relative risk aversion, $\gamma = 7.5$, an elasticity of intertemporal substitution, $\psi = 1.5$, and a time discount factor, $\delta = 0.99$.

Figure 5 shows the calibrated model-implied term structures of volatility and variance ratios for both output growth and dividend growth. Under full information, the term structure of output growth variance ratios is about flat because it was calibrated accordingly. This shows that the model dynamics are flexible enough to match the observed term structure of output growth variance ratios together with a 3% output growth volatility. Note that these flat variance ratios could also be obtained by assuming an i.i.d. process for output. However, the presence of the latent variables x_t and z_t in our model is key for asset pricing because the agent's estimates of these variables are priced risk factors in equilibrium. We note that, given our calibration, learning does not significantly affect the overall level of output growth volatility and dividend growth volatility. Indeed, the former lies in the interval 2.5-3%, while the latter lies in the interval 21-24%.

Importantly, learning alters the timing of output growth risk and dividend growth risk across different horizons. We observe that the variance ratios of both output growth and dividend growth are decreasing with the horizon under partial information. The reason is the following. The transitory process z is mean-reverting but not highly persistent under our calibration. This implies that the horizon at which the filtered volatility of z diverges the most from the true volatility is relatively short. As a result, long horizon variance ratios lie below unity under partial information. As shown in what follows, learning also affects the timing of equity risk premia via the impact of the priced risk factors $\widehat{x}$ and $\widehat{z}$.

We now perform an empirical exercise that provides support for the mechanism of the model. Our theoretical model predicts that empirically observable equilibrium outcomes are function of the latent factors filtered by the agent, $\widehat{x}$ and $\widehat{z}$. Two key equilibrium outcomes are the risk-free rate and the price-dividend ratio. At the same time, the filtered latent

Figure 5: Term Structures of Output Growth Risk and Dividend Growth Risk Calibrated to U.S. Data.



factors $\widehat{x}$ and $\widehat{z}$ are the drivers of the agent's belief about expected growth:

$$\frac{1}{dt}\mathbb{E}[d\log C_t|\mathcal{F}_t^o] = \mu + \widehat{x}_t - \lambda_z\,\widehat{z}_t.$$

We aim to verify whether the theoretical link between equilibrium outcomes and beliefs about expected growth finds support in U.S. data.

First, we use the time series of the risk-free rate and the S&P 500 log price-dividend ratio obtained from Robert Shiller's website to infer the model-implied time series of $\widehat{x}$ and $\widehat{z}$, as follows:[2]

$$\min_{\{\widehat{x}_t,\,\widehat{z}_t\}} \; \left|{r_t}^{\text{data}} - {r_t}^{\text{model}}\right|^2 + \left|{pd_t}^{\text{data}} - {pd_t}^{\text{model}}\right|^2, \quad \forall\, t.$$

Second, we construct the model-implied time series of expected growth (up to a constant), as follows:

$$\widehat{g}_t^{\;\text{model}} = \widehat{x}_t - \lambda_z\, \widehat{z}_t.$$

Third, we consider several measures to proxy for agents' beliefs about expected growth. Namely, we consider the mean and median forecasts of real GDP growth, industrial production growth, and corporate profits growth. Quarterly data available from Q4:1968 to Q2:2015 are obtained from the Survey of Professional Forecasters. We then regress these proxies of beliefs about expected growth on the model-implied time series, as follows:

$$\widehat{g}_t^{\;\text{survey}} = \alpha + \beta\, \widehat{g}_t^{\;\text{model}} + \epsilon_t.$$

Whether or not the data lend empirical support to the economic mechanism of the model depends on the sign and significance of the coefficient β. Indeed, if β is positive and significant, then the way the model predicts that beliefs about growth shape equilibrium prices is consistent with empirical evidence.

Table 1 reports the results from the regression. All of the six empirical measures of beliefs about expected growth are positively and significantly related to the model-implied beliefs inferred from actual prices. The coefficient β is statistically different from zero at the 1% level in all of the six cases and the R^2 ranges from 6% to 10%. Thus, we recover in actual data the model mechanism through which beliefs about growth drive prices.

[2]The risk-free rate and the log price-dividend ratio are affine in $\widehat{x}$ and $\widehat{z}$. Therefore, by observing the former we can infer the latter by simply solving a linear system.

Table 1: Model-Implied Growth Forecasts and Survey Data.
This table reports the regression coefficients, t-statistic, and R^2 from the regressions of several measures of expected growth forecasts on the model-implied expected growth:

$$\widehat{g}_t^{\text{ survey}} = \alpha + \beta\, \widehat{g}_t^{\text{ model}} + \epsilon_t.$$

The expected growth forecasts are either the cross-sectional mean or median from the Survey of Professional Forecasters (SPF) about real GDP, industrial production, and corporate profits. The data are quarterly from Q4:1968 to Q2:2015. The model-implied time series of expected growth is computed as

$$\widehat{g}_t^{\text{ model}} = \widehat{x}_t - \lambda_z\, \widehat{z}_t,$$

where $\widehat{x}_t$ and $\widehat{z}_t$ are extracted by minimizing the distance between the time series of the risk-free rate and the S&P 500 log price-dividend ratio obtained from Robert Shiller's website and their model-implied counterparts.

Growth Forecasts:	Gross Domestic Product		Industrial Production		Corporate Profits	
	mean	median	mean	median	mean	median
constant	0.029***	0.029***	0.035***	0.034***	0.094***	0.085***
t-stat	(13.88)	(13.65)	(8.31)	(8.09)	(8.53)	(7.89)
slope	0.133***	0.131***	0.243***	0.229***	0.762***	0.742***
t-stat	(4.06)	(3.90)	(3.64)	(3.40)	(4.35)	(4.33)
R^2 (%)	8.18	7.61	6.69	5.87	9.27	9.10

4.2 Term Structures

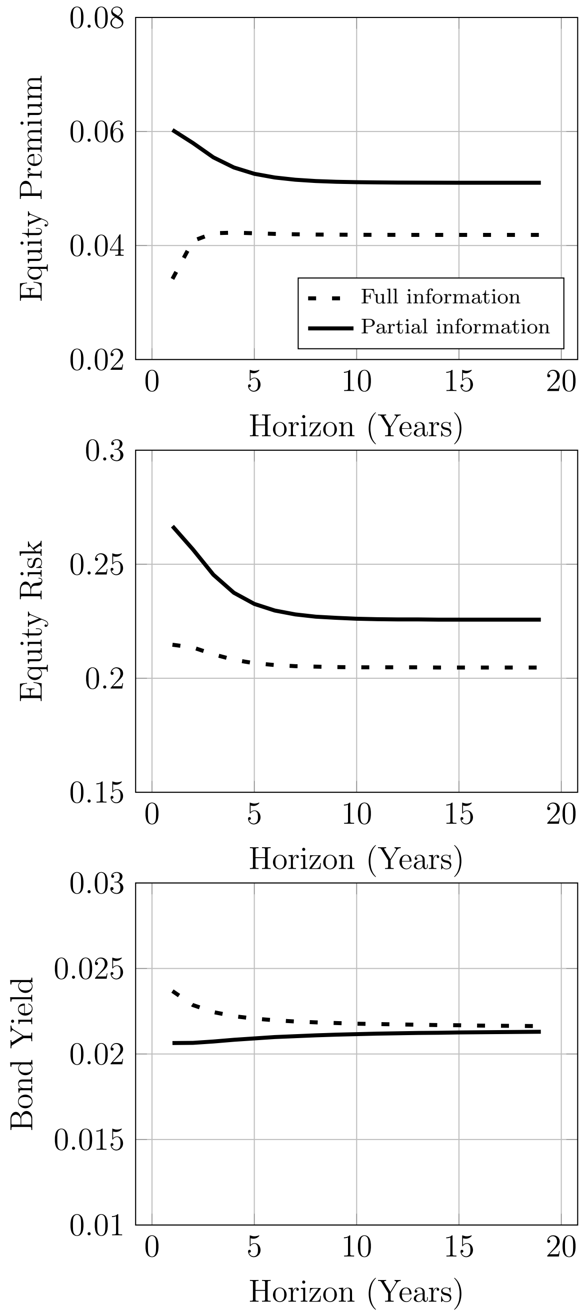
This section studies the term structures of equity risk, equity risk premia, and interest rates. The former are computed using the instantaneous return volatility and risk premium of dividend strips with different maturities. The latter is obtained by computing the yield of zero-coupon bonds with different maturities.

Figure 6 depicts the term structures under both full information and partial information with learning. The upper panel shows that, under full information, the risk premium for the dividend strip with short maturity (e.g. one-year) is close to zero. Risk premia increase sharply up to the three-year maturity and then are flat around a level of about 4.2%. As a consequence of learning, the behavior of the dividend strip risk premium is opposite under partial information. We note that the risk premium at short maturity (e.g. one-year) is about 6.2%. Risk premia decrease uniformly up to the ten-year maturity and then are almost flat around a level of about 5.0%. To summarize, the slope of the term structure of equity risk premia switches from positive in the full information economy to negative in the partial information economy with learning.

Since it is typically easier for investors to acquire accurate information in good times than in bad times (Bloom, 2009, 2014; Jurado et al., 2015), our economies with full and partial information can be interpreted as bounds of a single economy with counter-cyclical uncertainty. In good times uncertainty is low, and therefore investors are well informed. According to our model with full information, this predicts an upward-sloping term structure of risk premia. In bad times uncertainty is high, thereby forcing investors to actively learn. In this case, our model with partial information and learning predicts a downward-sloping term structure of risk premia. This reasoning shows that our model can explain why the slope of the term structure of equity risk premia is pro-cyclical in the data (van Binsbergen et al., 2013; Bansal et al., 2019).

The middle panel of Figure 6 depicts the dividend strip return volatility. Equity risk is downward-sloping under both full information and partial information with learning. In the

Figure 6: Term Structures of Equity Premia, Equity Risk, and Bond Yields.



latter case, the level of volatility is somewhat higher and the negative slope in the first five years is steeper than under full information. This result comes from the fact that the term structure of growth risk is downward-sloping under partial information (see Figure 5).

The lower panel of Figure 6 shows the zero-coupon bond yields. Under full information we observe that bond yields are downward-sloping, in contrast with the data. Under partial information we find that learning produces two results which make the model predictions conform with the data. First, the short-term bond yield is lower than under full information. Second, the slope of the term structure is positive, consistent with TIPS data. Indeed, between 2003 and 2018 the average yields on TIPS were 0.6%, 0.9%, 1.1%, and 1.4% at the 5-year, 7-year, 10-year, and 20-year maturities, respectively.

4.2.1 Why Does Learning Switch the Slope of the Term Structures?

Under full information the risk premium on the dividend strip with maturity τ is given by[3]

$$RP^{Full}(t,\tau) = \sum_{i=y,x,z} \Lambda_i \, \sigma_i \, w_i(\tau),$$

where $w_i(\tau) = \partial_i \log S(t,\tau)$, $i = \{y, x, z\}$. That is, the risk premium is a sum of price sensitivities ($w_i(\tau)$, the only terms depending on τ) weighted by the product of the fundamental volatilities (σ_i) and the corresponding prices of risk (Λ_i).

Under partial information the risk premium on the dividend strip with maturity τ is given by

$$RP^{Partial}(t,\tau) = \Lambda \sum_{i=\widehat{y},\widehat{x},\widehat{z}} \widehat{\sigma}_i \, w_i(\tau),$$

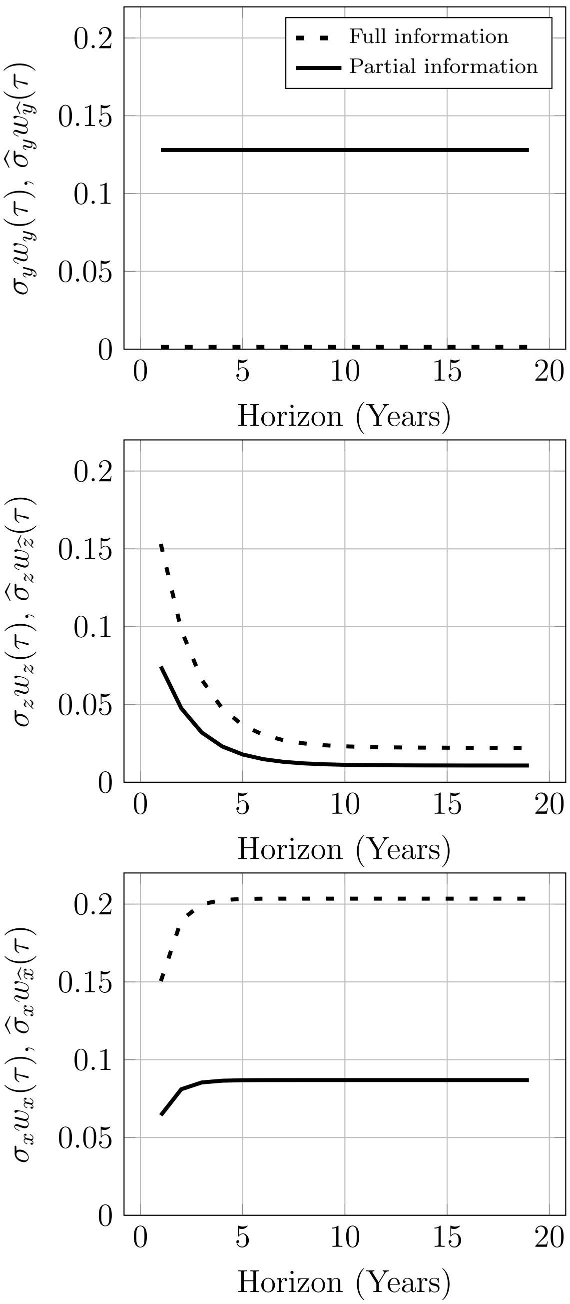
where $w_i(\tau) = \partial_i \log S(t,\tau)$, $i = \{\widehat{y}, \widehat{x}, \widehat{z}\}$. That is, the risk premium is the product of the unique price of risk in the economy (Λ) and a sum of price sensitivities ($w_i(\tau)$, the only terms depending on τ) weighted by the fundamental volatilities ($\widehat{\sigma}_i$).

Note that $w_y(\tau)$, $w_x(\tau)$, and $w_z(\tau)$ are respectively constant, increasing, and decreasing with the maturity τ. The same holds for $w_{\widehat{y}}(\tau)$, $w_{\widehat{x}}(\tau)$, and $w_{\widehat{z}}(\tau)$.

The positive slope of the term structure of equity risk premia under full information is due to the fact that the price of risk Λ_x, which rewards for bearing variations in x, is larger than the price of risk Λ_z, which rewards for bearing variations in z. Even if $w_z(\tau)$ is steeper

[3]Results pertaining to the full information model are provided in Appendix C.

Figure 7: Price Elasticities.



than $w_x(\tau)$, the price of risk Λ_x is large enough to dominate the impact of the transitory shock z. Therefore, the term structure of risk premia is upward-sloping. Note that such a positive slope obtains in the full information case, although the model has been calibrated to match the empirically observed flat term structure of growth risk.

Consider now the partial information case and the role of learning. While the processes y,

x, and z have independent increments, learning leads to an endogenous correlation structure between the dynamics of the filtered variables, $\widehat{y}$, $\widehat{x}$, and $\widehat{z}$. More precisely, learning implies perfect correlations among these filtered variables because the investor updates her beliefs by observing a single source of information (the history of output). In turn, the unique source of risk commands a unique price of risk Λ. Therefore, learning neutralizes the role of the prices of risk in determining the shape of the term structure. Hence, its slope is solely driven by the magnitude and the steepness of the price elasticities. In our case, the range of values taken by the price elasticity with respect to $\widehat{z}$ (i.e., $\widehat{\sigma}_z\, w_z(\tau)$) is substantially larger than the range of values taken by the price elasticity with respect to $\widehat{x}$ (i.e., $\widehat{\sigma}_x\, w_{\widehat{x}}(\tau)$). Formally, we have

$$\lim_{\tau\to 0} \widehat{\sigma}_z\, w_{\widehat{z}}(\tau) \; - \; \lim_{\tau\to\infty} \widehat{\sigma}_z\, w_{\widehat{z}}(\tau) \; > \; \lim_{\tau\to\infty} \widehat{\sigma}_x\, w_{\widehat{x}}(\tau) \; - \; \lim_{\tau\to 0} \widehat{\sigma}_x\, w_{\widehat{x}}(\tau)$$

when

$$\widehat{\sigma}_z > \frac{\widehat{\sigma}_x}{\lambda_x},$$

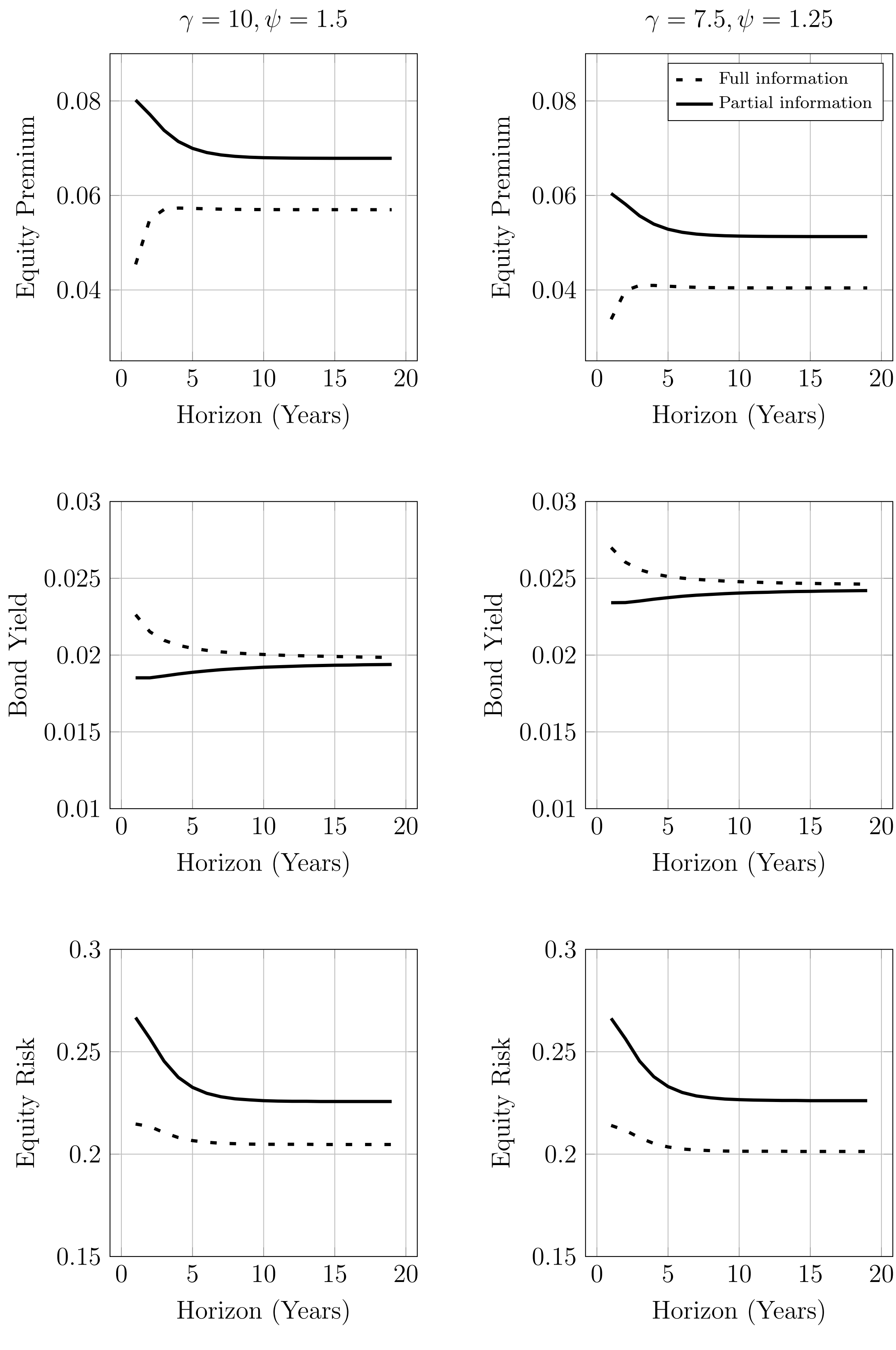
provided that $\phi > 1/\psi$. Therefore, learning leads to a downward-sloping term structure of equity risk premia via its impact on both the quantity of risk across horizons (driven by the perception of downward-sloping growth risk) and the price of risk (driven by the endogenous correlation structure among the filtered variables).

4.2.2 Robustness: The Impact of Preferences

For robustness, we investigate the impact of preference parameters on the term structures of equity risk premia, equity return volatility, and interest rates. We compare the baseline setting ($\gamma = 7.5, \psi = 1.5$) with two alternative parametrizations: ($\gamma = 10, \psi = 1.5$) and ($\gamma = 7.5, \psi = 1.25$). Note that, for each parametrization, the representative agent has a preference for early resolution of uncertainty.

Figure 8 depicts the term structures. An increase in relative risk aversion increases all the three prices of risk under full information as well as the unique price of risk under partial

Figure 8: The Impact of Preferences on the Term Structures.



information with learning. However, an increase in relative risk aversion does not alter the relative size of price elasticities. In turn, we observe an increase in the level of equity risk premia while preserving the sign switch of the slope of the term structure of equity risk premia implied by learning. That is, with a risk aversion $\gamma = 10$ the model predicts simultaneously a high equity premium (about 7%) and a downward-sloping term structure of dividend strip risk premia. This is a remarkable result of learning, which obtains in absence of any stochastic volatility in fundamentals; all fundamental processes were assumed to be homoscedastic for the sake of highlighting the impact of learning on the term structures.

An increase in relative risk aversion decreases slightly the level of the risk-free rate, but it does not alter the impact of learning. The slope of the term structure of bond yields switches from negative under full information to positive under partial information. That is, learning yields an increasing term structure of interest rates, as in the data. Finally, an increase in relative risk aversion does not affect substantially the term structure of dividend strip return volatility because the relative size of price elasticities are weakly sensitive to changes in risk aversion.

Consider now the case of a decrease in the elasticity of intertemporal substitution. Under full information, the price of risk for the permanent component x decreases and the price of risk for the transitory component z increases. Since the former is much larger than the latter in our calibration, the net effect reduces the level of the equity premium. Under partial information, the unique price of risk decreases. At the same time, price elasticities with respect to x and z move in opposite directions but with the former being smaller than the latter. In turn, learning still switches the sign of the term structure of equity risk premia.

A lower elasticity of intertemporal substitution leads bond yields to be more sensitive to the long-run growth rate. In turn, bond yields are slightly higher than in the baseline economy under both full information and partial information with learning. However, even in this case the effect of learning is preserved. Bond yields are decreasing under full information and increasing under partial information. Finally, the term structure of dividend strip return volatility is barely affected by a decrease in the elasticity of intertemporal substitution. The relative size of price elasticities is unaffected and the downward-sloping shape is driven by the price elasticity with respect to z under both full and partial information.

4.3 Asset Pricing Moments

This section highlights how partial information and learning affect the asset pricing moments.

Table 2 compares the average model-implied and empirical risk-free rate, equity risk premium, equity return volatility, and dividend yield.

Table 2: Model-Implied and Empirical Asset Pricing Moments.
The first and second columns use real S&P 500 prices and dividends as well as real interest rates. Monthly data from 1871 to 2018 are obtained from Robert Shiller's website. The first column uses pre- and post-war data, while the second column uses post-war data only. The last six columns provide the model-implied counterparts.

	Data		Preference Parameters					
	1871-2018	1946-2018	$\gamma = 7.5, \psi = 1.5$		$\gamma = 10, \psi = 1.5$		$\gamma = 7.5, \psi = 1.25$	
			Information		Information		Information	
			Full	Partial	Full	Partial	Full	Partial
Risk-free rate (%)	4.2	2.0	2.5	2.1	2.5	1.9	2.9	2.4
Equity premium (%)	3.5	5.5	4.1	5.2	5.6	6.9	4.0	5.2
Return volatility (%)	14.4	12.0	20.3	22.8	20.2	23.0	20.0	22.9
Sharpe ratio (%)	24.1	46.0	20.4	22.8	27.7	30.1	20.0	22.7
Dividend yield (%)	4.3	3.4	2.1	2.1	3.5	3.7	2.4	2.4

Under full information and using our benchmark parameters ($\gamma = 7.5, \psi = 1.5$), the risk-free rate, equity premium, and return volatility are 2.5%, 4.1%, and 20.3%, respectively. The risk premium and risk-free rate are respectively lower and higher than their post-war counterparts. As a result, the Sharpe ratio is lower that its empirical counterpart. Also, the dividend yield (2.1%) is lower than in actual data.

Under partial information, learning yields a lower risk-free rate (2.1%) and a higher equity premium (5.2%) than under full information. This corresponds to a decrease in the risk-free rate and an increase in the equity premium of about 25%. Both the equity premium and the risk-free rate are fairly close to their post-war counterparts under partial information, which shows that learning helps to solve the equity premium and risk-free rate puzzles.

An increase in risk aversion ($\gamma = 10, \psi = 1.5$) modifies the results as follows. The risk-free rate decreases slightly and the equity premium increases substantially, which implies that the Sharpe ratio increases substantially too. This result obtains under both full and partial information. In addition, the dividend yields are fairly close to their empirical counterparts.

A decrease in the elasticity of intertemporal substitution ($\gamma = 7.5, \psi = 1.25$) has minor effects on the model-implied moments. The risk-free rate increases marginally, while the equity premium is unaffected. The reason is that a lower elasticity of intertemporal substitution reduces the compensation associated with the permanent shock x but increases the compensation associated with the transitory shock z_t. That is, these two effects offset each other.

While our main goal is to study the role of learning on the term structures, it is important to note that the model provides a good fit to the main asset pricing moments. In particular, partial information and learning yield a lower risk-free rate, a higher equity premium, and a higher Sharpe ratio than in the full information economy.

5 Conclusion

This paper shows that rational learning about expected output growth helps explain the term structures of equity and interest rates. While the term structures of equity risk premia and interest rates would be respectively upward-sloping and downward-sloping under full information, rational learning implies a switch of sign in the slope of these term structures. The fact that the agent has to filter out unobservable economic fundamentals alters the way different sources of risk are priced in equilibrium. The term structure of equity risk premia becomes downward-sloping, while the term structure of interest rates becomes upward-sloping. This economic channel is simple and novel, and arises naturally given the latent nature of expected growth. Economic growth forecast data lends support to the model mechanism. Moreover, this mechanism can explain the pro-cyclical variation in the slope of the term structure of equity risk premia as incomplete information and the need for learning become more pronounced in bad times.

References

Abel, A. B. (1999, February). Risk premia and term premia in general equilibrium. *Journal of Monetary Economics 43*(1), 3–33.

Ai, H., M. M. Croce, A. M. Diercks, and K. Li (2018). News shocks and the production-based term structure of equity returns. *The Review of Financial Studies 31*(7), 2423–2467.

Bansal, R., D. Kiku, and A. Yaron (2012). An empirical evaluation of the long-run risks model for asset prices. *Critical Finance Review 1*, 183–221.

Bansal, R., S. Miller, D. Song, and A. Yaron (2019). The term structure of equity risk premia. Technical report, NBER Working Paper No. 25690.

Bansal, R. and A. Yaron (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *Journal of Finance 59*(4), 1481–1509.

Beeler, J. and J. Y. Campbell (2012). The long-run risks model and aggregate asset prices: An empirical assessment. *Critical Finance Review 1*, 141–182.

Belo, F., P. Collin-Dufresne, and R. S. Goldstein (2015). Dividend dynamics and the term structure of dividend strips. *Journal of Finance 70*(3), 1115–1160.

Bloom, N. (2009, 05). The impact of uncertainty shocks. *Econometrica 77*(3), 623–685.

Bloom, N. (2014). Fluctuations in uncertainty. *Journal of Economic Perspectives 28*(2), 153–76.

Brennan, M. J. and Y. Xia (2001). Stock price volatility and equity premium. *Journal of Monetary Economics 47*(2), 249 – 283.

Campbell, J., A. Lo, and C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton University Press.

Campbell, J. Y. and R. J. Shiller (1988). Stock prices, earnings, and expected dividends. *Journal of Finance 43*(3), 661–676.

Croce, M. M., M. Lettau, and S. C. Ludvigson (2015). Investor information, long-run risk, and the term structure of equity. *Review of Financial Studies 28*(3), 706–742.

Detemple, J. B. (1986). Asset pricing in a production economy with incomplete information. *Journal of Finance 41*(2), 383–91.

Dew-Becker, I. (2017). How risky is consumption in the long-run? benchmark estimates from a robust estimator. *The Review of Financial Studies 30*(2), 631–666.

Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica 60*(2), 353–94.

Dumas, B., A. Kurshev, and R. Uppal (2009). Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility. *The Journal of Finance 64*(2), 579–629.

Epstein, L. G. and S. E. Zin (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*(4), 937–69.

Eraker, B. and I. Shaliastovich (2008). An equilibrium guide to designing affine pricing models. *Mathematical Finance 18*(4), 519–543.

Gennotte, G. (1986). Optimal portfolio choice under incomplete information. *Journal of Finance 41*(3), 733–46.

Hasler, M. and M. Khapko (2018). The shape of the term structures. *Working Paper, University of Toronto*.

Hasler, M. and R. Marfè (2016). Disaster recovery and the term structure of dividend strips. *Journal of Financial Economics 122*(1), 116 – 134.

Jurado, K., S. C. Ludvigson, and S. Ng (2015). Measuring uncertainty. *The American Economic Review 105*(3), 1177–1216.

Kreps, D. M. and E. L. Porteus (1979). Temporal von neumann-morgenstern and induced preferences. *Journal of Economic Theory 20*(1), 81–109.

Liptser, R. S. and A. N. Shiryaev (2001). *Statistics of Random Processes II.* Springer Verlag, New York.

Lucas, R. E. (1978). Asset prices in an exchange economy. *Econometrica 46*(6), 1429–1445.

Marfè, R. (2017). Income insurance and the equilibrium term structure of equity. *The Journal of Finance 72*(5), 2073–2130.

Scheinkman, J. A. and W. Xiong (2003). Overconfidence and speculative bubbles. *The Journal of Political Economy 111*(6), 1183–1219.

van Binsbergen, J., M. Brandt, and R. Koijen (2012). On the timing and pricing of dividends. *American Economic Review 102*(4), 1596–1618.

van Binsbergen, J., W. Hueskes, R. Koijen, and E. Vrugt (2013). Equity yields. *Journal of Financial Economics 110*(3), 503 – 519.

van Binsbergen, J. H. and R. Koijen (2017). The term structure of returns: Facts and theory. *Journal of Financial Economics 124*(1), 1 – 21.

Veronesi, P. (1999). Stock market overreaction to bad news in good times: A rational expectations equilibrium model. *Review of Financial Studies 12*(5), 975–1007.

Veronesi, P. (2000, April). How does information quality affect stock returns? *Journal of Finance 55*(2), 807–837.

Weber, M. (2018). Cash flow duration and the term structure of equity returns. *Journal of Financial Economics 128*(3), 486 – 503.

Weil, P. (1989). The equity premium puzzle and the risk-free rate puzzle. *Journal of Monetary Economics 24*(3), 401–421.

Xia, Y. (2001). Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. *The Journal of Finance 56*(1), pp. 205–246.

Appendix

A Notation Summary

Parameter	Interpretation
μ	Average growth in log consumption
$\sigma_y > 0$	Volatility of the permanent component of consumption growth
$\lambda_x > 0$	Mean reversion of the stochastic drift of the permanent component
$\sigma_x > 0$	Volatility of of the stochastic drift of the permanent component
$\lambda_z > 0$	Mean reversion of the transitory component
$\sigma_z > 0$	Volatility of of the transitory component
β_d	Parameter that determines the growth rate of dividends
$\phi \geq 1$	Leverage
$\delta \in (0, 1)$	Rate of time preference
$\gamma > 0$	Relative risk aversion
$\psi > 0$	Elasticity of intertemporal substitution

B Proofs

B.1 Proposition 1

Proof. See Liptser and Shiryaev (2001).

The steady-state volatilities $\widehat{\sigma}_x$ and $\widehat{\sigma}_z$ satisfy

$$\widehat{\sigma}_x = \frac{\bar{\gamma}_x - \lambda_z \bar{\gamma}_{xz}}{\sqrt{v}}, \qquad \widehat{\sigma}_z = \frac{\sigma_z^2 + \bar{\gamma}_{xz} - \lambda_z \bar{\gamma}_z}{\sqrt{v}},$$

where the steady-state posterior variances $\bar{\gamma}_x$ and $\bar{\gamma}_z$, and the steady-state posterior covariance $\bar{\gamma}_{xz}$ solve the following system of equations

$$\begin{aligned}
0 &= \sigma_x^2 - 2\lambda_x \bar{\gamma}_x - v^{-1} \left(\bar{\gamma}_x - \lambda_z \bar{\gamma}_{xz}\right)^2, \\
0 &= \sigma_z^2 - 2\lambda_z \bar{\gamma}_z - v^{-1} \left(\sigma_z^2 - \lambda_z \bar{\gamma}_z + \bar{\gamma}_{xz}\right)^2, \\
0 &= -\left(\lambda_x + \lambda_z\right) \bar{\gamma}_{xz} - v^{-1} \left(\bar{\gamma}_x - \lambda_z \bar{\gamma}_{xz}\right) \left(\sigma_z^2 - \lambda_z \bar{\gamma}_z + \bar{\gamma}_{xz}\right).
\end{aligned}$$

□

B.2 Proposition 2

Proof. Using the moment generating function of consumption under the full information filtration and the definition of consumption volatility in (13), we can compute the annualized variance of consumption as

$$\sigma_C^2(t, \tau) = \sigma_y^2 + \frac{\sigma_x^2 e^{-2\lambda_x \tau} \left(e^{2\lambda_x \tau}(2\lambda_x \tau - 3) + 4e^{\lambda_x \tau} - 1\right)}{2\lambda_x^3 \tau}. \tag{20}$$

Partial derivatives with respect to the horizon τ, volatility σ_x, and the mean-reversion speed λ_x are as follows:

$$\begin{aligned}
\frac{\partial\sigma_C^2(t,\tau)}{\partial\tau} &= \frac{\sigma_x^2 e^{-2\lambda_x\tau}\left(1+3e^{2\lambda_x\tau}+2\lambda_x\tau-4e^{\lambda_x\tau}(\lambda_x\tau+1)\right)}{2\lambda_x^3\tau^2},\\
\frac{\partial\sigma_C^2(t,\tau)}{\partial\sigma_x} &= -\frac{\sigma_x\left(3+e^{-2\lambda_x\tau}-4e^{-\lambda_x\tau}-2\lambda_x\tau\right)}{\lambda_x^3\tau},\\
\frac{\partial\sigma_C^2(t,\tau)}{\partial\lambda_x} &= \frac{\sigma_x^2 e^{-2\lambda_x\tau}\left(2\lambda_x\tau+e^{2\lambda_x\tau}(9-4\lambda_x\tau)-4e^{\lambda_x\tau}(\lambda_x\tau+3)+3\right)}{2\lambda_x^4\tau}.
\end{aligned}$$

For $\tau>0$ the following holds: $\frac{\partial\sigma_C^2(t,\tau)}{\partial\tau}>0$, $\frac{\partial\sigma_C^2(t,\tau)}{\partial\sigma_x}>0$, and $\frac{\partial\sigma_C^2(t,\tau)}{\partial\lambda_x}<0$. The fist inequality holds since

$$\begin{aligned}
&1+3e^{2\lambda_x\tau}+2\lambda_x\tau-4e^{\lambda_x\tau}(\lambda_x\tau+1)\\
&=2\left(e^{\lambda_x\tau}-1\right)\left(e^{\lambda_x\tau}-2\lambda_x\tau-1\right)+\left(e^{2\lambda_x\tau}-2\lambda_x\tau-1\right)\\
&>\left(e^{\lambda_x\tau}-1\right)\left(e^{\lambda_x\tau}-2\lambda_x\tau-1\right)+\left(e^{2\lambda_x\tau}-2\lambda_x\tau-1\right)\\
&=2e^{\lambda_x\tau}\left(e^{\lambda_x\tau}-\lambda_x\tau-1\right)>0 \text{ for } \tau>0.
\end{aligned}$$

The second inequality holds since

$$\begin{aligned}
&e^{2\lambda_x\tau}\left(3+e^{-2\lambda_x\tau}-4e^{-\lambda_x\tau}-2\lambda_x\tau\right)\\
&=1-4e^{\lambda_x\tau}+e^{2\lambda_x\tau}\left(2-3\lambda_x\tau\right)\\
&<1-4e^{\lambda_x\tau}+e^{\lambda_x\tau}\left(2-3\lambda_x\tau\right)\\
&=1-e^{\lambda_x\tau}\left(1+\lambda_x\tau\right)<0 \text{ for } \tau>0.
\end{aligned}$$

The third inequality, $\frac{\partial\sigma_C^2(t,\tau)}{\partial\lambda_x}<0$, holds since $3+2\lambda_x\tau+e^{2\lambda_x\tau}(9-4\lambda_x\tau)-4e^{\lambda_x\tau}(\lambda_x\tau+3)<0$.

Similarly, using the moment generating function of consumption under the partial information filtration and the definition of consumption volatility in (14), we can compute the agent's estimate of the annualized variance of consumption as

$$\begin{aligned}
\widehat{\sigma}_C^2(t,\tau) = \frac{e^{-2\lambda_x\tau}}{2\lambda_x^3\tau}\Big(&2\lambda_x\sigma_y\left(\left(e^{\lambda_x\tau}-1\right)^2\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}+\lambda_x^2\sigma_y\tau e^{2\lambda_x\tau}-\lambda_x\sigma_y\left(e^{\lambda_x\tau}-1\right)^2\right)\\
&+\sigma_x^2\left(e^{2\lambda_x\tau}(2\lambda_x\tau-3)+4e^{\lambda_x\tau}-1\right)\Big).
\end{aligned} \tag{21}$$

Partial derivatives with respect to the horizon τ, volatility σ_x, and the mean-reversion speed

λ_x are as follows:

$$\begin{aligned}
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \tau} &= \frac{e^{-2\lambda_x \tau}}{2\lambda_x^3 \tau^2}\Big(2\lambda_x \sigma_y \left(e^{\lambda_x \tau}-1\right)\left(1+2\lambda_x\tau - e^{\lambda_x\tau}\right)\left(\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}-\lambda_x\sigma_y\right) \\
&\quad + \sigma_x^2\left(1+3e^{2\lambda_x\tau}+2\lambda_x\tau-4e^{\lambda_x\tau}(\lambda_x\tau+1)\right)\Big), \\
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \sigma_x} &= \frac{e^{-2\lambda_x\tau}\sigma_x}{\lambda_x^3\tau}\left(\frac{\lambda_x\sigma_y\left(e^{\lambda_x\tau}-1\right)^2}{\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}} + e^{\lambda_x\tau}\left(4+e^{\lambda_x\tau}(2\lambda_x\tau-3)\right)-1\right), \\
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \lambda_x} &= \frac{e^{-2\lambda_x\tau}}{2\lambda_x^4\tau\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}}\Big(4\lambda_x^4\sigma_y^3\tau\left(e^{\lambda_x\tau}-1\right)+3\sigma_x^2\left(-4e^{\lambda_x\tau}+3e^{2\lambda_x\tau}+1\right)\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2} \\
&\quad + 2\lambda_x^2\sigma_y\left(e^{\lambda_x\tau}-1\right)\left(\sigma_y\left(e^{\lambda_x\tau}-1\right)\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}+2\sigma_x^2\tau\right) \\
&\quad - 2\lambda_x\sigma_x^2\left(\tau\left(2e^{\lambda_x\tau}\left(e^{\lambda_x\tau}+1\right)-1\right)\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}+2\sigma_y\left(e^{\lambda_x\tau}-1\right)^2\right) \\
&\quad - 2\lambda_x^3\sigma_y^2\left(e^{\lambda_x\tau}-1\right)\left(2\tau\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}+\sigma_y\left(e^{\lambda_x\tau}-1\right)\right)\Big).
\end{aligned}$$

For $\tau > 0$ we have $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \tau} > 0$, $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \sigma_x} > 0$, and $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \lambda_x} < 0$. The first inequality holds since

$$\begin{aligned}
&2\lambda_x\sigma_y\left(\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}-\lambda_x\sigma_y\right)\left(e^{\lambda_x\tau}-1\right)\left(1+2\lambda_x\tau-e^{\lambda_x\tau}\right) \\
&\quad + \sigma_x^2\left(1+3e^{2\lambda_x\tau}+2\lambda_x\tau-4e^{\lambda_x\tau}(\lambda_x\tau+1)\right) \\
&\geq \min\left\{\lambda_x\sigma_y\left(\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}-\lambda_x\sigma_y\right), \sigma_x^2\right\}\Big(2\left(e^{\lambda_x\tau}-1\right)\left(1+2\lambda_x\tau-e^{\lambda_x\tau}\right) \\
&\quad + \left(1+3e^{2\lambda_x\tau}+2\lambda_x\tau-4e^{\lambda_x\tau}(\lambda_x\tau+1)\right)\Big) \\
&= \min\left\{\lambda_x\sigma_y\left(\sqrt{\lambda_x^2\sigma_y^2+\sigma_x^2}-\lambda_x\sigma_y\right), \sigma_x^2\right\}\left(e^{2\lambda_x\tau}-2\lambda_x\tau-1\right) > 0 \text{ for } \tau > 0,
\end{aligned}$$

where we use the fact that for $0 < a < b$ and $y > 0$ we have $ax + by \geq \min\{a,b\}(x+y)$. The second inequality holds since $4 + e^{\lambda_x\tau}(2\lambda_x\tau - 3) > 0$. Finally, by a lengthy and tedious calculation, one can show that $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \lambda_x} < 0$.

□

B.3 Proposition 3

Proof. Taking the limits of (20) and (21) as horizon τ approaches zero or infinity gives the result in (15). Furthermore,

$$\widehat{\sigma}_C^2(t,\tau) - \sigma_C^2(t,\tau) = \frac{e^{-2\lambda_x\tau}\left(e^{\lambda_x\tau}-1\right)^2\left(\sqrt{\sigma_y^2\left(\lambda_x^2\sigma_y^2+\sigma_x^2\right)}-\lambda_x\sigma_y^2\right)}{\lambda_x^2\tau} > 0. \tag{22}$$

The derivative of the difference in (22) with respect to horizon is

$$\frac{\partial\left(\widehat{\sigma}_C^2(t,\tau)-\sigma_C^2(t,\tau)\right)}{\partial\tau}=\frac{e^{-2\lambda_x\tau}\left(e^{\lambda_x\tau}-1\right)\left(1+2\lambda_x\tau-e^{\lambda_x\tau}\right)\left(\sqrt{\sigma_y^2\left(\lambda_x^2\sigma_y^2+\sigma_x^2\right)}-\lambda_x\sigma_y^2\right)}{\lambda_x^2\tau^2}.$$

The sign of this derivative depends on the sign of $1+2\lambda_x\tau-e^{\lambda_x\tau}$ and the result in (16) follows.

□

B.4 Proposition 4

Proof. Using the moment generating function of consumption under the full information filtration and the definition of consumption volatility in (13), we can compute the annualized variance of consumption as

$$\sigma_C^2(t,\tau)=\sigma_y^2+\frac{\sigma_z^2\left(1-e^{-2\lambda_z\tau}\right)}{2\lambda_z\tau}. \tag{23}$$

Partial derivatives with respect to the horizon τ, volatility σ_z, and the mean-reversion speed λ_z are as follows:

$$\begin{aligned}
\frac{\partial\sigma_C^2(t,\tau)}{\partial\tau}&=-\frac{\sigma_z^2e^{-2\lambda_z\tau}\left(e^{2\lambda_z\tau}-1-2\lambda_z\tau\right)}{2\lambda_z\tau^2},\\
\frac{\partial\sigma_C^2(t,\tau)}{\partial\sigma_z}&=\frac{\sigma_z\left(1-e^{-2\lambda_z\tau}\right)}{\lambda_z\tau}\\
\frac{\partial\sigma_C^2(t,\tau)}{\partial\lambda_z}&=-\frac{\sigma_z^2e^{-2\lambda_z\tau}\left(e^{2\lambda_z\tau}-1-2\lambda_z\tau\right)}{2\lambda_z^2\tau}.
\end{aligned}$$

For $\tau>0$ we have $\frac{\partial\sigma_C^2(t,\tau)}{\partial\tau}<0$, $\frac{\partial\sigma_C^2(t,\tau)}{\partial\sigma_z}>0$, and $\frac{\partial\sigma_C^2(t,\tau)}{\partial\lambda_z}<0$.

Similarly, using the moment generating function of consumption under the partial information filtration and the definition of consumption volatility in (14), we can compute the agent's estimate of the annualized variance of consumption as

$$\begin{aligned}
\widehat{\sigma}_C^2(t,\tau)=\frac{e^{-2\lambda_z\tau}}{2\lambda_z\tau}\Big(&\left(e^{\lambda_z\tau}-1\right)\left(2\sqrt{\sigma_y^2\left(\sigma_y^2+\sigma_z^2\right)}\left(e^{\lambda_z\tau}-1\right)+\sigma_z^2\left(e^{\lambda_z\tau}+1\right)\right)\\
&+2\sigma_y^2\left(e^{2\lambda_z\tau}(\lambda_z\tau-1)+2e^{\lambda_z\tau}-1\right)\Big).
\end{aligned} \tag{24}$$

Partial derivatives with respect to the horizon τ, volatility σ_z, and the mean-reversion speed

λ_z are as follows:

$$\begin{aligned}
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \tau} &= -\frac{e^{-2\lambda_z\tau}}{2\lambda_z\tau^2}\Big(2\sigma_y\left(\sqrt{\sigma_y^2+\sigma_z^2}-\sigma_y\right)\left(e^{\lambda_z\tau}-1\right)\left(e^{\lambda_z\tau}-2\lambda_z\tau-1\right)\\
&\quad+\sigma_z^2\left(e^{2\lambda_z\tau}-2\lambda_z\tau-1\right)\Big),\\
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \sigma_z} &= \frac{\sigma_z e^{-2\lambda_z\tau}}{\lambda_z\tau}\left(e^{\lambda_z\tau}-1\right)\left(\frac{\sigma_y^2\left(e^{\lambda_z\tau}-1\right)}{\sqrt{\sigma_y^2\left(\sigma_y^2+\sigma_z^2\right)}}+e^{\lambda_z\tau}+1\right),\\
\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \lambda_z} &= -\frac{e^{-2\lambda_z\tau}}{2\lambda_z^2\tau}\Big(2\sigma_y\left(\sqrt{\sigma_y^2+\sigma_z^2}-\sigma_y\right)\left(e^{\lambda_z\tau}-1\right)\left(e^{\lambda_z\tau}-2\lambda_z\tau-1\right)\\
&\quad+\sigma_z^2\left(e^{2\lambda_z\tau}-2\lambda_z\tau-1\right)\Big).
\end{aligned}$$

For $\tau > 0$ we have $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \tau} < 0$, $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \sigma_z} > 0$, and $\frac{\partial \widehat{\sigma}_C^2(t,\tau)}{\partial \lambda_z} < 0$. The second inequality is obvious, the first and last inequalities follow since

$$\begin{aligned}
&2\sigma_y\left(\sqrt{\sigma_y^2+\sigma_z^2}-\sigma_y\right)\left(e^{\lambda_z\tau}-1\right)\left(e^{\lambda_z\tau}-2\lambda_z\tau-1\right)+\sigma_z^2\left(e^{2\lambda_z\tau}-2\lambda_z\tau-1\right)\\
&\geq \min\left\{2\sigma_y\left(\sqrt{\sigma_y^2+\sigma_z^2}-\sigma_y\right),\sigma_z^2\right\}\Big(\left(e^{\lambda_z\tau}-1\right)\left(e^{\lambda_z\tau}-2\lambda_z\tau-1\right)+\left(e^{2\lambda_z\tau}-2\lambda_z\tau-1\right)\Big)\\
&= \min\left\{2\sigma_y\left(\sqrt{\sigma_y^2+\sigma_z^2}-\sigma_y\right),\sigma_z^2\right\}2e^{\lambda_z\tau}\left(e^{\lambda_z\tau}-\lambda_z\tau-1\right)>0 \text{ for } \tau>0,
\end{aligned}$$

where we use the fact that for $0 < a < b$ and $y > 0$ we have $ax + by \geq \min\{a, b\}(x + y)$.

□

B.5 Proposition 5

Proof. Taking the limits of (23) and (24) as horizon τ approaches zero or infinity gives the result in (17). Furthermore,

$$\widehat{\sigma}_C^2(t,\tau)-\sigma_C^2(t,\tau)=\frac{\left(\sqrt{\sigma_y^2\left(\sigma_y^2+\sigma_z^2\right)}-\sigma_y^2\right)e^{-2\lambda_z\tau}\left(e^{\lambda_z\tau}-1\right)^2}{\lambda_z\tau}>0. \tag{25}$$

The derivative of the difference in (25) with respect to horizon is

$$\frac{\partial\left(\widehat{\sigma}_C^2(t,\tau)-\sigma_C^2(t,\tau)\right)}{\partial\tau}=\frac{e^{-2\lambda_z\tau}\left(e^{\lambda_z\tau}-1\right)\left(1+2\lambda_z\tau-e^{\lambda_z\tau}\right)\left(\sqrt{\sigma_y^2\left(\sigma_y^2+\sigma_z^2\right)}-\sigma_y^2\right)}{\lambda_z\tau^2}.$$

The sign of this derivative depends on the sign of $1 + 2\lambda_z\tau - e^{\lambda_z\tau}$ and the result in (18) follows.

□

B.6 Proposition 6

Proof. This proof follows closely Eraker and Shaliastovich (2008). We conjecture that the log wealth-consumption ratio is affine in the state variables $X_t = (\widehat{y}_t, \widehat{y}_{d,t}, \widehat{x}_t, \widehat{z}_t)^\top$, where $\widehat{y}_t \equiv \log C_t - \widehat{z}_t$ and $\widehat{y}_{d,t} \equiv \phi \widehat{y}_t - \beta_d t$[4] , so that

$$wc_t \equiv \log \frac{W_t}{C_t} = A + B^\top X_t, \tag{26}$$

and use the fact that the state variables belong to the affine class, so that their dynamics can be written as:

$$\begin{aligned} dX_t &= \mu(X_t)dt + \Sigma(X_t)d\widehat{B}_t \\ \mu(X_t) &= \mathcal{M} + \mathcal{K}X_t \\ \Sigma(X_t)\Sigma(X_t)^\top &= h + \sum_{i=1}^{4} H^i X_t^i, \end{aligned}$$

where X_t is the vector of the filtered state variables, $\Sigma(X_t) \in \mathbb{R}^{4\times 1}$ encodes the diffusions of the state variables, $\mathcal{M} \in \mathbb{R}^4$, $\mathcal{K} \in \mathbb{R}^{4\times 4}$, $h \in \mathbb{R}^{4\times 4}$, $H \in \mathbb{R}^{4\times 4\times 4}$, and $\widehat{B}_t$ is a standard Brownian motion.

The dynamics of the state-price density then can be written as

$$d\log M_t = (\theta \log\delta - (\theta-1)\log k_1 + (\theta-1)(k_1-1)B'(X_t - \mu_X)dt - \Omega' dX_t, \tag{27}$$

where $X_t = (\widehat{y}_t, \widehat{y}_{d,t}, \widehat{x}_t, \widehat{z}_t)^\top$, $\mu_X = (0,0,0,0)^\top$, $\Omega = \gamma(1,0,0,1)^\top + (1-\theta)k_1 B$, and the coefficients $A \in \mathbb{R}$ and $B \in \mathbb{R}^4$ are the loadings defined in (26).

The coefficients $A \in \mathbb{R}$, $B \in \mathbb{R}^4$ solve the following system of equations

$$0 = \mathcal{K}^\top \chi - \theta(1-k_1)B + \frac{1}{2}\chi^\top H \chi, \tag{28}$$

$$0 = \theta(\log\delta + k_0 - (1-k_1)A) + \mathcal{M}^\top \chi + \frac{1}{2}\chi^\top h \chi, \tag{29}$$

and the linearization coefficient $k_1 \in \mathbb{R}$ satisfies

$$\theta \log k_1 = \theta(\log\delta + (1-k_1)B^\top \mu_X) + \mathcal{M}^\top \chi + \frac{1}{2}\chi^\top h \chi,$$

where $\chi = \theta\left((1 - \frac{1}{\psi})(1,0,0,1)^\top + k_1 B\right)$.

[4] Note that the dividend dynamics on the full filtration can be equivalently written as

$$d\log D_t = dy_{d,t} + \phi dz_t,$$

where

$$dy_{d,t} = (\mu_d + \phi x_t)\,dt + \phi\sigma_y dB_{y,t},$$

with $\mu_d \equiv \phi\mu - \beta_d$.

Solving (28) for the vector of loadings $B \in \mathbb{R}^4$ gives

$$B^\top = \left(0, 0, \frac{1 - 1/\psi}{1 - k_1\,(1 - \lambda_x)}, -\frac{\lambda_z(1 - 1/\psi)}{1 - k_1(1 - \lambda_z)}\right).$$

Plugging this solution in equation (29) allows to solve for the coefficient A.

From the arbitrage theory we know that the state-price density M_t satisfies

$$\frac{dM_t}{M_t} = -r_t dt - \Lambda_t^\top dB_t.$$

where r_t is the risk-free rate and Λ_t is the market price of risk vector.

Eraker and Shaliastovich (2008) show that from the expression for the state price density in (27), the risk free rate and market price of risk vector can be determined as follows:

$$\begin{aligned} r_t &= r_0 + r_1^\top X_t, \\ \Lambda_t &= \Sigma(X_t)^\top \Omega, \end{aligned}$$

where the vector $\Omega = \gamma(1, 0, 0, 1)^\top + (1-\theta)k_1 B$ and the coefficients $r_0 \in \mathbb{R}$ and $r_1 \in \mathbb{R}^4$ solve the system of equations

$$\begin{aligned} r_1 &= (1 - \theta)(k_1 - 1)B + \mathcal{K}^\top\Omega - \frac{1}{2}\Omega^\top H\Omega, \\ r_0 &= -\theta \log\delta + (\theta - 1)(\log k_1 + (k_1 - 1)B^\top \mu_X) + \mathcal{M}^\top\Omega - \frac{1}{2}\Omega^\top h\Omega. \end{aligned}$$

Solving for r_1, r_0 gives $r_1^\top = (0, 0, 1/\psi, -\lambda_z/\psi)$ and

$$r_0 = -\frac{1 - \gamma}{1 - 1/\psi}\log\delta + \frac{1/\psi - \gamma}{1 - 1/\psi}\log k_1 + \gamma\mu - \frac{1}{2}\Theta\left(\widehat{\sigma}_y, \widehat{\sigma}_x, \widehat{\sigma}_z\right),$$

where

$$\begin{aligned} &\Theta\left(\widehat{\sigma}_y, \widehat{\sigma}_x, \widehat{\sigma}_z\right) \equiv \\ &\frac{1}{\psi^2(k_1(\lambda_x - 1) + 1)^2(k_1(\lambda_z - 1) + 1)^2}(\gamma\psi((k_1(\lambda_z - 1) + 1)(k_1((\lambda_x - 1)\widehat{\sigma}_y + \widehat{\sigma}_x) + \widehat{\sigma}_y) \\ &- (k_1 - 1)\widehat{\sigma}_z(k_1(\lambda_x - 1) + 1)) + k_1\lambda_z\widehat{\sigma}_z(k_1(\lambda_x - 1) + 1) + k_1\widehat{\sigma}_x(k_1(-\lambda_z) + k_1 - 1))^2, \end{aligned}$$

where $\widehat{\sigma}_y \equiv \sqrt{v} - \widehat{\sigma}_z$, and $\widehat{\sigma}_x$, $\widehat{\sigma}_z$ are defined in Proposition 1. Similarly, market price of risk can be written as

$$\Lambda = \gamma\widehat{\sigma}_y + \left(\frac{(\gamma - 1/\psi)}{1/k_1 - (1 - \lambda_x)}\right)\widehat{\sigma}_x + \left(\gamma - \frac{\lambda_z(\gamma - 1/\psi)}{1/k_1 - (1 - \lambda_z)}\right)\widehat{\sigma}_z.$$

Finally, following Eraker and Shaliastovich (2008), the dynamics of the vector of state variables X_t under the risk neutral measure $\mathbb{Q}$ are given by

$$dX_t = (\mathcal{M}^{\mathbb{Q}} + \mathcal{K}^{\mathbb{Q}} X_t)dt + \Sigma(X_t)d\widehat{B}_t^{\mathbb{Q}},$$

where $\widehat{B}_t^{\mathbb{Q}} = \widehat{B}_t + \int_0^t \Lambda_s ds$ is a $\mathbb{Q}$-Brownian motion and the coefficients $\mathcal{M}^{\mathbb{Q}} \in \mathbb{R}^4$ and $\mathcal{K}^{\mathbb{Q}} \in \mathbb{R}^{4\times 4}$ satisfy

$$\mathcal{M}^{\mathbb{Q}} = \mathcal{M} - h\Omega, \tag{30}$$
$$\mathcal{K}^{\mathbb{Q}} = \mathcal{K} - H\Omega. \tag{31}$$

□

B.7 Proposition 7

Proof. Price of a zero-coupon bond can be determined from

$$B(t,\tau) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+\tau} r_s ds}\right) = e^{q_0(\tau)+q_1(\tau)X_t},$$

where $X_t = (\widehat{y}_t, \widehat{x}_t, \widehat{z}_t)^\top$. Eraker and Shaliastovich (2008) show that the functions $q_0(\tau) \in \mathbb{R}$ and $q_1(\tau) \in \mathbb{R}^4$ solve the following system of Ricatti equations

$$\frac{\partial}{\partial\tau} q_1(\tau) = -r_1 + \mathcal{K}^{\mathbb{Q}\top} q_1(\tau) + \frac{1}{2} q_1(\tau)^\top H\, q_1(\tau), \tag{32}$$
$$\frac{\partial}{\partial\tau} q_0(\tau) = -r_0 + \mathcal{M}^{\mathbb{Q}\top} q_1(\tau) + \frac{1}{2} q_1(\tau)^\top h\, q_1(\tau), \tag{33}$$

with boundary conditions $q_0(0) = 0$ and $q_1(0) = (0,0,0,0)^\top$. Coefficients $\mathcal{M}^{\mathbb{Q}} \in \mathbb{R}^4$ and $\mathcal{K}^{\mathbb{Q}} \in \mathbb{R}^{4\times 4}$ are characterized in (30)–(31).

Solving (32) gives

$$q_1(\tau)^\top = \left(0, 0, -\frac{1}{\lambda_x \psi}\left(1 - e^{-\lambda_x \tau}\right), \frac{1}{\psi}\left(1 - e^{-\lambda_z \tau}\right)\right).$$

Using these results in (33) allows to solve for function q_0.

□

B.8 Proposition 8

Proof. Price of a dividend strip can be determined from

$$S(t,\tau) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^{t+\tau} r_s ds} D_{t+\tau}\right) = e^{w_0(\tau)+w_1(\tau)X_t}.$$

Eraker and Shaliastovich (2008) show that the functions $w_0(\tau) \in \mathbb{R}$ and $w_1(\tau) \in \mathbb{R}^4$ solve the following system of Ricatti equations

$$\frac{\partial}{\partial\tau} w_1(\tau) = -r_1 + \mathcal{K}^{\mathbb{Q}^\top} w_1(\tau) + \frac{1}{2} w_1(\tau)^\top H\, w_1(\tau), \tag{34}$$

$$\frac{\partial}{\partial\tau} w_0(\tau) = -r_0 + \mathcal{M}^{\mathbb{Q}^\top} w_1(\tau) + \frac{1}{2} w_1(\tau)^\top h\, w_1(\tau), \tag{35}$$

with boundary conditions $w_0(0) = 0$ and $w_1(0) = (0, 1, 0, \phi)^\top$. Coefficients $\mathcal{M}^{\mathbb{Q}} \in \mathbb{R}^4$ and $\mathcal{K}^{\mathbb{Q}} \in \mathbb{R}^{4\times4}$ are characterized in (30)–(31).

Solving (34) gives

$$w_1(\tau)^\top = \left(0, 1, \frac{1}{\lambda_x\psi}(1 - e^{-\lambda_x\tau})(\phi\psi - 1), \frac{1}{\psi}\big(1 - e^{-\lambda_z\tau}(1 - \phi\psi)\big)\right).$$

Using these results in (35) allows to solve for function w_0.

□

B.9 Proposition 9

Proof. Following Eraker and Shaliastovich (2008) we consider an approximate equilibrium solution for the price-dividend ratio, which is obtained, as wealth-consumption ratio in Proposition 6, through the log-linearization of returns. Namely, the log equilibrium price–dividend ratio is linear in the state variables,

$$pd_t = \equiv \log \frac{P_t}{D_t} = A_d + B_d^\top X_t.$$

The coefficients $A_d \in \mathbb{R}$, $B_d \in \mathbb{R}^4$ solve the following system of equations

$$0 = \mathcal{K}^\top \chi_d + (\theta - 1)(k_1 - 1)B + (k_{1,d} - 1)B_d + \frac{1}{2}\chi_d^\top H \chi_d, \tag{36}$$

$$\begin{aligned} 0 = \theta \ln\delta - (\theta - 1)\left(\ln k_1 + (k_1 - 1)B^\top \mu_X\right) - \left(\ln k_{1,d} + (k_{1,d} - 1)B_d^\top \mu_X\right) \\ + \mathcal{M}^\top \chi_d + \frac{1}{2}\chi_d^\top h \chi_d, \end{aligned} \tag{37}$$

where $\chi_d = (0, 1, 0, \phi)^\top + k_{1,d}B_d - \Omega$ and $k_{1,d} \in \mathbb{R}$ is the linearization coefficient for the stock return.

Solving (36) for the vector of loadings $B_d \in \mathbb{R}^4$ gives

$$B_d^\top = \left(0, 0, \frac{\phi - 1/\psi}{1 - k_{1,d}(1 - \lambda_x)}, -\frac{\lambda_z(\phi - 1/\psi)}{1 - k_{1,d}(1 - \lambda_z)}.\right).$$

Plugging this solution in equation (37) allows to solve for $k_{1,d}$. Then we obtain the intercept

A_d as

$$A_d = \log \frac{k_{1,d}}{1 - k_{1,d}} - B_d^\top \mu_X.$$

□

C Asset Prices in the Full Information Economy

Proposition C.1. *The equilibrium state-price density in the full information economy has dynamics given by*

$$\frac{dM_t}{M_t} = -r_t dt - \Lambda^\top dB_t,$$

where $B_t = (B_{y,t}, B_{x,t}, B_{z,t})^\top$. The risk-free rate satisfies

$$r_t = r_0 + r_x x_t + r_z z_t,$$

with

$$\begin{aligned}
r_0 = & - \frac{1-\gamma}{1-1/\psi} \log \delta + \frac{1/\psi - \gamma}{1-1/\psi} \log k_1 + \gamma \mu \\
& - \frac{1}{2} \left(\gamma^2 \sigma_y^2 + \left(\frac{(\gamma - 1/\psi)}{1/k_1 - (1-\lambda_x)} \right)^2 \sigma_x^2 + \left(\gamma - \frac{\lambda_z (\gamma - 1/\psi)}{1/k_1 - (1-\lambda_z)} \right)^2 \sigma_z^2 \right), \\
r_x = & \frac{1}{\psi}, \\
r_z = & - \frac{\lambda_z}{\psi},
\end{aligned}$$

and the market price of risk vector is

$$\Lambda^\top = \left(\gamma \sigma_y,\ \frac{(\gamma - 1/\psi)}{1/k_1 - (1-\lambda_x)} \sigma_x,\ \left(\gamma - \frac{\lambda_z (\gamma - 1/\psi)}{1/k_1 - (1-\lambda_z)} \right) \sigma_z \right).$$

Proof. The proof is similar to the proof of analogous proposition for the partial information economy, Proposition 6. We conjecture that the log wealth-consumption ratio is affine in the state variables $X_t = (y_t, y_{d,t}, x_t, z_t)^\top$, where $y_{d,t} \equiv \phi y_t - \beta_d t$ and use the fact that the state variables belong to the affine class, so that their dynamics can be written as[5]:

$$\begin{aligned}
dX_t &= \mu(X_t) dt + \Sigma(X_t) dB_t \\
\mu(X_t) &= \mathcal{M} + \mathcal{K} X_t \\
\Sigma(X_t) \Sigma(X_t)^\top &= h + \sum_{i=1}^{4} H^i X_t^i.
\end{aligned}$$

[5]Note that in this appendix we use X to denote the vector of state variables in the full information economy and not the vector of their filter estimates as we did in Appendix B.

Moreover, following Eraker and Shaliastovich (2008), the dynamics of the vector of state variables X_t under the risk neutral measure $\mathbb{Q}$ are given by

$$dX_t = (\mathcal{M}^{\mathbb{Q}} + \mathcal{K}^{\mathbb{Q}} X_t)dt + \Sigma(X_t)dB_t^{\mathbb{Q}},$$

where the coefficients can be identified analogously to (30)-(31).

□

Proposition C.2. *The equilibrium price of the zero-coupon bond with time to maturity τ in the full information economy is given by*

$$B(t,\tau) = \mathbb{E}\left[\frac{M_{t+\tau}}{M_t} \,\middle|\, \mathcal{F}_t\right] = e^{q_0(\tau)+q_x(\tau)x_t+q_z(\tau)z_t},$$

where

$$\begin{aligned} q_x(\tau) &= -\frac{1}{\lambda_x\psi}\left(1-e^{-\lambda_x\tau}\right), \\ q_z(\tau) &= \frac{1}{\psi}\left(1-e^{-\lambda_z\tau}\right). \end{aligned}$$

and $q_0(\tau)$ solves

$$\frac{\partial}{\partial\tau} q_0(\tau) = -r_0 + \mathcal{M}^{\mathbb{Q}\top} q_1(\tau) + \frac{1}{2} q_1(\tau)^\top h\, q_1(\tau)$$

with $q_1(\tau)^\top \equiv (0, 0, q_x(\tau), q_z(\tau))$.

Proof. Analogous to the proof of Proposition 7.

□

Proposition C.3. *The equilibrium price of the dividend strip with time to maturity τ in the full information economy is given by*

$$S(t,\tau) = \mathbb{E}\left[\frac{M_{t+\tau}}{M_t} D_{t+\tau} \,\middle|\, \mathcal{F}_t\right] = e^{-\beta_d t+\phi y_t+w_0(\tau)+w_x(\tau)x_t+w_z(\tau)z_t},$$

where

$$\begin{aligned} w_x(\tau) &= \frac{1}{\lambda_x\psi}(1-e^{-\lambda_x\tau})(\phi\psi-1) \\ w_z(\tau) &= \frac{1}{\psi}\big(1-e^{-\lambda_z\tau}(1-\phi\psi)\big). \end{aligned}$$

and $w_0(\tau)$ solves

$$\frac{\partial}{\partial\tau} w_0(\tau) = -r_0 + \mathcal{M}^{\mathbb{Q}\top} w_1(\tau) + \frac{1}{2} w_1(\tau)^\top h\, w_1(\tau)$$

with $w_1(\tau)^\top = (0, 1, w_x(\tau), w_z(\tau))$ *and* $w_0(0) = 0$. *The return premium of the dividend strip with time to maturity* τ *is given by*

$$RP(t,\tau) = -\frac{1}{dt}\langle \frac{dM_t}{M_t}, \frac{dS(t,\tau)}{S(t,\tau)} \rangle = (\phi\sigma_y,\, w_x(\tau)\sigma_x,\, w_z(\tau)\sigma_z)^\top \Lambda.$$

The return volatility of the dividend strip with time to maturity τ *is given by*

$$Vol(t,\tau) = \sqrt{\frac{1}{dt}\langle \frac{dS(t,\tau)}{S(t,\tau)} \rangle} = \|(\phi\sigma_y,\, w_x(\tau)\sigma_x,\, w_z(\tau)\sigma_z)^\top\|,$$

where $\|\cdot\|$ *is the Euclidean norm.*

Proof. Analogous to the proof of Proposition 8.

□

Proposition C.4. *The equilibrium price of equity in the full information economy is given by*

$$P_t = \int_0^\infty \mathbb{E}_t\left[\frac{M_{t+\tau}}{M_t} D_{t+\tau} \,\middle|\, \mathcal{F}_t\right] d\tau \approx D_t e^{A_d + B_{x,d}x_t + B_{z,d}z_t},$$

where

$$B_{x,d} = \frac{\phi - 1/\psi}{1 - k_{1,d}(1-\lambda_x)},$$
$$B_{z,d} = -\frac{\lambda_z(\phi - 1/\psi)}{1 - k_{1,d}(1-\lambda_z)}.$$

and A_d *is satisfies*

$$A_d = \log\frac{k_{1,d}}{1-k_{1,d}} - B_d^\top \mu_X,$$

where $B_d^\top = (0, 0, B_{x,d}, B_{z,d})$ *and the linearization coefficient* $k_{1,d}$ *solves a full information analogue of* (37). *The return premium of equity is given by*

$$RP(t) = -\frac{1}{dt}\langle \frac{dM_t}{M_t}, \frac{dP_t}{P_t} \rangle = (\phi\sigma_y,\, B_{x,d}\sigma_x,\, (\phi + B_{z,d})\,\sigma_z)^\top \Lambda.$$

The return volatility of equity is given by

$$Vol(t) = \sqrt{\frac{1}{dt}\langle \frac{dP_t}{P_t} \rangle} = \|(\phi\sigma_y,\, B_{x,d}\sigma_x,\, (\phi + B_{z,d})\,\sigma_z)^\top\|,$$

where $\|\cdot\|$ *is the Euclidean norm.*

Proof. Analogous to the proof of Proposition 9.

□