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Abstract

I analyze the equilibrium level of liquidity and its relevance for the allocation of credit, when the notion of liquidity is related to private information. The general equilibrium analysis yields the following main implications: firstly, it provides an explanation of procyclical liquidity even in the presence of security endogeneity; secondly, it illustrates how government debt, by providing liquidity to an otherwise illiquid private market, encourages rather than "crowds out" private investment; thirdly, it offers a well defined notion of securities' value, the liquidity of which is endogenously enhanced by the arrangements within financial markets.

The approach jointly analyzes the three factors crucial to liquidity: (1) its level is endogenously determined through equilibrium pricing while entrepreneurs choose which security to issue; (2) the introduction of government debt has the two-fold effect of directly providing liquidity to entrepreneurs and indirectly influencing the type of securities they issue in equilibrium; (3) financial markets develop arrangements to allow the beneficial employment of securities, not only physical assets, as collateral (financial pyramiding).

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1 Introduction

The liquidity of an economy is considered one of its defining features. Liquidity shortages are often associated with poor economic performance and asset pricing is believed to be significantly affected by the liquidity available to investors[1]. It is therefore important to study what determines the equilibrium level of liquidity of an economy. This paper presents a general equilibrium characterization of the notion of liquidity and its determinants. The latter include the nature of the optimal security design in the private sector, the presence of government debt, and the available financial arrangements[2]. The result of this enquiry provides an explanation of the relation between liquidity and the allocation of credit, the correlation between liquidity and the state of economy and of the economic policies that affect it.

Following Eisfeldt (2004), the liquidity of a security is defined by the ease of translating its future value into current market prices. A security is thus more liquid the closer its market price to its actual (discounted) expected payment and - similarly - we say that the economy's liquidity increases with the liquidity of its securities[3]. In reality, there may be different reasons for the existence of a wedge between the actual - full information - value and the market price of a security[4]. In this work, I will focus *exclusively* on informational asymmetries.

The first distinctive feature of this economy is that agents have private information about the quality of the technologies they own when they are entrepreneurs. For simplicity there are only two kinds of technologies, whose outcomes are stochastic, and which can be ranked at the time investment is undertaken in terms of expected production. All agents are equal in terms of preferences and endowments but, in each period, some individuals receive an investment opportunity. When an agent faces an investment opportunity he becomes an entrepreneur: some entrepreneurs receive the better technology while the others receive the worse one. The agents who do not face an investment opportunity in a given period will be referred to as consumers. Each entrepreneur desires to borrow in order to increase his private investment above his own endowment and portfolio. In these economies, borrowing is channeled through the issuance of securities that are traded in anonymous capital markets.

By purchasing securities, consumers lend funds to entrepreneurs. The security price consumers are willing to pay is an increasing function of the security's expected payment. I

[1] See Eisfeldt (2004) for the former and Caballero and Krishnamurthy (2001), Holmstrom and Tirole (1998), (2001), Huang (2003), Allen and Gale (2003) for the latter.

[2] This study does not explicitly model financial intermediation, even though, since Diamond and Dybvig (1983), an ample literature has stressed the relation between banks and liquidity. Given our construct though, introducing banks would not make any difference unless they were endowed with a superior contractual/screening technology.

[3] Quoting holmstrom and Tirole (2001), footnote 1: "Liquidity [...] does not [only] refer to the ease with which assets can be resold, but rather to [...] the value of financial instruments used to transport wealth across time [...]."

[4] Johnson (2004) stresses that the finance literature highlights at least three distinct sources for (il)liquidity: search costs, inventory risks and asymmetric information. Here I focus on a reinterpretation of the last. Morris and Shin (2004) base their notion of liquidity on private information, though not about the value of the traded object.

assume that the expected delivery is in turn positively related to the quality of the technology of the issuer so that firms with better technologies issue securities with higher expected deliveries, *ceteris paribus*. Since entrepreneurs are privately informed about the quality of the technology they own, the payoff of a security becomes uncertain. This uncertainty is ultimately responsible for the difference between the actual value of a security and its market price, i.e. its liquidity. In conclusion, by affecting securities market prices, the entrepreneur's private information affects his ability to borrow and then to invest.

The second distinctive feature of this economy is that entrepreneurs choose what kind of security they issue. In reality firms issuing securities can offer a wide array of contingent payments[5]. This observation is reinterpreted here by allowing firms to design securities differing from each other by the likelihood and extent of their default. This construction establishes a large security space for entrepreneurs. When a security is issued, the entrepreneur decides how much "collateral" is attached to it[6]. The more collateral a security carries, the higher is its expected payment[7]. Deciding what kind of security to issue, entrepreneurs compare the relation between the market price and expected delivery (i.e. its liquidity) across each different security[8]. But since collateral is limited, increasing the *per security* collateral reduces the number of securities an entrepreneur can issue. Therefore there is a trade-off between the equilibrium liquidity of a security and the collateral it requires, the result of which determines the optimal security. Unlike most literature addressing liquidity - but in line with Demarzo and Duffie (1999), Geanakoplos (2003) and Demarzo (2003) - the asset structure thus arises endogenously.

I assume that consumers are able to observe the amount of collateral attached to each security - thus its defining feature - but are unable to observe its quality. This inability comes from the fact that the value of each unit of collateral is an increasing function of the quality of the underlying technology, which is private information. Each security price thus reflects the degree of uncertainty associated with the mix of entrepreneurs issuing a specific security. Different securities may therefore carry different degrees of uncertainty and their demands take this fact into account in the determination of equilibrium prices.

In fact, at one extreme of the security space, we have securities characterized by collateral level high enough to insure no default irrespective of whom the issuer is. A consumer choosing to buy such a security would face no relevant informational asymmetry and the equilibrium price would reflect it; if instead the collateral level implies some default for at least one kind of technology, then the buyer knows he may be purchasing a security issued

[5] Equity, bonds, indexation and default are only the main instances of how firms design contingent payments.

[6] This construction of the security space is similar to Geanakoplos and Zame (2002) and Geanakoplos (2003). For the moment it is convenient to keep the notion of collateral vague and focus on the properties we impose on it. Later on we will identify it with the capital that each entrepreneur is building at the time of investment.

[7] What we call collateral here is not some unambiguous stock of wealth (e.g. gold or dollars). It is rather a factor of production, as we will see, whose value is strictly related to the performance and productivity of a firm. Notice the similarity between this notion of collateral and Kiyotaki and Moore (1997), Krishnamurthy (2000) and Caballero and Krishnamurthy (2001).

[8] In the context of competitive markets, this comparizon is non trivial and will be addressed through the adoption of a tremble "on the market" argument as in Dubey and Geanakoplos (2003).

by entrepreneurs with low quality technologies. If entrepreneurs with different technologies issue identical securities - in terms of collateral - the market displays features of "Lemons market": the market price reflects the inability to screen across different qualities of securities. The resulting (liquidity) premium charged to the sellers of relatively better securities discourages the issuance of good quality securities while it encourages the issuance of bad quality ones[9].

It is therefore natural to identify the imperfect liquidity of the economy as a pooling equilibrium in the security space. When the equilibrium is pooling, every entrepreneur issues securities with the same level of collateral. This implies that low quality securities are subsidized since they are traded at a price strictly above their expected delivery. By the same token, I will identify a perfectly liquid economy by a separating equilibrium. If, anticipating the illiquidity of a pooling equilibrium, entrepreneurs with better technologies issue securities with relatively higher level of collateral, then each consumer can track back the quality of a security simply by observing the level of collateral attached to it. It is crucial to observe that, in the attempt to signal themselves, high quality entrepreneurs decrease the number of securities they can issue, their maximum leverage, and become more borrowing constrained. Therefore, in a separating equilibrium, entrepreneurs are willing to accept the cost of decreasing their maximum leverage in order to obtain a price closer to the expected delivery of their securities, i.e. increase their liquidity.

The trade-off between the level of collateral necessary to solve the informational asymmetry (the leverage effect) and the degree of subsidization entailed in the pooling equilibrium (the price effect) is ultimately responsible for the presence of one type of equilibrium over the other. But because of the informational asymmetry between entrepreneurs and consumers, in both pooling and separating equilibria the securities issued by entrepreneurs with better technologies face worse borrowing terms[10]. Therefore better technologies are adversely selected. Relating the liquidity of a security to its suitability in financing private investment, the equilibrium level of liquidity has become a crucial factor in the allocation of credit, and thus private investment, across different technologies.

The proposed general equilibrium perspective is then directed at analyzing the conditions sufficient to guarantee a pooling equilibrium in the security space. This equilibrium is central to our analysis because an illiquid economy is defined by it. In signalling games, pooling equilibria tend to be sensitive to how optimistic agents are about off-equilibrium choices (securities in our case)[11]. I address this issue and show that our pooling equilibria are in fact robust to optimistic off-equilibrium expectations, provided that the share of good

[9] Kiyotaki and Moore (2001) studies the relation between liquidity and asset pricing in an economy with both liquid and illiquid assets. Their notion of liquidity relates to limited commitment rather than private information. Their perspective allows to reinterpret asset pricing and monetary policy in interesting ways. This paper stresses instead the fact that, in the case of securities, the liquidity of an asset should be considered the result of an optimizing behavior and is, therefore, endogenous. This allows to discuss the relation between liquidity and the business cycle and reinterpret the role of government debt.

[10] In a pooling equilibrium, high quality securities "subsidize" the price of low quality ones. In a separating equilibrium, entrepreneurs issuing high quality securities can sell less of them since each entails strictly more collateral.

[11] See Dubey and Geanakoplos (2003) and Martin (2004) for a discussion about the "fragility" of pooling equilibria in the context of insurance models.

quality entrepreneurs in the population is high enough. The intuition behind the existence of pooling equilibria is simple, bearing the basic trade-off in mind. The pooling equilibrium entails a cost (the price effect) for entrepreneurs with the best technology. If this cost is high enough, then they deviate and a liquid separating equilibrium emerges. But the higher is the share of high quality firms, the higher is the pooling price and the lower is the cost of the pooling equilibrium they bear. Therefore a sufficiently high price makes high quality entrepreneurs not willing to deviate and thus supports the pooling equilibrium[12]. The robustness of pooling equilibria is ensured by the local failure of the "single crossing property" consequence of our setup. Good quality entrepreneurs may attempt to distinguish themselves by issuing securities with lower expected default (i.e. higher expected payments). But since the capital is more productive - and thus more valuable - the better is the technology, the expected delivery of securities issued by good quality entrepreneurs increases relatively faster than that of bad quality securities as the capital that guarantees repayment increases. The positive correlation between capital/collateral value and technology's quality - a reasonable assumption in our opinion - is ultimately responsible for the local failure of the "single crossing property". This local failure is sufficient to ensure robustness since it restrains good entrepreneurs from abandoning the pooling equilibrium, even in the presence of "optimistic" beliefs about off equilibrium securities.

Once the robustness of illiquid pooling equilibria has been established, we turn to the evidence suggesting that the liquidity of the economy tends to be procyclical. The provision of an empirically consistent explanation in the presence of security endogeneity is not at all obvious. One may be tempted to say that when the economic outlook looks grim, good quality firms want to avoid the additional liquidity cost connected to the pooling equilibrium. Thus a stronger incentive toward a liquid separating equilibrium would emerge. This would be an explanation at odds with the evidence. A simple reinterpretation of the proposed framework provides an empirically consistent explanation. When the economy is booming, the contingency with low production becomes less likely. Thus both production technologies tend to deliver more production, in expectation, at any given level of investment. Since the contingency with low production, where default is greater, becomes less likely, expected deliveries of different quality securities tend to become closer, even for securities characterized by low level of collateral. High quality firms are then encouraged to deviate out of the pooling since the cost of separation decreases. Eventually a separating equilibrium emerges when the economy booms and vice versa when the economy is in recession. Following this argument, it is possible to provide an explanation of procyclical liquidity where the economy switches from liquid separating equilibria during good times to illiquid pooling equilibria during bad times.

Moreover, the framework provides a novel interpretation of the role of government debt in crowding "in" private investment[13]. Since government bonds do not require the provision of collateral, and under the assumption that they provide the security with the least informational asymmetry in the economy, a positive amount of government debt raises aggregate

[12] Our methodology is very close to Dubey and Geanakoplos (2003).

[13] This view was already suggested by Woodford (1990) and Holmstrom and Tirole (1998). Gorton and Huang (2004) is different since it focuses on banks bailouts.

investment. Government bonds are ideal candidates to transfer wealth to the future, besides what is already provided by private securities. In fact, an economy such as the one described here faces two main problems: the first is the cost of using the available collateral to guarantee securities' repayment; the second problem is the lack of an instrument to increase the saving of current period consumers who may face an investment opportunity in the future. Current consumers may become entrepreneurs in the future. When this happens, they may find themselves borrowing constrained. They thus find it convenient to purchase government bonds today, as an additional store of value, in order to increase the funds at their disposal tomorrow. By relaxing the collateral constraint in an informationally efficient way, positive government debt crowds *in* private investment. I will show that the positive effect of government debt over equilibrium investment is potentially twofold. In fact the introduction of government debt, by relaxing the collateral constraint, not only increases private investment directly but builds an incentive to move the equilibrium of the economy toward separation. The *liquidity effect* of government debt on equilibrium securities is novel in the literature and underlines the different purposes that government debt serves in the economy.

Finally, I will show that an additional beneficial role is played by the possibility of using securities, besides physical capital, as collateral when sellers have superior information about their value. I will name this arrangement *financial pyramiding*. I argue that part of the extensive security creation we observe in financial markets arises to address the informational asymmetry present in the market for the securities resale. The reasoning is straightforward: if individuals use the securities they own as collateral instead of reselling them, they may avoid paying the liquidity premium the market imposes because of the private information. Allowing pyramiding reduces the cost of resale for the owner of good securities and therefore may increase the equilibrium level of investment. This is also reflected in the securities issued in equilibrium: since the asymmetric information in the resale market is less severe, the level of adverse selection in investment is thereby reduced. This interpretation of the role of financial markets can be traced back to the intuition of Arrow (1964): as financial markets arise to add missing markets or replace existing ones, so financial arrangements are interpreted as instruments tackling the imperfections imposed by the possibility of default.

1.1 Related Literature

The focus of this paper is macroeconomic though this line of enquiry has the benefit of bringing together different strands of the economic literature. Even though the study proposed here relates most closely to Woodford (1990), Holmstrom and Tirole (1998) and Eisfeldt (2004), the employed setup builds upon Geanakoplos and Zame (2002) and Geanakoplos (2003). Three branches of the literature therefore relate to this study.

The problem of how to embed an endogenous asset structure in general equilibrium is well known and has recently been addressed by Geanakoplos and Zame (2002). In their work, lending is subject to the provision of collateral and they highlight what inefficiencies are caused by the introduction of collateral requirements. Geanakoplos (2003) applies this perspective to the study of liquidity: he provides an explanation of margin/liquidity changes which depends on the kind of information that becomes available in financial market. While the asset structure employed here is based upon their work, their analysis abstracts from the role of informational asymmetries in the determination of liquidity. In the context of the insurance model, Dubey and Geanakoplos (2003) address the issue of informational asymmetry in competitive markets. Agents trade shares of different *pools* and each pool is characterized by a maximum number of shares each individual can purchase. Active pools are endogenously selected in equilibrium, thereby recasting the Rotschild-Stiglitz model. In doing so they propose an argument to establish the existence and robustness of separating equilibria in a competitive context where the usual off-equilibrium reasoning cannot be applied. In my work I tackle the same methodological difficulty and thus adapt their argument to show that (illiquid) pooling equilibria are robust to optimistic perturbations in the spirit of Dubey and Geanakoplos (2003).

In this paper I leave moral hazard aside and focus on the role of private information. In this way the argument naturally evolves towards adverse selection, outlined as crucial to firm's financing since the seminal contributions of Leland and Pyle (1977), Myers and Majluf (1978) and Stiglitz and Weiss (1981). Since these studies, entrepreneur's private information is recognized as an important limitation in the financing of profitable investment opportunities. Leland and Pyle (1977), Myers and Mayluf (1984) focus on the choice of whether or not to issue new securities when entrepreneurs/managers undertake valuable projects on which they are privately informed. They argue that constraining the issuance of securities, or just refraining from issuance, signals the quality of the available investment opportunity. This same perspective has been recently revived and enhanced by Duffie and DeMarzo (1999), DeMarzo (2003) in the context of security design. When the issuer of a security has better information about its underlying value, the design problem becomes non trivial. Issuers typically face two issues: the design of the security they wish to sell and the decision regarding how much should be sold. Different kinds of securities convey different information and hedge in different ways against the potential "Lemons market" problem. Here I follow the same intuition but I focus on the case of a production economy. Because of the investment opportunity they face, entrepreneurs prefer "money" today to "money" tomorrow. In order to raise funds, they take into account their private information and issue the security that provides them the best trade-off between the price they receive, net of the information-based liquidity cost, and the number of securities of that kind they are able to issue.

The positive relation between the liquidity of an economy and its performance has recently received attention. The important contribution of Eisfeldt (2004) is probably the first attempt in this direction. In her paper, agents are infinitely lived and can finance the investment opportunity they face selling claims over future production of the initiated projects they own. Since there are two kinds of projects in the economy -let's call them good and bad- but claims are indistinguishable in capital markets, claims over good projects are underpriced while claims over bad projects are overpriced. Since bad claims are always completely sold and good claims are partially sold because of the cost they entail, the share of good claims increases when the economy faces a positive productivity shock and viceversa. The most natural criticism to this approach is that entrepreneurs may change the security they issue depending on the state of the economy. Our framework addresses precisely this point.

The last branch of literature connected to this work refers to the role of government debt in fostering private investment. Woodford (1990) and Holmstrom and Tirole (1998) present this argument in the context of imperfect capital markets (the former) and moral hazard (the latter). Their conclusion is the illustration of when and how the supply of government debt tackles the illiquidity of the economy beside what is already possible through private capital markets. The intuition behind their work - which can be tracked back to Cass and Yaari (1966) - is that government debt provides a transfer technology that becomes particularly useful when markets are illiquid. This work recasts the same intuition but with an additional twist: in a setting where firms may choose the security they issue, the introduction of government debt does not only foster private investment but also affects the type of securities that are issued in equilibrium.

Outline. Section 2 lays out the setup and equilibrium definition. Sections 3 discusses the equilibrium. Section 4 develops the relation between business cycle and the equilibrium level of liquidity. Sections 5 and 6 extend the baseline introducing, respectively, government debt and financial arrangements. Section 7 finally concludes.

2 Baseline Economy: Setup with Physical Collateral

This setup formalizes the discussion above. The reader can refer to Figure 1 throughout the presentation:

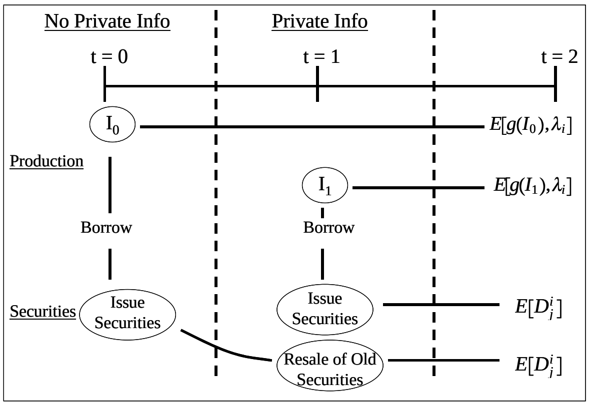

Figure 1

- **Time:** Assume an economy lasting three periods: $t = 0, 1, 2$, and where two contingencies, $s = \{G(ood), B(ad)\}$, can realize at $t = 2$ with probability $\{p, 1-p\}$ respectively;

- **Commodity Space:** there is a single perishable consumption/capital good in each period. Let $c_t^h(s)$ denote the amount consumed by agent h at time t in state s;

- **Agents and Endowment:** there is a continuum of agents $h \in [0,1]$, uniformly distributed. Each agent has an equal endowment profile $(w, w, 0) \in \Re_+^3$ where w

denotes individual endowment measured in units of the unique good;

- **Investment Opportunities:** every agent has the same ex ante probability of facing an investment opportunity and knows that he can own *at most* one project in his lifetime. At $t = 0$ each agent encounters an investment opportunity (and thus becomes an entrepreneur) with probability π_0. By the law of large numbers π_0 is also the measure of $t = 0$ entrepreneurs. At $t = 1$ the probability of becoming an entrepreneur is π_1 and the measure of $t = 1$ entrepreneurs is $0 < [\pi_1 \cdot (1 - \pi_0)] < 1$. Also, $[\pi_0 + \pi_1 \cdot (1 - \pi_0)] < 1$. Each h becomes aware of his technology's productivity λ_i only at the time when he faces the investment opportunity.

  For reasons that will become clear later, we choose π_0 and π_1 such that consumers have positive consumption in each period, after they lend to entrepreneurs. All agents are ex ante identical, but they differ ex post if they face an investment opportunity. The individual tree is illustrated in Figure 2:

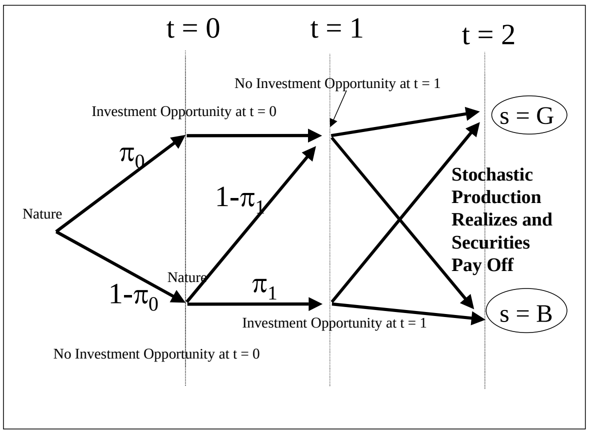

Figure 2

- **Preferences:** in order to focus on liquidity only, it is convenient to assume that agents are not concerned by risk:

$$V^h\left[c_0^h, c_1^h, c_2^h = \left\{c_2^h(G), c_2^h(B)\right\}\right] = E_{\pi_0}\left[c_0^h\right] + E_{\pi_1}\left[c_1^h\right] + E_s\left[c_2^h(s)\right]$$
$$s \in \{G, B\} \text{ at } t = 2 \text{ only}$$

- **Information Structure**

  At $t = 0$ each and every agent can observe the actual value of any investment project in the economy and the expected delivery of each security is observable. Therefore, lending involves no asymmetric information

  At $t = 1$ every agent is privately informed about the actual value of the securities in his portfolio and about the productivity of his investment project (if he is an entrepreneur). Given the way securities are constructed - as detailed below - entrepreneurs are privately informed about the actual quality of the securities they are issuing. Because of asymmetric information, buyers can only form expectations about the average value of each security and pay a price consistent with it. Summarizing:

$t = 0$	Symmetric information about investment projects/collateral
$t = 1$	Asymmetric information about investment projects/collateral and individual portfolios

This information structure may seem contrived. For the sake of our argument there is no need to assume this exact structure. We could equivalently assume that asymmetric information also characterizes $t = 0$ and all the results of this paper would still hold. In fact, $t = 0$ would be an almost identical copy of $t = 1$ and we could therefore extend to $t = 0$ the analysis and conclusion of $t = 1$, given some minor adaptations.

We have chosen this construction because it highlights more thoroughly the different effects of private information. The structure we will use in the paper shows that even the mere anticipation of $t = 1$ asymmetric information has an effect on the equilibrium prices of $t = 0$ symmetric information economy.

- **Technology:** there are 2 different types of investment projects, $i = (H(igh\ variance),\ L(ow\ variance))$ both available at $t = 0$ and $t = 1$. $t = 2$ production is stochastic and is labelled by $g_i(I_t; s)$:

$$g_i(I_t; s) = \begin{cases} (1+\lambda_i)g(I_t) & if\ s = G \\ (1-\lambda_i)g(I_t) & if\ s = B \end{cases}$$

$$\lambda_i \in \{\lambda_H, \lambda_L\}$$

$$1 > \lambda_H > \lambda_L > 0$$

$$g(0) = 0,\ g'(.) > 0,\ g''(.) < 0$$

$$\lim_{x \longrightarrow 0} g'(x) = +\infty$$

The function $g(.)$ is a standard strictly concave neoclassical production function, while I_t is the units of capital invested. The ranking of technologies is based on expected production, i.e.:

$$E_s[g_i(I_t; s)] = [p(1+\lambda_i) + (1-p)(1-\lambda_i)]\, g(I_t) = \\ = [1 + (2p-1)\lambda_i]\, g(I_t)$$

Therefore their ranking depends upon p being smaller or larger than (1/2). Moreover I will write:

$$E_s\left[g_i'(I_t; s)\right] = [1 + (2p-1)\lambda_i] \frac{dg(I_t)}{dI_t}$$

Notice finally that we assume **perfect positive correlation** (i.e. aggregate uncertainty) in technology's payoff[14]. This is equivalent to assume that there two aggregate

[14] This assumption is made to simplify notation and without loss of generality.

states but that each production process is subject to technology specific state contingent shock. The two technologies are distributed in the aggregate according to measure $\eta(.)$:

$$\lambda_i \backsim (\eta(H), \eta(L) = 1 - \eta(H))$$
$$1 > \eta(H) > 0$$

- **Security Structure**[15]: lending is mediated through the exchange of securities that are traded in anonymous financial markets. Securities' payments are guaranteed by the provision of collateral[16]. Here the only available collateral is the amount of the single good built into invested capital. (Finitely) many financial contracts (securities) can be generated by simply adjusting the level of collateral, as we now illustrate.

  Each security is characterized by a vector D_j^i of state contingent deliveries, $D_j^i(s)$, which depends on the level of collateral j and the quality of technology i where the collateral is employed. The collateral is *physical* units of capital and the number of units of capital attached to a security is labelled by j. All financial contracts, issued either at $t = 0$ or $t = 1$ with collateral j in technology i, translate into deliveries at $t = 2$ in the following way:

$$D_j^i = \begin{bmatrix} D_j^i(G) = \min\{1, (1+\lambda_i)j\} \\ D_j^i(B) = \min\{1, (1-\lambda_i)j\} \end{bmatrix} \tag{1}$$

  By construction, each financial contract pays 1 unit of the consumption good if there is no default and, respectively, $[(1+\lambda_i)j]$ or $[(1-\lambda_i)j]$ otherwise[17]. The delivery vector (D_j^i) is thus affected by both the level of collateral (j) *and* the quality of the technology (i) in which it is employed. Notice that the collateral value is an increasing function of technology's productivity and contingent production. In fact we have:

$$E\left[D_j^i\right] = \begin{cases} [p(1+\lambda_i) + (1-p)(1-\lambda_i)] \cdot j & \text{if} \quad 0 \le j \le \frac{1}{1+\lambda_i} \\ p + (1-p)(1-\lambda_i) \cdot j & \text{if} \quad \frac{1}{1+\lambda_i} \le j \le \frac{1}{1-\lambda_i} \\ 1 & \text{if} \quad j \ge \frac{1}{1-\lambda_i} \end{cases} \tag{2}$$

  where

$$p = \Pr(s = G);\ (1-p) = \Pr(s = B)$$
$$\lambda_i \in \{\lambda_H, \lambda_L\}$$
$$1 > \lambda_H > \lambda_L > 0$$

[15] Similar to Geanakoplos and Zame (2002) and Geanakoplos (2002).

[16] The majority of lending is subject to collateral requirements, in particular a large share of credit to the productive sector. In Italy, 20 % of credit to firms is guaranteed by the provision of real collateral only. Notice this is likely to be a large understatement of the importance of collateral. This is due to the difficulty of measuring other kind of collateral relevant to lending decision (e.g. financial collateral or "implicit" collateral).

[17] To ensure measurability in the budget constraint we have to restrict the collateral level j into the following grid $\{k/10^{10} : k \in N \text{ and } 1 \le k \le 10^{100}\}$.

Graphically:

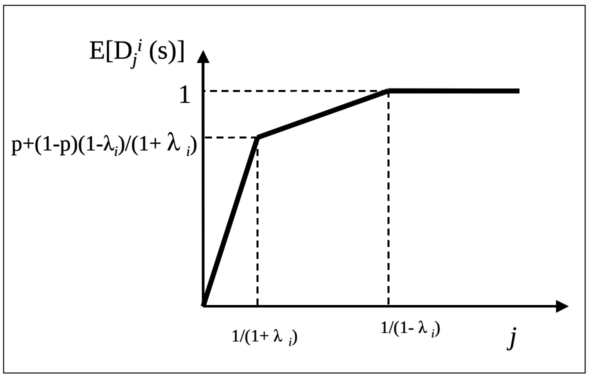

Figure 3

Each security/financial contract is characterized by a delivery vector, the fundamental determinant of its price[18]. By assumption, the seller of a security always knows the quality of the technology where the collateral is invested. $A\ fortiori$ he is always aware about the actual delivery of the security he is selling.

Given the assumed information structure, securities issued at $t = 0$ are identified by the pair (j, i). Thus the market for each security is indexed by (j, i) and this notation denotes the fact that $t = 0$ securities with collateral level j are traded on two separate i-specific market.

Securities issued at $t = 1$ instead are indexed by the level of collateral (j) only since buyers do not know the technology where the collateral is invested. Each security market is denoted by (j) only.

This setup provides a simple microfoundation of a security's value. I assume that this depends upon the units of collateral (i.e. units of capital invested) attached to it and the quality of the technology where these units are invested. The assumptions that the value of collateral is a increasing function of production and that it is directly related to the technology specific parameter λ_i play a central role in our arguments. A number of stories could provide an economic rationale for it. In fact this is no different than assuming that the more productive a firm is, the more valuable its "plants" (units of capital) are, and that the better is the economic performance of a firm - its production here - the more valuable its capital becomes.

- **Security Holdings:** agent h may hold $a_t^h(j, i)$ units of the financial contract (j) with the standard convention that $a_t^h(j, i) < 0$ when he sells and $a_t^h(j, i) > 0$ when he buys. It will be apparent below that, because of collateral requirements, it is convenient to

[18] Notice that the concavity of the production function together with the linearity of the collateralization rule imply that future production is not completely seizable by lenders at $t = 2$. In particular, standing our assumptions, the share of production that can be seized is an increasing function of the investment size, I, given the concavity of $g(.)$ and is constant across contingency, given I.

adopt the following notation:

$$\theta_t^h(j,i) = \max\left\{a_t^h(j,i), 0\right\}$$
$$\varphi_t^h(j,i) = \min\left\{a_t^h(j,i), 0\right\}$$

Therefore, h's portfolio is denoted by the vectors:

$$\theta_t^h;\ \varphi_t^h;\ \left(\theta_1^h\right)^R \in \Re^{N\cdot 10^{100}},\ \forall t$$

where $\left(\theta_1^h\right)^R$ refers to the securities issued at $t = 0$ that are purchased at $t = 1$.

- **Security Pricing:** I will denote by $q_0(j,i)$ the price of a security issued at $t = 0$ with collateral j invested in technology i, by $q_1(j)$ the price of a security issued at $t = 1$ with collateral j and by $q_1^R(j)$ the price of a security issued at $t = 0$ and traded at $t = 1$ the resale market. Thus prices are denoted by the vectors:

$$q_0;\ q_1;\ q_1^R \in \Re^{N\cdot 10^{100}}$$

- **Individual Budget Constraint**[19]

Since the issue of securities requires the provision of collateral and collateral coincides with capital, only entrepreneurs can issue new securities. So the issue of securities has to satisfy:

Collateral Constraint (**Physical Collateral**)

$$\sum_{\varphi_t^h(j,i)} \varphi_t^h(j,i) j \leq I_t^h \qquad \text{(CC(t))}$$

Intuitively, the total collateral employed by one borrower can not exceed the units of capital he builds. Since investment, and therefore capital, is endogenous, the borrowing constraints are endogenous too[20]. Agent h's $t = 1$ long positions can only be resold at $t = 0$ and so each agent satisfies:

Resaleability Constraint:

$$0 \leq \gamma_{ji}^h \leq 1 \text{ iff } \theta_0^h(j,i) > 0 \qquad \text{(RC)}$$

where γ_{ji}^h is the share of security i purchased at $t = 0$ and resold at $t = 1$ by h. Intuitively, no one can resell at $t = 1$ more securities than he bought at $t = 0$.

In this case, agent h faces the following budget constraint $B^h(q)$. At $t = 0$, if h is a consumer:

$$c_0^h \leq w - \sum_i \sum_j q_0(j,i)\theta_0^h(j,i)$$

[19] The reader may refer to the Figure 2 for a better grasp of the budget constraint.

[20] Geanakoplos (2002) defines the liquidity of an economy "by how closely the collateral budget set comes to attaining the general equilibrium (with incomplete markets) budget set". The intuition is similar to what we have here though his does not explicitly relate to asymmetric information as the conceptual basis of liquidity.

while if h is an entrepreneur:

$$c_0^h + I_0^h \leq w + \sum_i \sum_j q_0(j,i) \left[\varphi_0^h(j,i) - \theta_0^h(j,i) \right]$$

At $t = 1$, if h is a consumer:

$$c_1^h \leq w + \sum_j \left\{ q_1^R(j) \left[\sum_i \left(\gamma_{ji}^h \right) \theta_0^h(j,i) - \left(\theta_1^h(j) \right)^R \right] + q_1(j)\theta_1^h(j) \right\}$$

If h is an entrepreneur, we have:

$$c_1^h + I_1^h \leq w + \sum_j \left\{ q_1^R(j) \left[\sum_i \left(\gamma_{ji}^h \right) \theta_0^h(j,i) - \left(\theta_1^h(j) \right)^R \right] - q_1(j) \left(\theta_1^h(j) - \varphi_1^h(j,i) \right) \right\}$$

At $t = 2$, if h has always been a consumer:

$$\begin{array}{c} c_2^h(s) \leq \sum_i \sum_j \left\{ D_j^i(s) \left[(1-\gamma_{ji}^h)\theta_0^h(j,i) \right] \right\} + \\ + \sum_j \left\{ D_j^{i^*}(s)\theta_1^h(j) - D_j^i(s)\varphi_1^h(j,i) + \left[D_j^{i^*}(s) \right]^R \left[\theta_1^h(j) \right]^R \right\} \end{array}$$

while if h has been an entrepreneur we have:

$$\begin{array}{c} c_2^h(s) \leq g_i\left(I_t^h; s \right) + \sum_i \sum_j \left\{ D_j^i(s) \left[(1-\gamma_{ji}^h)\theta_0^h(j,i) - \varphi_0^h(j,i) \right] \right\} + \\ + \sum_j \left\{ D_j^{i^*}(s)\theta_1^h(j) - D_j^i(s)\varphi_1^h(j,i) + \left[D_j^{i^*}(s) \right]^R \left[\theta_1^h(j) \right]^R \right\} \end{array}$$

where i*is the average collateral quality of the securities purchased at $t = 1$. $t = 2$ securities' payoff depends on the *actual* value of collateral. Since consumers are all identical, they hold the same portfolio so the actual value of collateral is nothing else than the average one, given any j.

The budget constraint is notationally intense but standard in the interpretation. Consumers allocate their endowment between consumption and the purchase of securities, while entrepreneurs issue securities, trade their portfolio and allocate their endowment between consumption and investment.

- **Notation:** I adopt the following convention in labelling a generic variable x:

$$x_{t,k}^h(s)$$

where $k \in \{\pi_0, (1-\pi_0), \pi_1, (1-\pi_1)\}$. This convention allows to distinguish between individuals, at a given time t, that face an investment opportunity $\{\pi_0, \pi_1\}$ and individuals who do not $\{(1-\pi_0), (1-\pi_1)\}$. For instance, $\varphi_{1,\pi_1}^h(j)$ denotes the number of units of security j sold by an entrepreneur at $t = 1$.

In order to keep notation intuitive, I will often use the superscript (^{i}) instead of (^{h}) to denote that a variable relates to the entrepreneurs owning technology i. I_t^i thus denotes individual investment undertaken at time t by an entrepreneur endowed with technology i. Moreover, for the sake of simplicity, I will often use the superscript $\bar{i}$ ($\underline{i}$) to denote best (worst) technologies/securities.

2.1 Definition of the Equilibrium

An equilibrium in this economy is defined by consumption allocations $c^h = \left[c_0^h,\ c_1^h,\ c_2^h = \left\{c_2^h(G), c_2^h(B)\right\}\right]$, asset holdings $a^h = [\theta_0^h(j,i),\ \varphi_0^h(j,i),\ \theta_1^h(j),\ \varphi_1^h(j,i),\ \left(\theta_1^h(j)\right)^R],\ \forall h$, and asset prices $q = \left(q_0, q_1, q_1^R\right) \in \Re^{N\cdot 10^{100}}$ satisfying the following set of equations:

1. **Individual Optimum**

$$\left(a^h, c^h\right) \in \arg\max \left\{ \begin{array}{c} V^h\left[c^h\right] \\ \text{s.t.} \ \left[B^h\left(q\right)\right] \end{array} \right\} \text{ at given } q, \forall h$$

2. **Market Clearing Conditions**

$$\int_0^1 \left[\theta_0^h(j,i) - \varphi_0^h(j,i)\right] dh = 0 \ \forall j, i \text{ at } t = 0 \qquad (t = 0 \text{ securities})$$

$$\int_0^1 \left[\theta_1^h(j) - \varphi_1^h(j,i)\right] dh = 0 \ \forall j \text{ at } t = 1 \qquad (t = 1 \text{ securities})$$

$$\int_0^1 \left[\gamma_{ji}^h \theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R\right] dh = 0 \ \forall j \text{ at } t = 1 \qquad (\text{Resale Market})$$

$$\begin{array}{c} \int_0^1 \left[c_t^h + I_t^h - w_t^h\right] dh = 0,\ t = 0,1 \\ \int_0^1 \left[c_2^h(s) - g_i\left(I_0^h; s\right) - g_i\left(I_1^h; s\right)\right] dh = 0,\ t = 2,\ s \in \{G, B\} \end{array} \qquad (\text{Goods Market})$$

3 Equilibrium in the Baseline Economy

I will first illustrate the symmetric information benchmark and then I will discuss at length the properties of the production economy where private information is present. Before doing so, it is necessary to analyze in detail the relationship among the payoffs of securities of different types. This will identify the security space in which our analysis is made. We turn to this question now.

3.1 Security Payoffs: Collateral and the Delivery Vector

In equilibrium, each security is priced according to the *expected* consumption it delivers. This expectation is clearly affected by the presence of asymmetric information. In this subsection I show the implications of choosing the level of collateral for the relation between the payoffs of different securities. This is the prerequisite to characterizing, in terms of collateral level, the security a firm will optimally issue. For the entire discussion in Section 3 I assume:

(A.1) *The probability that the Good state realizes is* $p < 1/2$.

(A.1) implies that technology L is ranked superior, in expected production, to technology H (since $\lambda_H > \lambda_L$). As a consequence of the ranking across technologies, the expected delivery of securities issued by technology H and L behave according to Figure 4.

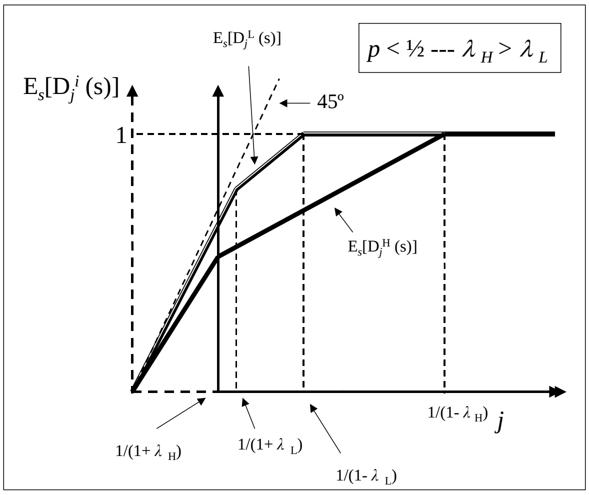

Figure 4

(A.1) will be relaxed later when the case $p > 1/2$ will be addressed in section 4 where I apply our framework to the analysis of the business cycle.

Without loss of generality, recalling (2), we can restrict our attention to the following security space:

$$j \in \widehat{J} = \left\{ \underline{j} = \frac{1}{1+\lambda_H} \leq j \leq \frac{1}{1-\lambda_H} = \overline{j} \right\} \tag{3}$$

Since the ratio $\frac{E_s\left[D_j^L\right]}{E_s\left[D_j^H\right]}$ is equalized for securities $0 < j \leq \frac{1}{1+\lambda_H}$, we can restrict attention to $j = \frac{1}{1+\lambda_H}$.

3.2 The Economy with Symmetric Information

In this section I discuss the *symmetric* information benchmark of the economy with entrepreneurial private information. Collateral is still required to issue securities and borrow. Since the value of collateral is "transparent" (i.e. anyone can check its worth at any time) lenders always know the actual delivery of the security they are purchasing. Since preferences are risk neutral, a security's price is a linear function of its deliveries:

$$q_0(j,i) \geq E_s\left[D_j^i(s)\right];\ t = 0,\ \forall i$$
$$q_1(j,i) = E_s\left[D_j^i(s)\right];\ t = 1,\ \forall i$$

Notice that the security price may be strictly larger than its expected delivery at $t = 0$, since $t = 0$ securities serve as a source of financing at $t = 1$ and thus relax the collateral constraint even further. This service is beneficial to $t = 1$ entrepreneurs, when the collateral constraint binds, and thus is positively priced at $t = 0$. In order for the problem to be meaningful we assume that access to the credit market is beneficial to entrepreneurs:

$$E_s\left[g_i'(w; s)\right] >> 1, \ \forall i \tag{4}$$

Let

$$R_t^i(j) = \frac{E_s\left[D_j^i(s)\right]}{q_t(j)}$$

be the gross rate of return (or interest factor) for collateral of quality i in security j. $R_t^i(j)$ is just the ratio between expected delivery, $E_s\left[D_j^i(s)\right]$, and the market price, $q_1(j)$, of security j issued by entrepreneur with technology i.

Solving for the individual problem and rearranging the FOCs[21] with respect to $\varphi(j, i)$ and I_t^i:

$$E_s\left[g_i'\left(I_t^i; s\right)\right] + \beta_t^i = \frac{E_s\left[D_j^i(s)\right]}{q_t(j, i)} + \beta_t^i \frac{j}{q_t(j, i)} \tag{5}$$

where β_t^i is the multiplier on the collateral constraint CC(t) and measures the benefit of relaxing it. The interpretation is straight forward: the level of investment, I_1^i, is determined at the point where the marginal benefit of investing - marginal productivity, $E_s\left[g_i'\left(I_t^i; s\right)\right]$, and the benefit of relaxing the collateralization constraint, β_1^i, - is equal to the marginal cost - the rate of return of the issued security, $R_t^i(j)$, and the cost in terms of collateral (controlling for the price) of the issued security, $\beta_1^i \frac{j}{q_t(j)}$.

It is immediate to observe that technology i holder issues the security that minimizes the collateral relative to price ($\frac{j}{q_t(j,i)}$), i.e. security $j(i) = 1/\left(1 + \lambda_i\right)$. For simplicity I assume that w is large enough so that lenders maintain positive consumption (net of lending) at $t = 0$ (and thus at $t = 1$)[22].

Entrepreneurs, by their individual budget constraint, can invest up to the point where the collateralization constraint is binding:

$$I_0^i \leq w + \frac{I_0^i}{j} q_0(j, i)$$
$$I_1^i \leq \widetilde{w}^i + \frac{I_1^i}{j} q_1(j, i)$$

where $\widetilde{w}^i = w + q_1^R(j)\left[\sum_{i=H,L}\left(\gamma_{ji,\pi_1}^h \theta_{0,1-\pi_0}^h(j, i)\right)\right]$, i.e. the sum of the entrepreneur's private endowment and what he can raise reselling his portfolio. Rearranging:

$$I_0^i \leq w\left(\frac{j}{j - q_0(j,i)}\right)$$
$$I_1^i \leq \widetilde{w}^i\left(\frac{j}{j - q_1(j,i)}\right)$$

[21] Appendix

[22] Alternatively: entrepreneurs are a sufficiently small share of the total population.

Thus we have, since $q_t(j,\bar{i}) > q_t(j,\underline{i})$, that $I_t^{\bar{i}} > I_t^{\underline{i}}$. Furthermore, by construction, we have that $I_0^i < I_1^i$ which implies:

$$E_s\left[g_i'\left(\left(I_0^i\right)^*;s\right)\right] \geq E_s\left[g_i'\left(\left(I_1^i\right)^*;s\right)\right];\ \forall i \tag{6}$$

where $\left(I_t^i\right)^*$ denotes the equilibrium level of investment for the economy with symmetric information. Notice that, in equilibrium, the rates of return of securities issued by different entrepreneurs are equalized:

$$R_t^{\bar{i}}(j^{\bar{i}}) = R_t^{\underline{i}}(j^{\underline{i}})$$

j^i is the security issued by entrepreneur i

otherwise no agent would purchase the security that pays the smaller rate of return. Since different technologies pay the same rate of return, different quality entrepreneurs borrow at the same terms and no adverse selection takes place. The following proposition summarizes:

Proposition 1 *In the economy without private information, the followings hold:*

1. *a technology i entrepreneur issues that security which minimizes the ratio of collateral to market price ($\frac{j}{q_t(j,i)}$), i.e. security $j^i = 1/(1+\lambda_i)$;*
2. *equilibrium investment is increasing in the quality of technology: $\left(I_t^{\bar{i}}\right)^* > \left(I_t^{\underline{i}}\right)^*$, $\forall t$;*
3. $R_t^{\bar{i}}(j^{\bar{i}}) = \frac{E_s\left[D_j^{\bar{i}}(s)\right]}{q_t(j,\bar{i})} = \frac{E_s\left[D_j^{\underline{i}}(s)\right]}{q_t(j,\underline{i})} = R_t^{\underline{i}}(j^{\underline{i}})\forall t$, *i.e. the rate of return is equalized across technologies;*
4. *the equilibrium is constrained pareto optimal*[23].

3.3 The Economy with Asymmetric Information

3.3.1 Market Equilibrium at $t = 1$

We determine the equilibrium of our economy proceeding by backward induction. At $t = 2$, the final period, financial contracts are settled and production realizes. Each entrepreneur either defaults or he does not. In the event of default, he surrenders the collateral underlying his short positions. If he does not default, he pays his promises. Agents holding long positions thus receive either payments or the value of consumption corresponding to the collateral they are entitled to[24]. Finally everyone consumes his net wealth: no other choices are made.

At $t = 1$ measure $[\pi_1(1-\pi_0)]$ of agents face an investment opportunity and become entrepreneurs. Entrepreneurs can raise funds in two ways through anonymous financial

[23] Constrained here refers to the collateralization constraint imposed on borrowers.

[24] See page 10.

markets: either selling some of the securities they previously acquired or issuing new securities using the capital they are building as collateral. In both cases private information plays a crucial role:

(A.2) (Collateral Unobservability) *At $t = 1$ the purchaser of a security cannot observe the actual value of the collateral underlying it.*

(A.3) (Private Information) *At $t = 1$ the issuer/reseller of a security is privately informed about its actual value.*

At $t = 1$, the buyer of a security can *only* observe the number of units of collateral attached to the security he is purchasing. At any given level of collateral, he can only form expectations about the value of collateral for the average traded security. This expectation is the crucial determinant of $t = 1$ market prices,. $q_1(j)$ and $q_1^R(j)$.

Agents selling their long positions, in the resale market, bear the burden of private information. This results from the assumption that a security's quality is unobservable, given the level of collateral. Since sellers of "bad" securities wish to be viewed as sellers of "good" ones, a single market price emerges for any given collateral level. This price in turn reflects, by rational expectations, the expected value of the average security resold in that market.

A similar problem is faced by issuers of new securities at $t = 1$. They take into account the role of private information when they decide how much collateral is attached to the securities they are issuing. As pointed out before, the central question becomes how much separation across technologies, if any, will be present in equilibrium, i.e. whether the equilibrium in the security space is going to be pooling or separating. Therefore, in order to address the issue of liquidity, one has to establish the kind of equilibrium that will prevail: I turn to this now.

3.3.2 Equilibrium Securities at $t = 1$: the economic intuition

In equilibrium, each financial contract is priced according to the expected consumption it delivers. Because of asymmetric information, buyers do not know the actual delivery of the security they are buying and so they must form expectations about it. In principle the relation between market price and expected delivery may change across different securities, providing different incentives for the sellers of less valuable securities ($\underline{i}$) to mimic sellers of more valuable ones. In general, the less costly it is to reproduce the behavior of the good technology entrepreneurs, the more depressed is the market price, $q_1(j)$, and the higher the illiquidity for securities with good collateral, ($\bar{i}$).

Optimally, holders of good technologies would need to put up less collateral (since their collateral is more valuable) in order to guarantee some given delivery vector. But the holders of bad technologies, i.e. providers of low quality collateral, may try to mimic them in order to deceive the market and sell overpriced securities. At the same time the owners of good technologies, anticipating this "shading", tend to increase the amount of collateral in the security they issue. However this increase bears a trade-off: on the one hand, higher collateral per security unambiguously increases the delivery and then the market price; on the other hand, it reduces the number of security units (and thus leverage) an entrepreneur may sell.

The buyer of a financial asset thus faces a "Lemons market" problem: he knows that, given the market price, all agents with securities whose actual value is below the market price will sell and realize a capital gain. This drives down the equilibrium market price of traded financial contracts. The lower market price implicitly imposes a premium charged upon those entrepreneurs with good technologies issuing securities backed by good collateral. Notice that the "Lemons" problem arises because the holders and issuers of bad securities always sell to the market relatively more shares than the holders of good ones.

Notice also the importance of (A.2): since the productivity of investment opportunities at $t = 1$ is private information, each entrepreneur would rather issue new collateralized securities than sell good securities in his portfolio. Finally, in a pooling equilibrium, holders of good securities find it optimal to undersell their securities and pay the liquidity premium in order to fund their (more profitable) investment opportunities. Since all agents are risk neutral furthermore, the (liquidity) premium is due *only* to asymmetric information and not to risk.

3.3.3 Equilibrium Securities at $t = 1$: a formal analysis

Equilibrium and Off-Equilibrium Pricing The first step necessary to characterize the equilibrium of our economy consists in the definition of pricing. Within the proposed competitive analysis in the presence of private information, this raises some important issues. By the FOCs with respect to $\theta^h_{1,1-\pi_1}(j)$ and $\left(\theta^h_{1,1-\pi_1}(j)\right)^R$ in the appendix - the prices of newly issued securities and of securities traded in the resale market respectively are:

$$\begin{aligned} q_1(j) &= E_i\left[E_s\left[D^i_j(s)\right]\middle|\,\varphi_1(j,\underline{i}),\varphi_1(j,\bar{i})\right] \\ q^R_1(j) &= E^R_i\left[E_s\left[D^i_j(s)\right]\middle|\,\gamma_{\underline{i}j}\varphi_0(j,\underline{i}),\gamma_{\bar{i}j}\varphi_0(j,\bar{i})\right] \end{aligned} \tag{7}$$

Since buyers can *only* observe the amount of collateral attached to the security they are purchasing, the price is indexed by j only. The equations above formalize the assumption of rational expectations: the equilibrium price must be equal to the expected delivery (quality) of the *average* security[25] traded at collateral level j. The average security in turn depends on the relative shares of good investment across technologies.

In order to choose the optimal security, entrepreneurs must be able to compare the price for the possible securities they may issue. Although establishing prices for equilibrium securities is conceptually simple, rational expectations provide no guidance in the determination of prices for non traded securities. This may lead to a paradox: every agent may expect the price of all off-equilibrium securities to be zero, simply because no one is trading them. In order to get around this problem, I refer to the recent work by Dubey and Geanakoplos (2003) in the context of the insurance model. They address this feature by imposing a tremble "on the market": introducing an external agent forced to trade securities that would otherwise be absent, they are able to precisely define off equilibrium prices.

[25] It could not be different since, if $q_1(j) > E_i\left[E_s\left[D^i_j(s)\right]\middle|\,I^i_t\right]$, no one would buy security j and, if $q_1(j) < E_i\left[E_s\left[D^i_j(s)\right]\middle|\,I^i_t\right]$, everyone would buy security j and an excess demand would appear.

Here I adapt Dubey and Geanakoplos (2003)'s reasoning in a closely related argument. I establish an *external agent* of measure $\varepsilon = \{\varepsilon_j\}_{j\epsilon\widehat{J}}$ to issue every off-equilibrium security as if he were a good quality entrepreneur. The price of off-equilibrium security j' would be equal to $E_s\left[D_{j'}^{\bar{i}}\right]$, if only the external agent were to issue it. Therefore the external agent pins down prices different from zero in all markets. I denote the economy where the external agent has been introduced as an "ε-economy".

In order to determine the equilibrium of the ε-economy, one has to check whether entrepreneurs find it profitable to issue the same original security or to deviate to another. In practice we are asking each agent whether an entrepreneur would "change his mind", once the external agent is introduced. Taking into account the optimizing behavior of all agents, one can, by rational expectations, compute the security prices for the given ε-economy - i.e. the equilibrium prices of the economy where the external agent is forced to issue some specific securities and everyone else optimizes. In the same spirit of (7), we have:

$$q_1(j) = \frac{\eta(\underline{i})\varphi_1\left(j,\underline{i}\right)^{\varepsilon_j} E_s\left[D_j^{\underline{i}}(s)\right] + \left(\eta(\bar{i})\varphi_1\left(j,\bar{i}\right)^{\varepsilon_j} + \varepsilon_j\right) E_s\left[D_j^{\bar{i}}(s)\right]}{\eta(\underline{i})\varphi_1\left(j,\underline{i}\right)^{\varepsilon_j} + \left(\eta(\bar{i})\varphi_1\left(j,\bar{i}\right)^{\varepsilon_j} + \varepsilon_j\right)}$$

Thus, rearranging section 2.1, the equilibrium of the ε-economy is defined:

Definition 1 *An equilibrium of the ε-economy is defined by consumption allocations* $c_\varepsilon^h = [c_{0\varepsilon}^h,\ c_{1\varepsilon}^h,\ c_{2\varepsilon}^h = \left\{c_{2\varepsilon}^h(G), c_{2\varepsilon}^h(B)\right\}]$, *asset holdings* $a_\varepsilon^h = [\theta_{0\varepsilon}^h(j,i),\ \varphi_{0\varepsilon}^h(j,i),\ \theta_{1\varepsilon}^h(j),\ \varphi_{1\varepsilon}^h(j,i),\ \left(\theta_{1\varepsilon}^h(j)\right)^R]$, $\forall h$, *and asset prices* $q_\varepsilon = (q_{0\varepsilon},\ q_{1\varepsilon},\ q_{1\varepsilon}^R) \in \Re^{N\cdot 10^{100}}$ *satisfying:*

1. ***Individual Optimum***

$$\left(a_\varepsilon^h, c_\varepsilon^h\right) \in \arg\max\left\{\begin{array}{c} V^h\left[c_\varepsilon^h\right] \\ s.t.\ \left[B^h\left(q_\varepsilon\right)\right] \end{array}\right\} \text{ at given } q_\varepsilon, \forall h$$

2. ***Market Clearing Conditions***

$$\int_0^1 \left[\theta_0^h(j,i) - \varphi_0^h(j,i) - \varepsilon_j\right] dh = 0\ \forall j, i\ \ at\ t = 0 \qquad (t = 0 \text{ securities})$$

$$\int_0^1 \left[\theta_1^h(j) - \varphi_1^h(j,i) - \varepsilon_j\right] dh = 0\ \forall j\ \ at\ t = 1 \qquad (t = 1 \text{ securities})$$

$$\int_0^1 \left[\gamma_{ji}^h \theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R\right] dh = 0\ \forall j\ \ at\ t = 1 \qquad (\text{Resale Market})$$

$$\begin{array}{c} \int_0^1 \left[c_t^h + I_t^h - w_t^h\right] dh = 0,\ \ t = 0, 1 \\ \int_0^1 \left[c_2^h(s) - g_i\left(I_0^h; s\right) - g_i\left(I_1^h; s\right)\right] dh = 0,\ \ t = 2,\ \ s \in \{G, B\} \end{array} \qquad (\text{Goods Market})$$

Finally, one takes the measure of external agent to zero, i.e. $\varepsilon(n) \to 0$ for $n \longrightarrow +\infty$. If, as the external agent gets smaller and smaller, more and more entrepreneurs leave off-equilibrium securities and return back to issue the original security/ies, we say that the original security/ies is/are an equilibrium surviving the “tremble”. In principle, the larger is the set of expected deliveries of external agent for which an equilibrium survives, the more robust we will say it is. For the sake of simplicity, we have explicitly considered only the case most likely to break any equilibrium, where the external agent behaves as a good quality entrepreneur. If an equilibrium survives this "optimistic" tremble, it survives any other tremble where the external agent behaves as an entrepreneur of lower quality. Thus we will say that an equilibrium is robust if it survives the tremble in which the external agent behaves as a good quality entrepreneur. Formally:

Definition 2 *An equilibrium is robust if it is the limit, for $\varepsilon_j(n) \to 0$ when $n \longrightarrow +\infty$, of a sequence of ε-economies in which the external agent issues $\varepsilon_j(n)$ shares paying $E_s\left[D_j^{\bar{i}}(s)\right]$ in each off-equilibrium security.*

Reinterpreting the notion of *pool* in Dubey and Geanakoplos (2003) as our notion of *security*, we can appeal to their Theorem 1 and be ensured that a robust equilibrium always exists. In the following section I turn to the conditions under which *pooling* equilibria are robust.

Liquidity Premium and Optimal Security In the discussion I will keep on referring to the FOCs of the individual problem and refer to the appendix for details. In an equilibrium where entrepreneurs endowed with different technologies issue different securities j^i, i.e. in a separating equilibrium, the ratio $R_1^i(j^i) = 1, \forall i$. Instead, if, in equilibrium, all entrepreneurs issue the same security j^{pool}, i.e. if the equilibrium is pooling, it must be that:

$$R_1^{\bar{i}}(j^{pool}) > 1 > R_1^{\underline{i}}(j^{pool})$$

Since our focus is economy wide liquidity, we will use the following measure of liquidity at each time t:

Definition 3 *The liquidity premium is defined as:*

$$LP_t = R_t^{\bar{i}}(j^{\bar{i}}) - R_t^{\underline{i}}(j^{\underline{i}})$$

i.e. as the difference between the rates of return of the security issued by entrepreneurs with better technology, $j^{\bar{i}}$ and of the security issued by entrepreneurs with worse technology, $j^{\underline{i}}$.

Therefore, in a pooling equilibrium $LP_t > 0$ and the economy is said *imperfectly liquid* (or *illiquid*), while in a separating equilibrium $LP_t = 0$ and the economy is in fact *liquid*.

Because of rational expectations, (7) ensures that security prices reflect the average quality. The better a technology is the higher (weakly) is the interest rate it pays. Therefore, in equilibrium, marginal productivity is smaller and investment higher, the worse is the technology. This is the intuition behind the instance of adverse selection that will appear here. Equation (5) states the reference criterion by which the entrepreneur chooses the optimal security to issue. Since this is also relevant in the case of asymmetric information, I repeat it here:

$$E_s\left[g_i'\left(I_1^h;s\right)\right]+\beta_1^i=\frac{E_s\left[D_j^i(s)\right]}{q_1(j)}+\beta_1^i\frac{j}{q_1(j)}$$

where β_1^i is the multiplier on the collateral constraint CC(1). The reader is referred to (5) for the relevant interpretation. Here it suffices to note that (5) defines equality between marginal benefit and marginal cost.

In order for our problem to be interesting, the collateral must be binding. Let us notice that:

Remark 1 *If the function $g(.)$ is sufficiently unproductive (or endowment w is sufficiently large), entrepreneurs do not use all the capital they build as collateral. Given this technology, the collateral constraint is slack, i.e. $\beta_1^i=0$, $\forall i$, and the equilibrium security is $j=\overline{j}$. In this case the optimal level of investment $\left(I_t^i\right)^*$ is small enough and the equilibrium security entails no default: the informational asymmetry is immaterial and **no** adverse selection takes place.*

Therefore we say that liquidity problems are a feature of sufficiently productive economies. The interesting phenomena arise for the class of economies in which, because of asymmetric information, the collateralization constraint is binding in equilibrium for at least one type of technology:

$$\sum_{\varphi_t^i(j)}\varphi_t^i(j)j=I_t^h\Leftrightarrow\beta_1^i>0 \tag{8}$$

Therefore, since the existence of a pooling equilibrium is the foundation of our analysis of liquidity, it is necessary to characterize its sufficient conditions. As stated above, pooling equilibria in the security space $\widehat{J}$ are defined by the fact that a single security j^{pool} is optimal for any entrepreneur, i.e. $\forall i$. The entrepreneurs with the worst technology are better off in the pooling equilibrium since they pay a low interest rate and issue securities with low collateral. The entrepreneurs with the best technology face instead a trade-off: on one hand, they pay a higher interest rate than they would in a separating equilibrium; on the other, they issue securities carrying less collateral than the ones in a separating equilibrium. Therefore, the only agents with an incentive to abandon the pooling equilibrium are the good quality entrepreneurs wishing to differentiate themselves. They can do so increasing the level of collateral of the security they are issuing. But this is costly since, increasing the collateral per unit, an entrepreneur decreases the number of securities he can issue by (8). Therefore differentiation comes at a cost: increasing the collateral level (weakly) increases $E_i\left[D_j^i(s)\right]$ but reduces the number of security units (and thus leverage) an entrepreneur may issue.

This trade-off is formally characterized rearranging (5):

$$E_s\left[g_i'\left(I_1^i\right);s\right]q_1(j)=E_s\left[D_j^i(s)\right]+\beta_1^i\left(j-q_1(j)\right) \tag{9}$$

where $(j - q_1(j)) > 0$ under (A.1)[26]. Thus security j is issued by entrepreneur i, i.e. $\varphi_1(j,i) > 0$, if it minimizes:

$$R_1^i(j) + \beta_1^i \left(\frac{j - q_1(j)}{q_1(j)} \right)$$

In the appendix I prove that the minimization of the equation above leads to the Theorem below about the existence of robust pooling equilibria. Since in the discussion that follows I maintain (A.1), technology L (low variance) is ranked superior to technology H (high variance)[27].

Theorem 1 (Unique Pooling Equilibrium) *Under the maintained setup and assumptions (A.1), (A.2) and (A.3), economies characterized by a sufficiently large share, $\eta(L)$, of better technology entrepreneurs, i.e. $1 \geq \eta(L) \geq (\eta(L))^*$, display a unique robust pooling equilibrium $\underline{j} = j^{pool}$ in the security space. These economies are illiquid since they display a positive liquidity premium at $t = 1$:*

$$LP_1 > 0$$

Proof. Appendix. ■

Theorem 2 has the following corollary:

Corollary 1 *In the pooling equilibrium $j^{pool} = \underline{j}$ each technology holder i issues security $\underline{j}$ only.*

Theorem 2 has a somewhat complicated statement but a simple interpretation. It states that a pooling equilibrium in which only the security with the lowest collateral in $\widehat{J}$, $\underline{j}$, is issued by every entrepreneur emerges if the population of entrepreneurs is characterized by a sufficiently high share of good quality technologies. The intuition is simple: good quality entrepreneurs choose the pooling security weighing two factors. On one hand, they bear the difference between the market price and the actual value of the security they are selling - the "price effect"; on the other hand, they benefit from the additional leverage allowed by the fact that the pooling security requires less collateral, per unit, than any alternative security they may decide to issue- the "quantity effect" or "leverage effect". The theorem above is only saying that the cost of “sticking” to the pooling equilibrium security for good quality entrepreneurs increases with the share of bad quality entrepreneurs in the population, while the cost of increasing the level of collateral *per* security stays the same. It is therefore natural to argue that, as we increase the share of entrepreneurs endowed with the bad quality technology, the increase in the costly "price effect" may eventually offset the beneficial "quantity effect" - the additional leverage - that the pooling security $\underline{j}$ entails.

The condition on the relative share between good and bad quality entrepreneurs, though only sufficient, delivers a pooling equilibrium robust to perturbations in which any mix of good and bad entrepreneurs is forced to issue off equilibrium securities. This result, at

[26] See Figure 4.

[27] $\lambda_2 < \lambda_1$ by assumption.

odds with the implications of Dubey and Geanakoplos (2003)[28], is of interest in its own right. The intuition behind the robustness of pooling equilibria in our framework comes from a local failure of the "single crossing property" necessary to separation in signalling models. Here good quality entrepreneurs may attempt to distinguish themselves by issuing securities with higher collateral level. The problem is that, since their collateral is more valuable, their expected delivery increases relatively faster than that of bad quality securities as the collateral level increases. Therefore, the positive correlation between collateral value and technology's quality is ultimately responsible for the local failure of the "single crossing property". This failure is sufficient to restrain good entrepreneurs from abandoning the pooling equilibrium, even in the presence of "optimistic" beliefs about the average quality of the other securities, hence the robustness result.

The robustness of our pooling equilibrium has important macroeconomic implications. It shows that not only liquidity shortages may arise in the economy, but it states that imperfect liquidity, generated by asymmetric information, is robust to the different expectations that entrepreneurs may form about the prices of off-equilibrium securities. The only condition required for this result to hold, beside the fact that there are "enough" good quality entrepreneurs, is that better entrepreneurs employ more valuable collateral in guaranteeing their security.

Equilibria with Partial Resale: An Instructive Case It is instructive to study the equilibrium that arises in this economy when a pooling equilibrium emerges at $j = j_0$ in $t = 0$ security space and entrepreneurs resell only a part of the high quality securities, $\bar{i}$, they own. $g(.)$ and $\eta(\bar{i})$ can be chosen so that these two features are equilibrium outcome.

Theorem 2 above focuses on the conditions sufficient for $t = 1$ entrepreneurs to issue the same security j^{pool}. In general the level of investment I_1^i is equal to the sum of three distinct sources:

$$I_1^i = \underbrace{w}_{\text{Private Funds}} + \underbrace{q_1^R(j)\left[\sum_{i=\bar{i},\underline{i}}\left(\gamma^h_{j_0 i,\pi_1}\theta^h_{0,1-\pi_0}(j,i)\right)\right]}_{\text{Portfolio Resale}} + \underbrace{q_1(j)\varphi_1^i(j,i)}_{\text{New Securities}} \tag{10}$$

Since entrepreneurs borrow as much as they can since, we can exploit Theorem 2 and the Corollary 3 to find that the maximum level of investment with asymmetric information. By the budget constraint of entrepreneur i at $t = 1$:[29]

$$I_1^i = \left\{ w + q_1^R(j)\left[\sum_{i=\bar{i},\underline{i}}\left(\gamma^h_{j_0,i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right]\right\}\left(\frac{j}{j-q_1(j)}\right) \tag{11}$$

The collateral constraint, CC(1), still binds (i.e. $\varphi_1^h(j,i) = \frac{I_1^i}{j}$). (11) says that the optimal

[28] Dubey and Geanakoplos (2003) argue that, organizing security trade through "pools", separating equilibria always exist and are the only one robust to "optimistic" off-equilibrium expectations.

[29] Subsection 1 of the appendix.

level of investment is an increasing function of endowment and resources raised in the resale market multiplied by the maximum leverage factor $\left(\frac{j}{j-q_1(j)}\right)$.

So far I have taken as given the funds raised in the resale market, $q_1^R(j)\left[\sum_{i=\bar{i},\underline{i}}\left(\gamma^h_{j_0,i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right]$, but I have not derived how entrepreneurs choose which shares, $\gamma^h_{j_0,i,\pi_1}$, they are reselling. I turn to this now. Observing:

Proposition 2 *Suppose that at $t=0$ the equilibrium in the security market is pooling at collateral level $j=j_0$, asymmetric information implies:*

$$E_s\left[D^{\bar{i}}_{j_0}(s)\right] > q_1^R(j_0) = E_i^R\left[E_s\left[D^i_{j_0}(s)\right]\middle| I_0^i\right] \geq E_s\left[D^{\underline{i}}_{j_0}(s)\right]$$

where security $\bar{i}$ has better quality than $\underline{i}$.

The economic interpretation is that the market price in the resale market is (weakly) above the expected delivery of the worse security. Thus every entrepreneurs completely resells the worse security $\underline{i}$, i.e. $\gamma^h_{j_0,\underline{i},\pi_1}=1$. Proposition 4 above implies that consumers never resell good quality securities purchased at $t=0$. Otherwise they would incur a capital loss. Different is the case for $t=1$ entrepreneurs. By (11), investment is different from the symmetric information constrained pareto efficient level if $I_1^{\bar{i}}<\left(I_t^{\bar{i}}\right)^*$. The first reason for the difference was outlined above and is due to asymmetric information in the market for newly issued securities: the market price is below (above) the expected delivery of $\bar{i}$ ($\underline{i}$). The second reason is asymmetric information in the resale market: investors resell only part of their good securities since they pay a liquidity cost whenever they do so.

Under the technological assumption of this case, entrepreneurs are borrowing constrained at the point where the equilibrium level of investment $I_1^{\bar{i}}$ satisfies[30]:

$$E_s\left[g'_{\bar{i}}\left(I_1^{\bar{i}};s\right)\right]+\beta_1^{\bar{i}}=\left[R_1^{\bar{i}}(j_0)\right]^R>1$$

In this case, investors resell high quality security up to the point where the equation is satisfied, in other words:

$$\gamma^h_{j_0\bar{i},\pi_1}<1 \tag{12}$$

Thus, by rearranging the FOCs with respect to the number of security j issued by an entrepreneur with technology i, $\varphi_1^h(j,i)$, his investment level, I_1^i, and the share of good securities he resold, $\gamma^h_{j_0\bar{i},\pi_1}$, we can write the following equation:

$$\begin{aligned}\left[R_1^{\bar{i}}(j_0)\right]^R-R_1^i(j)=\\ =\frac{j}{q_1(j)}\left[\left[R_1^{\bar{i}}(j_0)\right]^R-E_s\left[g'_i\left(I_1^i;s\right)\right]\right]\end{aligned} \tag{13}$$

[30] This condition is determined rearranging the FOCs with respect to I_1^i and γ_{ij} bearing in mind that entrepreneurs with the best technology do not resell the entirity of high quality secuirities they own in this equilibrium.

The interpretation of this condition is simple. First, observe that $R_1^i(j)$ is the rate of return or, better, the rate of repayment of security j. It is the ratio between what an agent is paying in the future, the expected delivery, and what he gets today through the market price. Given optimal securities, (13) says that the difference in the cost of raising money through the resale market and the new issuance market, $\left[\left[R_1^{\bar{i}}(j_0)\right]^R - R_1^i(j)\right]$, must be equal to the difference between the cost of reselling and marginal productivity, $\left[\left[R_1^{\bar{i}}(j_0)\right]^R - E_s\left[g_i'\left(I_1^i; s\right)\right]\right]$, corrected by the cost of collateralization specific to security j, $\left[\frac{j}{q_1(j)}\right]$.

Notice that the investment level is different from the symmetric information benchmark because of (12). The resale market is only a residual way to finance investment, since it is used below its full information potential. Therefore, when the equilibrium in the security space at $t = 0$ is pooling we have:

Proposition 3 $\left[\left[R_1^{\bar{i}}(j_0)\right]^R - R_1^{\bar{i}}(j)\right] > 0$, *i.e. the resale market has a higher degree of adverse selection than the market for newly issued securities. Therefore entrepreneurs with better technologies,* $\bar{i}$, *will use the resale market as a source of funding only once the issuance of new securities is limited by the collateral available.*

Proof. By rearranging the FOCs with respect to $\varphi_1^h(j,i)$ and $\gamma^h_{j,\bar{i},\pi_1}$:

$$\left[\left[R_1^{\bar{i}}(j_0)\right]^R - R_1^{\bar{i}}(j)\right] =$$

$$= \left[\frac{E_s\left[D_{j_0}^{\bar{i}}(s)\right]}{q_1^R(j_0)} - \frac{E_s\left[D_j^{\bar{i}}(s)\right]}{q_1(j)}\right] = \beta_1^i \frac{j}{q_1(j)} > 0$$

■

The remark above allows us to define:

Definition 4 *The "resale" premium of security j_0 issued at $t = 0$ is* $\left[\left[R_1^{\bar{i}}(j_0)\right]^R - R_1^i(j)\right] > 0$.

The definition above simply points out that the early liquidation of a security comes at a cost that exceeds the one associated with newly issued securities. This is equivalent to say that the average quality in the resale market is lower. This result reflects optimal decision making: entrepreneurs accept to pay higher liquidity premia in the resale market because reselling securities not only provides extra funds for investment but, relaxing the collateralization constraint, allows the issuance of even more new securities. The reader is reminded that the resale market trades securities that had been issued at $t = 0$ but deliver at $t = 2$.

Equilibrium Investment at $t = 1$: Adverse Selection Keep in mind that, in a pooling equilibrium, we have:

$$E_s\left[D_j^{\bar{i}}(s)\right] > q_1(j) = E_i\left[E_s\left[D_j^i(s)\right] \middle| I_1^i\right] > E_s\left[D_j^{\underline{i}}(s)\right]$$

Therefore the rates of return for good and bad quality entrepreneurs/securities are in the following relation:

$$R_1^{\bar{i}}(j) > 1 > R_1^{\underline{i}}(j); \forall j < \bar{j}$$

In a separating equilibrium , we have already observed that $R_1^{\bar{i}}(j^{\bar{i}}) = R_1^{\underline{i}}(j^{\underline{i}})$, although high quality entrepreneurs issue securities with more collateral and are subject to a negative *leverage effect*. These observations imply that better technologies entrepreneurs always borrow under worse terms with asymmetric information.

The difference between the terms of borrowing for high and low quality entrepreneurs, absent in the symmetric information benchmark, is responsible for the instance of the adverse selection in investment we have here[31].

3.4 Market Equilibrium at $t = 0$

This section studies $t = 0$ equilibrium of this economy and so concludes the discussion of the baseline economy. $t = 0$ consumers lend to entrepreneurs and consumes the rest of their endowment w. The fraction of entrepreneurs in the total population at $t = 0$ is π_0, which coincides with the ex ante probability of becoming an entrepreneur at $t = 0$. Access to the credit market is still beneficial to entrepreneurs because of (4).

The discussion in section 3.1 assumed that different entrepreneurs chose the same security at $t = 0$. This implied that the securities traded in the resale market are illiquid. It is now necessary to prove that this assumption is indeed an equilibrium result. In order to do so I will briefly recast the reasoning presented above in the context of $t = 0$ security choice. I will then appeal to Theorem 2 and say that, under analogous conditions, we have a $t = 0$ robust pooling equilibrium in the security space.

Equilibrium Asset Pricing Each financial contract is priced at $t = 0$ according to the expected consumption it delivers and the liquidity premium it implies. Equilibrium pricing of a $t = 0$ security can be determined by the FOC with respect to $\theta^h_{0,1-\pi_0}(j, i)$ in the individual problem. Assuming that entrepreneurs are a sufficiently small portion of the population, i.e. (π_0) small, lenders have positive consumption at $t = 0$ and $t = 2$, and so:

$$q_0(j, i) = \sum_{k=\pi_1, 1-\pi_1} (k) \left(\mu_1^h(k)\gamma_{ji}^h(k)q_1^R(j) + (1 - \gamma_{ji}^h(k))E_s\left[D_j^i(s)\right]\right) \tag{14}$$

where $\mu_1^h(k)$ is the multiplier of the budget constraint at $t = 1$ in contingency k. Each individual faces two contingencies at $t = 1$: $k = \pi_1$ when the agent faces a project and

[31] The reader should observe that adverse selection in investment is not an exclusive feature of pooling equilibria. In fact, even in the presence of a separating equilibrium, entrepreneurs with high quality technology would borrow under worse terms by issuing securities with strictly more collateral.

$k = 1 - \pi_1$ when he does not. Notice that $\mu_1(1 - \pi_1) = 1$ and $\mu_1(\pi_1) > 1$. This is intuitive: the multiplier $\mu_1(k)$ measures the advantage of relaxing the budget constraint in contingency k at $t = 1$. Thus, if an individual has an investment opportunity, i.e. $k = \pi_1$, then the advantage of increasing his endowment is strictly bigger than marginal utility ($= 1$); if agent h does not face an investment opportunity, i.e. $k = 1 - \pi_1$, then the benefit of relaxing the constraint is just his marginal utility.

In this framework, securities issued at $t = 0$ not only entitle the owner to $t = 2$ payment but provide him with a "precautionary saving" instrument for transferring wealth to $t = 1$, the time when he could be borrowing constrained because of an investment opportunity. Therefore $t = 0$ issuers of securities benefit from the *twofold* service of their securities because prices are raised *above* expected delivery.

As was the case for $t = 1$, different liquidity levels correspond to pooling and separating equilibria in the security space at $t = 0$. When the equilibrium is separating, then high quality securities are characterized by higher collateral level and are perfectly liquid in the resale market (i.e. they are traded at $t = 1$ for a price equal to their expected delivery). Thus, in a separating equilibrium high quality and low quality securities both have the same rate of return and the economy is perfectly liquid. By (14) the price for security (j, i) issued at $t = 0$, reflecting this liquidity, is strictly above its expected delivery:

$$q_0^{sep}(j, i) > E_s\left[D_j^i(s)\right]$$

When the equilibrium is pooling, I have already pointed out that every entrepreneur sells in $t = 1$ resale market all the securities $\underline{i}$, the ones backed by low value collateral. This is natural since low quality securities are traded at a price weakly larger than their expected delivery when $t = 1$ entrepreneurs are borrowing constrained. Observe that low quality securities are the cheapest way to finance private investment at $t = 1$, weakly better than private endowment.

In the case of securities $\bar{i}$, i.e. high quality ones, the matter is more subtle. If, because of their endowment and technology's productivity, entrepreneurs sell part of securities $\bar{i}$ even in the case of a pooling equilibria (condition (12) above), then the price in the resale market $q_1^R(j)$ rises strictly above $E_s\left[D_j^{\underline{i}}(s)\right]$. This increase has two effects: (1) securities $\bar{i}$ become more liquid since the resale price gets closer to the expected delivery but (2) all security $\underline{i}$ are sold to realize a capital gain. But when all securities $\bar{i}$ are resold, the resale market is "flooded" by low quality securities and a Lemons market arises. In a pooling equilibrium at $t = 0$, the equilibrium prices can be shown to be[32]:

$$q_0^{pool}(j, i) = \begin{cases} q_0(j, \bar{i}) = E_s\left[D_j^{\bar{i}}(s)\right] & \text{if } j < \bar{j} \\ q_0(j, \underline{i}) = q_1^R(j) > E_s\left[D_j^{\underline{i}}(s)\right] & \\ q_0(\bar{j}, \underline{i}) = q_0(\bar{j}, \bar{i}) > 1 & \text{if } j = \bar{j} \end{cases} \tag{15}$$

There is one main comment about (15): the asymmetric information that buyers face in case of early liquidation of their securities (at $t = 1$) benefits entrepreneurs with the worse

[32] Rearranging FOCs with respect to $\theta_0^h(j, i)$, γ_{ji}^h in the Appendix.

technology $\underline{i}$ and imposes a cost on entrepreneurs with the better technology $\bar{i}$. Since this cost is present no matter what kind of equilibrium prevails at $t = 0$, adverse selection appears in $t = 0$ investment. The reader should notice that, although the price of securities $\bar{i}$ is equal to the expected delivery, a cost is borne by better technology owners. In fact issuers of securities $\bar{i}$ do not benefit from the fact that they provide an instrument for "precautionary saving" to potential entrepreneurs. Because of the asymmetric information in the resale market, only issuers of securities $\underline{i}$ gather this benefit being paid a market price $q_0(j, i)$ for their securities which exceeds their delivery $E_s\left[D_j^i(s)\right]$.

I conclude the discussion of this subsection observing that a pooling equilibrium for $t = 0$ securities emerges under similar conditions about the average entrepreneurial quality of Theorem 2. That argument could simply be recast here and I avoid rephrasing that same reasoning in the present context. Given the assumption about sufficiently high average quality in the entrepreneurial population, pooling equilibria are robust. Nonetheless notice that supporting a pooling equilibrium in the security space at $t = 0$ can be done for a larger parameter space than at $t = 1$. In fact, looking at the problem for entrepreneurs with better technology $\bar{i}$, one can observe that the cost of providing collateral - here the cost of differentiation - is the same as before but the benefit - the fall in the rate of repayment $R_0^i(j)$- is smaller since prices already equalize expected delivery for high quality entrepreneurs, by (15).

Notice finally, that adverse selection in $t = 0$ investment is still present here by the same reasoning of $t = 1$.

This concludes the discussion of the baseline economy. I have studied the conditions under which pooling equilibria exist. These equilibria are the foundation for the analysis of liquidity I propose. I have shown that liquidity is an important factor in the allocation of credit across different technologies and, in particular, even though illiquidity may be an equilibrium phenomenon it nevertheless decreases the efficiency of private investment adversely selecting technologies. I will now develop this analysis extending it to take into account the role of financial markets and government debt. But before doing so I will apply the proposed framework to the relation between the equilibrium level of liquidity and the business cycle.

4 Business Cycle and Liquidity

There is a developing line of research in the macroeconomic literature aimed at explaining the relation between the business cycle and liquidity of the economy[33]. The empirical literature seems to point out that market liquidity tends to positively covary with the state of the economy. In an important theoretical contribution in this line of research, Eisfeldt (2004) takes the same conceptual perspective we adopted here regarding the interpretation of liquidity. A summary of her results is as follows: as the productivity of technology changes along the business cycle, entrepreneurs have the incentive to raise more funds when productivity is higher. The only way open to entrepreneurs to finance their investment is

[33] Eisfeldt (2004), Rampini (2003) and Rampini and Eisfeldt (2003).

selling claims over the future production of projects previously *initiated.* Since the incentive to invest is stronger for entrepreneurs endowed with better technologies, when the economy faces a high productivity shock, they tend to sell more claims than the owner of worse technologies. Therefore the quality of the average traded security increases, the market price increases and liquidity turns higher when the economy faces a higher productivity shock. This is consistent with previous empirical work showing the positive correlation between the liquidity of an economy and its performance.

This explanation, however, relies on a crucial restriction in the security market, a restriction that is not present in the model considered here: entrepreneurs can only issue equity-like claims over future production. They cannot issue any other kind of security. This restriction is central to the argument and may, in principles, undermine the explanation of procyclical liquidity proposed in Eisfeldt (2004). In fact, if different entrepreneurs were to issue different kinds of securities, these would always be liquid and there would be no change in equilibrium liquidity as a consequence of productivity shocks. In reality, firms have different instruments resulting in a variety of securities they may issue. For instance, securities may differ by the likelihood of default and its extent, thereby enlarging the security space even further. This is precisely the standpoint we adopted in this work. I am going to show how we can apply it to the study of the relation between liquidity and the business cycle.

At least since Kydland and Prescott (1984), strong persistence in the growth rate of aggregate output is recognized as a characteristic feature of the US economy. I translate this simple fact by adjusting the probability attached to the G(ood) contingency, when technologies deliver high production. In the earlier discussion I assumed (A.1.) ($p < 1/2$), i.e. technology L was ranked superior to technology H. I now change this assumption and consider the alternative case $p > 1/2$. I define this case as an *expanding economy* since, with $p > 1/2$, the "high production" contingency becomes relatively more likely. Notice that this parameter change ranks technology H superior to L in terms of expected production. This assumption changes the behavior of expected delivery as a function of the collateral level. Putting the two expected delivery mappings in the same graph under the assumption $p > 1/2$ we have:

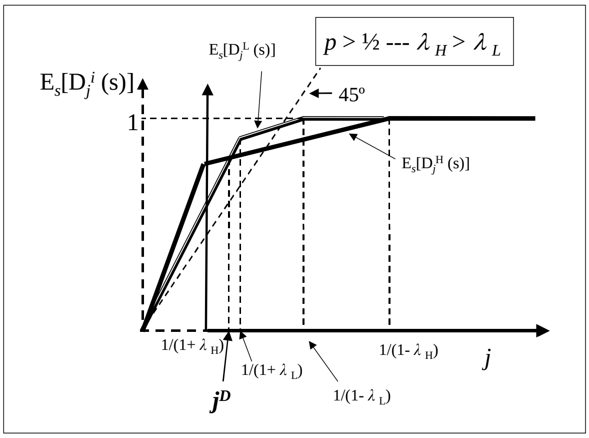

Figure 5

If expected deliveries are as in the relation described by the graph above, it is immediate to observe that there is a security, label it $j_D << \overline{j} = \frac{1}{1-\lambda_H}$, at which both expected deliveries coincide. In fact, security j^D entails no asymmetric information. This is an interesting feature of the model because j^D is perfectly liquid since the expected delivery is unambiguous, irrespective of the technology owned by the issuer. Therefore the price is equal to the expected delivery. It is therefore immediate to state the following proposition, given without proof:

Proposition 4 (Boom and Separating Equilibrium) *If the economy is "booming", i.e. $p > 1/2$, then it is "liquid", in the sense that the equilibrium in the security market is separating and $LP_t = 0$. Entrepreneurs with technology H issue security j^D while entrepreneurs with technology L issue security $\underline{j}$.*

Therefore, in the context of this paper, the increase in liquidity during a booming economy is due to the change in the equilibrium security more than to the increase in the average quality of the traded ones as in Eisfeldt (2004). This hypothesis seems also appealing for its testability.

By similar reasoning, I reinterpret a stagnating economy as the one in which our original assumption holds, i.e., $p < 1/2$[34]. In this case, the expected deliveries of the two technologies can be graphed as in Figure 4 above, where the analysis discussed throughout Section 3 holds. I already pointed out that, in this case, the collateralization cost borne by the better technology to avoid subsidizing worse securities is higher. Therefore, respecting the conditions of Theorem 2, entrepreneurs decide to issue securities whose liquidity is less than perfect (i.e. the equilibrium is pooling).

Therefore in order to replicate the procyclical behavior of liquidity - when securities are endogenous - that the empirical literature highlights , it suffices to observe that there is a (possibly large) class of economies for which the equilibrium in the security space delivers liquid securities, i.e. a separating equilibrium, during booms and the illiquid $\underline{j}$ security during recessions. This explanation does not rely upon a restriction in the security space that the recent literature has adopted, but is nonetheless consistent with the available evidence of procyclical liquidity. The basic mechanism here consists in the fact that firms adapt the choice of the security they issue to the state of the economy. This choice of entrepreneurs - I argue - is ultimately responsible for the level of liquidity of the economy.

5 Government Debt as Credit Provider

The work of Woodford (1990) and Holmstrom and Tirole (1998) recognizes the beneficial role of government debt in economies with imperfect liquidity. The importance of government bonds reached special attention in the recent policy debates when the US public debt seemed headed toward full repayment[35]. This attention resulted from the underlying conviction that government debt provides financial markets with some important services, often identified

[34] Figure 4 as reference.

[35] Reinhart and Sack (2000).

under the term liquidity. Here I propose an interpretation of this service that is richer than what is already advocated in the literature in at least one respect: the introduction of government debt may not only improve welfare but it also affects the securities issued at the equilibrium[36].

An economy such as the one we described faces two main problems: the first one is the cost of using the available collateral to guarantee repayments; the second one is the difficulty in stimulating saving for investment opportunities that may occur in the future. When economic agents face asymmetric information in their securities - as in section 3 - private investment is bounded away from the optimal allocation. Given these informational frictions, government bonds are ideal candidates to transfer wealth *beyond* what is provided by private financial contracts.

The beneficial role of government debt is therefore rooted in the observation that there is limited asymmetric information (one is tempted to say none) about it. This makes government bonds highly liquid casting them as an additional instrument to transfer wealth from the "unconstrained" present to the "constrained" future. Since future entrepreneurs (i.e. present consumers/lenders) will be borrowing constrained, they find it convenient to purchase government bonds as an (informationally efficient) store of value. Government bonds are not needed for consumption smoothing but only because they provide an extra store of wealth: they imply a substitution between current consumption and future investment, a substitution that current consumers (i.e. future entrepreneurs) are willing to accept. We will show that the stock of government debt has two effects on the entrepreneurs choice: firstly, it provides additional sources of funding, qualitatively comparable to private endowment, thus allowing the issuance of more securities (*leverage effect)*; second, it builds an incentive for entrepreneurs with the better technology $\bar{i}$ to abandon the pooling equilibrium issuing more liquid securities (*liquidity effect).*

The intuition for the liquidity effect of government debt relates to the fact that entrepreneurs with better quality technology face a binding collateral constraint. Remark 1 observed that when the collateralization constraint is not binding, there is no reason why good quality entrepreneurs would stick to the pooling. If government debt is enough to release the collateral constraint of good quality entrepreneurs, then the pooling equilibrium breaks down and liquid separating equilibria emerge. Therefore, in the presence of asymmetric information, government debt has two effects on the investment decisions of firms, the second of which is a novel contribution of this work.

6 Government Debt: Formal Analysis

It was argued earlier that it is in the interest of $t = 1$ entrepreneurs (i.e. $t = 0$ lenders) to move their wealth from the unconstrained $t = 0$ to the constrained $t = 1$. I will now formally illustrate how government debt (GB) provides an additional instrument to *crowd in* private investment. The existence of asymmetric information is the reason why the arguments here are at odds with Diamond (1965).

[36] The reader may recognize the similarity between my argument and Cass and Yaari (1966).

Government borrows issuing bonds with the following two crucial characteristics: (1) there is no associated collateral, (2) there is no informational asymmetry about GB (for simplicity but without loss of generality). Thus the government issues D_0 bonds at $t = 0$ and repays them at $t = 2$ raising lump sum taxes (T). Each unit of GB pays 1 unit of the consumption good in both contingencies at $t = 2$. At $t = 1$, the government does not issue new bonds and agents can exchange the GB they hold. I denote by D_1^h the units of GB purchased by an agent at $t = 1$. D_t^h and T^h have to be viewed as, respectively, agent h's GB and tax payment. D_0 is the aggregate GB issued. Observe that GB is purchased only by non entrepreneurs at both $t = 0$ and $t = 1$. Finally, $t = 0$ purchase of GB reflects the desire of potential entrepreneurs to transfer wealth to $t = 1$, thus relaxing the collateral constraint they then face.

In order to formally illustrate the argument, I will refer to the basic model introduced in Section 3. All the previous assumptions are maintained - in particular (A.1), i.e. the probability of the G(ood) contingency is $p < 1/2$ - but for the individual budget constraint, $\left[B^h(q)\right]^D$, which now becomes:

$$\begin{gathered}
\left[B^h(q)\right]^D = \\
t = 0: \\
c_0^h + (+I_0^h) \leq w_0 - \textstyle\sum_i \sum_j q_0(j,i) \left(\theta_0^h(j,i) - \varphi_0^h(j,i)\right) - q_0^D D_0^h \\
t = 1: \\
c_1^h + (+I_1^h) \leq w_1 + q_1^D(\gamma_D^h D_0^h - D_1^h) + \\
+ \textstyle\sum_j \left\{ q_1^R(j) \left[\sum_i \left(\gamma_{ji}^h\right) \theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R \right] - q_1(j) \left(\theta_1^h(j) - \varphi_1^h(j,i)\right) \right\} \\
t = 2: \\
c_2^h(s) \leq \textstyle\sum_i \sum_j \left\{ D_j^i(s) \left[(1 - \gamma_{ji}^h) \theta_0^h(j,i) - \varphi_0^h(j,i) \right] \right\} + \left(1 - \gamma_D^h\right) D_0^h + D_1^h + \\
+ \textstyle\sum_j \left\{ D_j^{i*}(s) \theta_1^h(j) + \left[D_j^{i*}(s) \right]^R \left[\theta_1^h(j)\right]^R \right\} - \sum_j D_j^i(s) \varphi_1^h(j,i) - T^h + \left(+g_i\left(I_t^h; s\right)\right)
\end{gathered} \tag{16}$$

where γ_D^h represents the share of h holdings of GB that are resold at $t = 1$; it satisfies:

$$0 \leq \gamma_D^h \leq 1 \text{ iff } D_0^h > 0 \tag{17}$$

since individuals can not resell more GB than they have.

The equilibrium is defined as in Section 2.2 but for the additional equilibrium conditions in the GB market:

$$D_0 = \int_0^1 D_0^h dh, \text{ at } t = 0$$

$$\int_0^1 \left(\gamma_D^h D_0^h\right) dh = \int_0^1 D_1^h dh, \text{ at } t = 1$$

Moreover the government repays the bonds it issued, i.e. it holds a balanced budget at $t = 2$:

$$D_0 = \int_0^1 T^h dh \tag{18}$$

We then recall that (1) individual endowment w is large enough to guarantee consumers with positive consumption after the purchase of private securities; moreover we assume (2) whatever the government raises at $t = 0$ through the issuance of bonds is immediately discarded, i.e. public expenditure is completely wasteful. The first assumption is aimed at ruling out that the simple introduction of GB *per se* decreases private investment. The second assumption is made for simplicity: with asymmetric information equation (6) says that investment is constrained at a point where marginal productivity is greater than 1 for all good quality entrepreneurs. Therefore, increasing investment means increasing the expected value of aggregate endowment in the economy. But then, to the extent that the introduction of GB is able to increase the aggregate level of investment chosen by the private sector, it is always possible to design an optimal transfer scheme across individuals such that welfare is in fact improved. In order to simplify our argument we will abstract from such transfer scheme and we will talk about GB neutrality and non neutrality simply focusing on its effect on private investment. Nonetheless, the reader should keep in mind that increasing investment is, in this context, sufficient to increase welfare, once the correct transfer scheme is assumed.

The derivation of equilibrium prices for GB is simple[37]. By the provided informational assumption on GB, the $t = 1$ GB price is:

$$q_1^D = 1$$

while the $t = 0$ GB price is proven in the appendix to satisfy:

$$q_0^D > 1$$

since GB provides not only a payoff at $t = 2$ but a liquidity service at $t = 1$. This service is positively priced because the possession of GB at $t = 1$ allows entrepreneurs to relax the borrowing constraint. The price is thus strictly larger than 1, the expected delivery of GB.

First it is necessary to clarify that the beneficial role of GB is due to the presence of entrepreneurs private information. In order to clarify this fact, I prove that positive GB has no effect on the symmetric information benchmark economy at the beginning of section 3. Let $\left(c^h_{0,1-\pi_0}\right)_{NoGB}$ and $\left(c^h_{1,1-\pi_1}\right)_{NoGB}$ denote, respectively, the individual consumption of consumers (purchasers of securities) at $t = 0$ and $t = 1$ and recall that that they both are strictly positive by assumption on π_0 and π_1. Moreover we assume that the endowment, w, is such that the equilibrium level of investment, $\left(I_1^i\right)^*$, in the symmetric information economy without GB satisfies:

$$E_s\left[g_i'\left(\left(I_1^i\right)^*; s\right)\right] = 1;\ \forall i$$

So we can state:

Proposition 5 (Government Debt Neutrality) *If the issuance of GB satisfies:*

$$\frac{D_0}{1-\pi_0} < q_0^D \frac{D_0}{1-\pi_0} \leq \left(c^h_{0,1-\pi_0}\right)_{NoGB}$$
$$1 - \pi_0 = \textit{ measure of } t = 0 \textit{ consumers}$$

[37] As usual I refer the reader to the relevant part in the appendix for details.

$$q_1^D \frac{\left(\int_0^{\pi_1(1-\pi_0)} D_0^h dh\right)}{1-\pi_1(1-\pi_0)} = \frac{\left(\int_0^{\pi_1(1-\pi_0)} D_0^h dh\right)}{1-\pi_1(1-\pi_0)} \leq \left(c_{1,1-\pi_1}^h\right)_{NoGB}$$
$$\pi_1 (1-\pi_0) = \textit{ measure of } t=1 \textit{ entrepreneurs}$$
$$1-\pi_1 (1-\pi_0) = \textit{ measure of } t=1 \textit{ consumers}$$

then GB has no effect on private investment in the economy with symmetric information.

Proof. We observe: (1) at $t=0$ the effect of GB on investment is nil if this is smaller than private consumption, as we assume; (2) at $t=1$ $\left(I_1^i\right)^*$ is such that marginal productivity is equal to 1 without GB. Since the GB rate of repayment, delivery over market price, is equal to one, its addition does not increase the incentive to invest of entrepreneurs. ■

The intuition is simple: in the symmetric information economy GB does not improve the terms at which entrepreneurs may finance investment. Therefore they have no incentive to increase the level of investment beyond the level $\left(I_0^i\right)^*$ and $\left(I_1^i\right)^*$.

The case of the economy where entrepreneurs are privately informed about the technology and security they own is of interest here. In order to illustrate the beneficial role of GB for private investment under this scenario, it is convenient to start from the case in which the government issues a small amount of GB, i.e. $D_0 = \eta > 0$, with η small. The introduction of GB has two effects at $t = 1$[38]. First, it relaxes the collateralization constraint since it provides entrepreneurs with sources of funding qualitatively comparable to private endowment, thus decreasing the use of less convenient ways to fund investment like the resale market (*leverage effect)*; second, to the extent that the GB is able to make the collateralization constraint not binding, Remark 1 ensures that GB provide an incentive for entrepreneurs with the better technology $\bar{i}$ to abandon the pooling equilibrium and issue liquid securities. In the latter case the economy moves toward a liquid separating equilibrium and we define this effect of GB *liquidity effect.* In fact, adapting (10) to the account for (17), individual investment at $t=1$ becomes:

$$I_1^h = w + q_1^R(j) \sum_{i=\bar{i},\underline{i}} \gamma_{j_0,i,\pi_1}^h \theta_{0,1-\pi_0}^h(j_0,i) + q_1(j)\varphi_1^i(j,i) + \underbrace{D_0^h}_{\text{GB Resale}} \qquad \text{since } \gamma_D^h(\pi_1) = 1 \tag{19}$$

Before the introduction of GB, two scenarios were possible. In the first scenario, entrepreneurs resell only low quality securities in their portfolio, i.e. $\gamma_{j_0,\bar{i},\pi_1}^h = 0$, and thus adding GB D_0^h unambiguously increases the investment level I_1^i, given the issuance of security j at $t=1$; in the second scenario, entrepreneurs resell all their low quality securities and part of their high quality ones, i.e. $0 < \gamma_{j_0,\bar{i},\pi_1}^h \leq 1$. In the latter case the introduction of GB implies that entrepreneurs substitute funds raised by the costly resale of private securities, with funds raised reselling GB: since all entrepreneurs act similarly, the price of private securities in the resale market falls with an overall ambiguous effect on the level of investment. Notice however that the effect on the level of investment is guaranteed to be positive if GB is sufficiently large:

$$D_0^h > q_1^R(j)\gamma_{j_0,\bar{i},\pi_1}^h \theta_{0,1-\pi_0}^h(j_0,\bar{i}) + \Delta q_1^R(j)\theta_{0,1-\pi_0}^h(j_0,\underline{i}) \tag{20}$$

[38] The reader should keep in mind that if the issue of GD is smaller than individual private consumption before the introduction of GD, i.e. $q_0^D \frac{D_0}{1-\pi_0} \leq c_{0,1-\pi_0}^h$, there is no effect over $t=0$ investment.

i.e. if the funds raised by reselling GB are larger than the fall in the amount raised by selling low quality securities and the total amount raised by reselling the given share of high quality securities. Since D_0^h is an exogenous policy variable, one can always set it to satisfy (20) as long as D_0 satisfies the conditions of Proposition 7. This illustrates the reasoning behind the beneficial *leverage effect* of GB.

The second effect of GB, the *liquidity effect*, identifies the pressure that the introduction of GB creates over optimal security design. To illustrate this argument in a formal way, it suffices to increase continuously GB. As more GB is introduced, the $t = 1$ collateralization constraint is eventually relaxed. For sufficiently high level of GB, we will have:

$$\begin{gathered} w + q_1^R(j)\theta_{0,1-\pi_0}^h(j_0, \underline{i}) + q_1(j)\varphi_1^i(j,i) + D_0^h = \left(I_1^{\bar{i}}\right)^* \\ \text{where } \varphi_1^i(j,i) = \frac{\left(I_1^i\right)^*}{j} \\ \text{and } j = \bar{j} \end{gathered}$$

where entrepreneurs with the best technology, $\bar{i}$, achieve the same investment level of the symmetric information benchmark economy simply reselling low quality securities and GB and then issuing liquid securities carrying high collateral[39]. This is due to the fact that, thanks to GB, entrepreneurs have enough wealth to issue securities carrying sufficient collateral to be distinguishable and to finance the symmetric information optimal level of investment. I summarize the discussion above by the following proposition:

Proposition 6 (Government Debt Non Neutrality) *If* $\left(c_{0,1-\pi_0}^h\right)_{NoGB} > 0$, $\left(c_{1,1-\pi_1}^h\right)_{NoGB} > 0$*, then, even in the presence of entrepreneurs' private information, there is a (possibly large) class of economies for which a positive level of GB enhances* $t = 1$ *private investment, leaving* $t = 0$ *investment unchanged.*

One should take the argument that GB increases private investment with some care. Naturally the stock of GB that relaxes the collateral constraint must be compatible with the conditions of Proposition 7. If this conditions are not satisfied, it is not always possible to issue the stock of GB that achieve the optimal level investment at $t = 1$ without decreasing the level of investment at $t = 0$. Not withstanding this issue, one can nonetheless observe that there is a large class of economies where the introduction of GB increases investment and thus, under the optimal transfer scheme, welfare.

7 Liquidity and Financial Arrangements

This section addresses the role that financial markets have in fostering the equilibrium level of liquidity of the economy. In particular, I will focus on the beneficial role of a well known feature of developed financial system: *financial pyramiding*, or simply pyramiding. Financial pyramiding is the financial arrangement allowing an agent to borrow using not only his physical assets but also his financial assets, as collateral. This possibility - as I will argue - is especially relevant for firms seeking to raise funds through competitive financial

[39] *A fortiori* do entrepreneurs endowed with the worse technology $\underline{i}$.

markets. I hereby illustrate how pyramiding can have an important role in affecting the equilibrium level of investment.

The issue revolves around this simple fact: it is impossible for a security purchaser to observe the real value of the securities held by each entrepreneur. When agents are privately informed about the value of the securities they hold, pyramiding has a wide scope: in fact it works as an instrument to reduce the cost of reselling high quality securities and allows entrepreneurs to shift up the level of investment. Pyramiding, by fostering securities creation, ameliorates the informational asymmetry present in the resale market, where the equilibrium price is between the expected delivery of the worse and better securities and so high quality securities are, in our terminology, illiquid. When the illiquidity of good quality securities is sufficiently poor, I will show that owners of high quality securities are better off over-collateralizing new securities using old ones as collateral and forego the liquidity premium high quality securities are charged in the resale market.

In the present context I allow entrepreneurs to use the securities they own as collateral for the issuance of new securities - now with financial collateral- instead of just reselling them in the market.

7.1 Equilibrium with Financial Pyramiding: the Formalization

The core of the issue lies in the liquidity premium borne by any agent who wishes to resell high quality securities, $\bar{\imath}$, at $t = 1$. This premium is positive only if a pooling equilibrium is present in the security market at $t = 0$. Otherwise, there is no benefit from financial pyramiding as defined here.

Allowing the issuance of new securities backed by financial collateral does not, by itself, solve the informational asymmetry. In fact, by (A.2) and (A.3), the buyer of a security remains unable to screen the quality of collateral, even if this financial. But now, when at $t = 2$ the actual value of the financial collateral is revealed, the securities backed by financial collateral pay off the minimum between 1 and the value of the financial collateral. This means that anytime the value of the securities used as collateral exceeds the face value of the promise they back, the difference is retained by the borrower, not the lender. It goes without saying that this improves the use of good quality securities as an instrument to raise funds at $t = 1$. Financial pyramiding differs from simple resale: when an agent sells the high quality securities in his portfolio he transfers the "value" of a security at a price strictly smaller than the security's expected delivery.

Decreasing the burden of asymmetric information on agent's portfolio, financial pyramiding improves the funds available to $t = 1$ entrepreneurs and reduces the severity of adverse selection. In order to illustrate the argument formally, I will refer to the basic model introduced in Section 3. All the previous assumptions are maintained - in particular A.1, i.e. the probability of the contingency G(ood) is $p < 1/2$ - but for two: the security structure and the individual budget constraint.

- The **security structure** is the same as before but for the fact that at $t = 1$ securities where collateral is a *financial asset* can now be issued. I will identify each of these new securities by $j^p(j)$, where $j^p(j)$ denotes the number of units of the financial contract

j used as collateral, the financial contract with $j^p(j)$ collateral delivers:

$$D_{j^p(j)} = \left[\begin{array}{c} D_{j^p(j)}(G) = \min\left\{1, D_j^i(G)j^p(j)\right\} \\ D_{j^p(j)}(B) = \min\left\{1, D_j^i(B)j^p(j)\right\} \end{array} \right]; \\ j^p(j) \in \left\{k/10^{10} : k \in N \text{ and } 1 \leq k \leq 10^{100}\right\}$$

When the collateral is "financial", the structure is very similar to the case when the collateral is physical: the delivery in contingency s ($D_{j^p(j)}(s)$) is the minimum between the face value (1) and the actual delivery of the securities used as collateral ($D_j^i(s)j^p(j)$).

- The **budget constraint**, $\left[B^h(q)\right]^P$, when financial pyramiding is permitted, is:

$$\begin{array}{c} \left[B^h(q)\right]^P = \\ t=0: \\ c_0^h(+I_0^h) \leq w - \sum_i \sum_j q_0(j,i)\left(\theta_0^h(j,i) - \varphi_0^h(j,i)\right) \\ t=1: \\ c_1^h(+I_1^h) \leq w + \sum_j \left[q_1^R(j)\left[\sum_i \left(\gamma_{ji}^h\right)\theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R\right]\right] + \\ -\sum_j \left[q_1(j)\left(\theta_1^h(j) - \varphi_1^h(j,i)\right)\right] + \\ -\sum_{j^p(j)}\sum_j \left\{q_1(j_p(j))\left[\theta_1^h(j_p(j)) - \varphi_1^h(j_p(j,i))\right]\right\} \\ t=2: \\ c_2^h(s) \leq \sum_i \sum_j \left[D_j^i(s)\left[(1-\gamma_{ji}^h)\theta_0^h(j,i) - \varphi_0^h(j,i)\right]\right] + \\ +\sum_j \left[D_j^{i*}(s)\theta_1^h(j) + \left[D_j^{i*}(s)\right]^R\left[\theta_1^h(j)\right]^R\right] - \sum_j D_j^i(s)\varphi_1^h(j,i) + \\ +\sum_{j^p(j)}\sum_j \left[D_{j^p(j)}(s)\left[\theta_1^h(j_p(j)) - \varphi_1^h(j_p(j,i))\right]\right]\left(+g_i\left(I_t^h;s\right)\right) \end{array} \tag{21}$$

where $\theta_1^h(j_p(j))$, $\varphi_1^h(j_p(j,i))$ denote, respectively, the long and short positions of h while $q_1(j_p(j))$ labels the market price of the security having j_p units of security j as collateral. It is important to observe that the only change in the budget constraint takes place for $t=1$ and $t=2$. This is natural since the resale market is present at $t=1$ only and at $t=2$ all securities pay off.

Similarly to the case of securities with physical collateral, the corresponding collateral constraint for financial contracts backed by securities is:

$$\begin{array}{c} \text{Collateral Constraint } (\textbf{Financial Collateral}) \\ \sum_{\varphi_t^h(j^p(j))}\left[\varphi_t^h(j^p(j),i)\cdot j^p(j,i)\right] \leq \left[\left(1-\gamma_{ji}^h\right)\cdot\theta_0^h(j,i)\right] \text{ iff } \theta_0^h(j,i) > 0, \forall j \end{array} \tag{FC}$$

The interpretation is simple: no agent can use as financial collateral more securities than the ones he previously purchased and does not resell at $t=1$. It is obvious that an entrepreneur can not resell the same security he is using as collateral.

In section 3.4 I have argued that, under fairly general conditions, $t=0$ equilibrium is pooling in the security space, i.e. different entrepreneurs issue securities with the same collateral level. This case only is of interest here. At $t = 0$ we have proved that, if the equilibrium is pooling, each entrepreneur issues security $\underline{j} = \frac{1}{1+\lambda_H}$ and only one security is present in the resale market at $t = 1$.

Let us observe that a security collateralized by $\left(\frac{1+\lambda_H}{1-\lambda_H}\right)$ units of any security never defaults. Security $\underline{i}$ with physical collateral $\underline{j} = \frac{1}{1+\lambda_H}$ pays, in contingency $s = B$, $\frac{1-\lambda_H}{1+\lambda_H}$. But then, even if security $j_p(\underline{j}) = \frac{1+\lambda_H}{1-\lambda_H}$ was backed only by securities $(\underline{j}, \underline{i})$, it would always pay 1 and would never default. Therefore, security $j_p(\underline{j}) = \frac{1+\lambda_H}{1-\lambda_H}$ is constructed in a so that it is payoff invariant to the quality of the underlying collateral. Thus securities $\underline{i}$ will always be resold and never be used as collateral to other securities. An agent who wishes to raise funds through good quality securities $(\bar{i})$ has the opportunity to use financial pyramiding instead of the resale market. In this case, for each security $\bar{i}$, he gets:

$$\begin{gathered} q_1(j_p(\underline{j}))\left(\frac{1}{j_p(\underline{j})}\right) = \left(\frac{1-\lambda_H}{1+\lambda_H}\right) < q_1^R(\underline{j}) \\ \text{since } q_1(j_p(\underline{j})) = 1 \text{ at } j_p(\underline{j}) = \frac{1+\lambda_H}{1-\lambda_H} > 1 \end{gathered} \tag{22}$$

Equation (22) states that the price obtained reselling one unit of the high quality security, i.e. $q_1^R(\underline{j})$, is larger than what can be raised using the same security as collateral for the new security $j_p(\underline{j}) = \frac{1+\lambda_H}{1-\lambda_H} > 1$, i.e. $q_1(j_p(\underline{j}))\left(\frac{1}{j_p(\underline{j})}\right)$. This does not mean though that resale is preferred to financial pyramiding since resale, contrary to pyramiding, implies that the seller is losing the difference $\left[E_s\left[D_{\underline{j}}^{\bar{i}}\right] - q_1^R(\underline{j})\right] > 0$ for each security.

The possibility of financial pyramiding is inconsequential for consumers since they have no need to reallocate their consumption between today and tomorrow. It is instead important to collateral constrained entrepreneurs because it allows a more efficient use of the wealth they own through securities $\bar{i}$. Without loss of generality we focus therefore on entrepreneurs in order to establish when financial pyramiding is beneficially employed to increase investment. Readapting (10), we have:

$$I_1^h = w + q_1^R(\underline{j}) \sum_{i=\bar{i},\underline{i}} \gamma_{\underline{j},i,\pi_1}^h \theta_{0,1-\pi_0}^h(\underline{j}, i) + q_1(j)\varphi_1^i(j,i) + \underbrace{q_1(j_p(\underline{j}))\varphi_1^h(j_p(\underline{j}, \bar{i}))}_{\text{Financial Pyramiding}} \tag{23}$$

Equation (23) highlights financial pyramiding as an alternative way of funding but it can not be used to prove that financial pyramiding increases $t = 1$ private investment. In order to do so it suffices to observe what share of high quality securities, $\gamma_{\underline{j},\bar{i},\pi_1}^h$, is resold in the absence of financial pyramiding. If this share is small (e.g. $\gamma_{\underline{j},\bar{i},\pi_1}^h \approx 0$), it means that the resale of securities $\bar{i}$ was perceived by entrepreneurs as too costly. In this case the introduction of financial pyramiding allows to use all the high quality securities that had no role before and then increases private investment. One can check this intuition of how the introduction of financial pyramiding increases private investment simply recasting (23) and

(10) when $\gamma^h_{\underline{j},\bar{i},\pi_1} \approx 0$, which implies $q^R_1(\underline{j}) \approx E_s\left[D^{\underline{i}}_{\underline{j}}\right]$, and keeping in mind that $\gamma^h_{\underline{j},\underline{i},\pi_1} = 1$:

$$I^h_1 = w + E_s\left[D^{\underline{i}}_{\underline{j}}\right]\theta^h_{0,1-\pi_0}(\underline{j},\underline{i}) + q_1(j)\varphi^i_1(j,i) <$$
$$< \left(I^h_1\right)^P = w + E_s\left[D^{\underline{i}}_{\underline{j}}\right]\theta^h_{0,1-\pi_0}(\underline{j},\underline{i}) + \underbrace{q_1(j_p(\underline{j}))\varphi^h_1(j_p(\underline{j},\bar{i}))}_{\text{Financial Pyramiding}}$$
$$\text{where } \varphi^h_1(j_p(\underline{j},\bar{i})) = \frac{\theta^h_{0,1-\pi_0}(\underline{j},\bar{i})}{j_p(\underline{j},\bar{i})}$$

Then, at $t = 1$, reselling high quality security $(\underline{j},\bar{i})$ is strictly dominated by using them as financial collateral. Therefore, there is a (possibly large) class of economies in which entrepreneurs are restrained from reselling their high quality securities but use them as financial collateral whenever possible. To summarize, this class of economies presents the following two features in the equilibrium with financial pyramiding: first, only low quality securities are traded in the resale market and thus the market price remains equal to their expected delivery, i.e. $q^R_1(j) = E_s\left[D^{\underline{i}}_{\underline{j}}(s)\right]$; second, all high quality securities are used as collateral in the issuance of securities $j_p(\underline{j}) = \frac{1+\lambda_H}{1-\lambda_H}$ which are liquid since their price is equal to their expected delivery. Since entrepreneurs leverage themselves as much as they can - rearranging (11) - the maximum amount invested now becomes:

$$I^h_1 = \left\{w + E_s\left[D^{\underline{i}}_{\underline{j}}(s)\right]\theta^h_{0,1-\pi_0}(\underline{j},\underline{i}) + \frac{\theta^h_{0,1-\pi_0}(\underline{j},\bar{i})}{\frac{1+\lambda_H}{1-\lambda_H}}\right\}\left(\frac{j}{j-q_1(j)}\right) \tag{24}$$

which is strictly larger than (11) under the assumption that $\gamma^h_{\underline{j},\bar{i},\pi_1} \approx 0$ in the absence of pyramiding.

Notice finally the two main implications of our discussion. First, financial pyramiding decreases adverse selection at $t = 0$: increasing the liquidity of high quality securities at $t = 1$, their price at $t = 0$ also increases. In particular the price of securities issued by better technology $\bar{i}$ will increase while the price of securities issue by worse technology will decrease, ameliorating adverse selection. Second, financial pyramiding enhances the robustness of a pooling equilibrium at $t = 0$ since it decreases the liquidity cost of high quality securities. Because of the possibility provided by financial pyramiding, the market is able to screen and distinguish good over bad quality securities. In fact, anyone reselling a security at $t = 1$ - instead of borrowing using it as collateral - must be trading a low quality financial asset while high quality securities will always be used as collateral in the successful attempt to differentiate themselves.

8 Conclusions

The main purpose of this paper has been to propose an integrated answer to the question of what determines the equilibrium level of liquidity of an economy. This answer has been advanced through a general equilibrium characterization of the notion of liquidity and its determinants. The latter include the nature of the optimal security design in the private sector, the presence of government debt, and the available financial arrangements. I measured the liquidity of a security via an informational asymmetry concerning the value of the collateral attached to it. The informational asymmetry is responsible for the illiquidity of a security because it may generate a discrepancy between current market prices and actual (discounted) future values. Then I related the liquidity of a security to its suitability in financing private investment. In this way the equilibrium level of liquidity became a crucial factor in the allocation of credit, and thus private investment, in a production economy.

The general equilibrium analysis yields the following main implications. First, it shows that illiquidity is an equilibrium phenomenon of the economy. Firms take into account the state of the economy when they decide what kind of security to issue: how often their securities default and how serious this default is. Even allowing this freedom, this study argues that firms may tend to issue very similar securities. So a pooling equilibrium emerges in the security space. Pooling equilibria become moreover robust to off-equilibrium expectations when the share of good entrepreneurs in the economy is sufficiently large. The existence of pooling equilibria is identified with the illiquidity of the economy.

Applying our study to the positive relation between the business cycle and liquidity, we argue that security endogeneity should not be ignored to construct a realistic analysis. This aspect, sidestepped in the literature, seems to play an important role in a world where firms actively react to the state of the economy. I show in Section 4 how it is possible to reconcile an explanation of procyclical liquidity with the fact that firms optimally select the security they issue.

Secondly, I reinterpret government bonds, via their ability to provide liquidity to the economy, as encouraging rather than crowding *out* private investment. This seems important given the special attention government debt has received in the recent policy debates when the US public debt seemed headed toward full repayment. This attention results from the underlying conviction that government debt provides financial markets with some important services and, among these, the enhancement of the level of "liquidity" in the economy. Here I have provided a precise interpretation of this notion arguing that the stock of government debt has two effects on private investment: first, it provides an additional store of value, besides private securities, which, releasing the collateral constraint of entrepreneurs, allows the issuance of more securities (*leverage effect)*; second, it builds an incentive for entrepreneurs with the better technology to issue securities with higher collateral level and abandon the pooling equilibrium so moving the economy toward liquid securities (*liquidity effect).* Therefore, in the presence of asymmetric information, I have showed that government debt may increase private investment through two channels, the second of which is, to our understanding, a novel contribution.

Thirdly, given our well defined notion of collateral, I have argued that the liquidity of securities retraded in the resale market is endogenously enhanced by some of the arrangements available within financial markets. Financial markets are not passive and develop arrangements trying to enhance the level of liquidity of an economy: allowing for the possibility of using other financial contracts as collateral, they tackle the informational asymmetry present anytime securities are retraded. Therefore they provide entrepreneurs with a more convenient way to "mobilize" the wealth stored in the private securities of their portfolio.

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9 Appendix

9.1 Individual Maximization

9.1.1 Baseline Model: Resale only

h maximizes:

$$V^h\left[c_0^h, c_1^h, c_2^h = \left\{c_B^h, c_R^h\right\}\right] = E_{\pi_0}\left[c_0^h\right] + E_{\pi_1}\left[c_1^h\right] + E_s\left[c_2^h(s)\right]$$

s.t.

$$\begin{gathered} c_0^h(+I_0^h) \leq w + \textstyle\sum_i \sum_j q_0(j,i)\left[\varphi_0^h(j,i) - \theta_0^h(j,i)\right] \\ c_1^h\left(+I_1^h\right) \leq w + \textstyle\sum_j \left\{q_1^R(j)\left[\sum_i \left(\gamma_{ji}^h\right)\theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R\right] - q_1(j)\left(\theta_1^h(j) - \varphi_1^h(j,i)\right)\right\} \\ c_2^h(s) \leq g_i\left(I_t^h; s\right) + \textstyle\sum_i \sum_j \left\{D_j^i(s)\left[(1-\gamma_{ji}^h)\theta_0^h(j,i) - \varphi_0^h(j,i)\right]\right\} + \\ + \textstyle\sum_j \left\{D_j^{i^*}(s)\theta_1^h(j) - D_j^i(s)\varphi_1^h(j,i) + \left[D_j^{i^*}(s)\right]^R\left[\theta_1^h(j)\right]^R\right\} \end{gathered} \tag{BC}$$

$$\sum_{\varphi_t^h(j)} \varphi_t^h(j) j \leq I_t^h \tag{PC}$$

$$0 \leq \gamma_{ji}^h \leq 1 \text{ iff } \theta_0^h(j,i) > 0 \tag{RC}$$

This problem is equivalent to the following where the subsctipt π_t accounts for the different timing in investment opportunities. One must keep in mind that, stating the individual problem, agents distinguish between contingency π_0 (π_1) at $t = 0$ ($t = 1$) when they face an investment opportunity and contingency $(1 - \pi_0)$ $[(1 - \pi_1)]$ when they do not. By the same token, the same distinction must be accounted for when one writes the individual problem for $t = 2$.

agent h may face:

$\Gamma^h = E_{\pi_0}\left[c_0^h\right] + E_{\pi_1}\left[c_1^h\right] + E_s\left[c_2^h(s)\right] +$
$+\mu_{0,\pi_0}^h\left[w_0 - \sum_i \sum_j q_0(j,i)\left(\theta_{0,\pi_0}^h(j,i) - \varphi_{0,\pi_0}^h(j,i)\right) - I_{0,\pi_0}^h - c_{0,\pi_0}^h\right] +$
$+\mu_{0,1-\pi_0}^h\left[w_0 - \sum_i \sum_j q_0(j,i)\left(\theta_{0,1-\pi_0}^h(j,i) - \varphi_{0,1-\pi_0}^h(j,i)\right) - c_{0,1-\pi_0}^h\right] +$
$+\mu_{1,\pi_1}^h\{w_1 + \sum_j \left[q_1^R(j)\left[\sum_i\left(\gamma_{ji,\pi_1}^h\theta_{0,1-\pi_0}^h(j,i)\right) - \left(\theta_{1,\pi_1}^h(j)\right)^R\right] - q_1(j)\left(\theta_{1,\pi_1}^h(j) - \varphi_{1,\pi_1}^h(j,i)\right)\right] +$
$-I_{1,\pi_1}^h - c_{1,\pi_1}^h\} + \mu_{1,1-\pi_1}^h\{w_1 + \sum_j [q_1^R(j)\left[\sum_i\left(\gamma_{ji,1-\pi_1}^h\right)\theta_{0,\pi_0}^h(j,i) - \left(\theta_{1,1-\pi_1}^h(j)\right)^R\right] +$
$-q_1(j)\left(\theta_{1,1-\pi_1}^h(j) - \varphi_{1,1-\pi_1}^h(j,i)\right)] - c_{1,1-\pi_1}^h\} +$
$+\sum_s \mu_{2s,\pi_1}^h\{g_i\left(I_t^h; s\right) + \sum_j\left[\sum_i D_j^i(s)\left[1 - \gamma_{ji,,\pi_1}^h\right]\theta_{0,1-\pi_0}^h(j,i) - D_j^i(s)\varphi_{0,1-\pi_0}^h(j,i)\right] +$
$+\sum_j\left[\left[D_j^{i*}(s)\right]^R\left(\theta_{1,\pi_1}^h(j)\right)^R + \left[D_j^{i*}(s)\right]\theta_{1,\pi_1}^h(j) - D_j^i(s)\varphi_{1,\pi_1}^h(j,i)\right] +$
$+g_i\left(I_{1,\pi_1}^h; s\right) - c_{2,\pi_1}^h(s)\} +$
$+\sum_s \mu_{2s,1-\pi_1}^h\{\sum_j\left[\sum_i D_j^i(s)\left[1 - \gamma_{ji,1-\pi_1}^h\right]\theta_{0,1-\pi_0}^h(j,i) - D_j^i(s)\varphi_{0,\pi_0}^h(j,i)\right] +$
$+\sum_j\left[\left[D_j^{i*}(s)\right]^R\left(\theta_{1,1-\pi_1}^h(j)\right)^R - D_j^i(s)\varphi_{1,1-\pi_1}^h(j,i) + \left[D_j^{i*}(s)\right]\theta_{1,1-\pi_1}^h(j)\right]$
$+g_i\left(I_{0,\pi_0}^h; s\right) - c_{2,1-\pi_1}^h(s)\} + \sum_{t,s;k=\pi_t,1-\pi_t}\nu_t^h c_{t,k}^h(s) + \sum_{t=0,1;k=\pi_t,1-\pi_t}\beta_t^h\left(I_t - \sum_{\varphi_t^h(j)}\varphi_{t,k}^h(j)j\right) +$
$+\sum_{k=\pi_1,1-\pi_1}\left[\sum_j\sum_i \overline{\delta}_{ji,k}^h(1 - \gamma_{ji,k}^h) + \sum_j\sum_i \underline{\delta}_{ji,k}^h\gamma_{ji,k}^h\right]$

where

$$\mu_t^h = \text{multiplier for period } t \text{ budget constraint}$$
$$\nu_t^h = \text{multiplier } c_t^h \text{ non negativity constraint}$$
$$\beta_t^h = \text{multiplier for CC(t)}$$
$$\overline{\delta}_{ji}^h = RC \text{ multiplier upper bound for contract } j, \text{ technology } i$$
$$\underline{\delta}_{ji}^h = RC \text{ multiplier lower bound for contract } j, \text{ technology } i$$

Accounting for the fact that when agent h has an investment opportunities he borrows and invest[40] and when he does not produce he lends and consumes, one obtains the following FOCs:

[40]Notice the importance of risk neutrality in this observation.

$$\mu^h_{t,s} = 1 + \nu^h_{t,s} \qquad (c^h_{t,s})$$

$$q^R_1(j)\theta^h_0(j,i) \cdot \mu^h_1(k) = \textstyle\sum_s \mu^h_{2k}(s) D^i_j(s)\theta^h_0(j,i) + \overline{\delta}^h_{ji,k} - \underline{\delta}^h_{ji,k} \quad k = \pi_1, 1-\pi_1 \qquad (\gamma^h_{ji,k})$$

$$\sum_s \left[\mu^h_{2s}(1-\pi_1) g'_i\left(I^h_0; s\right)\right] + \beta^h_0 = \mu^h_0 \qquad (I^h_0)$$

$$\sum_s \left[\mu^h_{2s}(\pi_1) g'_i\left(I^h_1, s\right)\right] + \beta^h_1 = \mu^h_1 \qquad (I^h_1)$$

$$q_0(j,i) \cdot \mu^h_{0,1-\pi_0} = \sum_{k=\pi_1,1-\pi_1} k\left(\mu^h_{1,k}\gamma^h_{ji}(k) q^R_1(j) + \sum_s \mu^h_{2,k}(s) D^i_j(s)(1-\gamma^h_{ji}(k))\right) \qquad (\theta^h_{0,1-\pi_0}(j,i))$$

$$q_0(j,i) \cdot \mu^h_{0,\pi_0} = \left(\sum_s \mu^h_{2,1-\pi_1}(s) D^i_j(s)\right) + \beta^h_0 \cdot j \qquad (\varphi_0(j,i))$$

$$q_1(j) \cdot \mu^h_{1,\pi_1} = \sum_s \mu^h_{2,\pi_1}(s) D^i_j(s) + \beta^h_1 \cdot j \qquad (\varphi^h_{1,\pi_1}(j,i))$$

$$q_1(j) \cdot \mu^h_{1,1-\pi_1} = \sum_s \mu^h_{2,1-\pi_1}(s) D^i_j(s) \qquad (\theta^h_{1,1-\pi_1}(j))$$

$$q^R_1(j) \cdot \mu^h_{1,1-\pi_1} = \sum_s \mu^h_{2,1-\pi_1}(s) D^i_j(s) \qquad (\left[\theta^h_{1,1-\pi_1}(j)\right]^R)$$

Observe that $\mu^h_{2s}(k)$, $k = \pi_1, 1-\pi_1$, is equal to $Pr(s)$, irrespective of k, since consumption is positive in each contingency of $t = 2$ and $\forall h$. Equilibrium investment at $t = 1$ for the entrepreneur is given by:

$$I^i_1 = w_1 + q^R_1(j)\left[\sum_{i=\bar{i},\underline{i}} \left(\gamma^h_{j_0 i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right] + q_1(j)\left(\frac{I^i_1}{j}\right)$$

and thus:

$$I^i_1 = \left\{w_1 + q^R_1(j)\left[\sum_{i=\bar{i},\underline{i}} \left(\gamma^h_{j_0 i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right]\right\}\left(\frac{j}{j - q_1(j)}\right)$$

9.1.2 Theorem 2: Existence of Robust Pooling Equilibrium $j^{pool} = \underline{j}$

The proof of the theorem is divided into few intermediate steps.

The general argument is organized as follows: first I characterize what conditions must be satisfed by the optimal security/ies; second, I characterize the set of pricing functionals consistent with the existence of pooling equilibria; finally, I show that there is a pricing functional in this set robust to any off-equilibrium perturbation.

Lemma 1 *The optimal security for entrepreneur endowed with technology i is the one solving the following problem:*

$$\begin{gathered}\min_{j\in\widehat{J}} \Phi\left[i,j\right] = \\ = \tfrac{E_s\left[D^i_j(s)\right]}{q_1(j)}\widetilde{w}^i + \tfrac{j}{q_1(j)}\left[\left(I^i_1 - \widetilde{w}^i\right)E_s\left[g'_i\left(I^i_1;s\right)\right] - D^i_1\right]\end{gathered}$$

where $\widetilde{w}^i = w + q^R_1(j_0)\left[\sum_{i=\bar{i},\underline{i}}\left(\gamma^h_{j_0 i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right]$.

Proof. Let us start rearranging the FOCs with respect to I^i_1 and $\varphi_1(j,i)$:

$$E_s\left[g'_i\left(I^i_1;s\right)\right]q_1(j) = E_s\left[D^i_j(s)\right] + \beta^i_1\left(j - q_1(j)\right) \tag{25}$$

The expression above has to be satisfied by any security issued in equilibrium. This condition allows to rank different securities from the point of view of entrepreneur i: the optimal security/ies minimize the RHS of the previous equation, i.e. its marginal cost. To determine the value of β^i_1, one multiplies both sides by $\varphi^i_1(j)$ and then adds over all $\varphi^i_1(j)$ of i:

$$\begin{gathered}E_s\left[g'_i\left(I^i_1;s\right)\right]\sum_{\varphi^i_1(j)} q_1(j)\varphi^i_1(j) = \\ = \sum_{\varphi^i_1(j)}\varphi^i_1(j)E_s\left[D^i_j(s)\right] + \beta^i_1\sum_{\varphi^i_1(j)}\left(j - q_1(j)\right)\varphi^i_1(j)\end{gathered}$$

Since $\sum_{\varphi^i_1(j)} j\varphi^i_1(j) = I^i_1$ and $\sum_{\varphi^i_1(j)} q_1(j)\varphi^i_1(j) = I^i_1 - \widetilde{w}^i$, where $\widetilde{w}^i = w + q^R_1(j_0)\left[\sum_{i=\bar{i},\underline{i}}\left(\gamma^h_{j_0 i,\pi_1}\theta^h_{0,1-\pi_0}(j_0,i)\right)\right]$ we finally have:

$$\beta^i_1 = \frac{\sum_{\varphi^i_1(j)}\left(q_1(j)E_s\left[g'_i\left(I^i_1;s\right)\right] - E_s\left[D^i_j(s)\right]\right)\varphi^i_1(j)}{\widetilde{w}^i} \tag{26}$$

Substituting (26) into (25) and dividing by $q_1(j)$, one has:

$$\begin{gathered}E_s\left[g'_i\left(I^i_1;s\right)\right] = \\ = \frac{E_s\left[D^i_j(s)\right]}{q_1(j)} + \frac{\left(I^i_1 - \widetilde{w}^i_1\right)E_s\left[g'_i\left(I^i_1;s\right)\right] - \sum_{\varphi^i_1(j)}E_s\left[D^i_j(s)\right]\varphi^i_1(j)}{\widetilde{w}^i}\left(\frac{j - q_1(j)}{q_1(j)}\right)\end{gathered}$$

Thus, defining $\sum_{\varphi^i_1(j)} E_s\left[D^i_j(s)\right]\varphi^i_1(j) = D^i_1$:

$$\begin{gathered}E_s\left[g'\left(I^i_1\right),\lambda_i;s\right]I^i_1 + D^i_1 = \\ = \tfrac{E_s\left[D^i_j(s)\right]}{q_1(j)}\widetilde{w}^i + \tfrac{j}{q_1(j)}\left[\left(I^i_1 - \widetilde{w}^i\right)E_s\left[g'_i\left(I^i_1;s\right)\right] - D^i_1\right]\end{gathered} \tag{27}$$

In conclusion, finding the optimal security for entrepreneur i is equivalent to minimize the RHS in (27):

$$\begin{aligned} &\min_{j \in \widehat{J}} \Phi\left[i,j\right] = \\ &= \frac{E_s\left[D_j^i(s)\right]}{q_1(j)}\widetilde{w}^i + \frac{j}{q_1(j)}C^i \end{aligned} \tag{28}$$

where $C^i = \left(I_1^i - \widetilde{w}^i\right) E_s\left[g_i'\left(I_1^i; s\right)\right] - D_1^i$ ■

Then notice:

Lemma 2 *When $p < 1/2$, technology L is ranked superior to technology 1. Given the production function $g(.)$, it is always possible to find an open set of economies characterized by a sufficiently large share, $\eta(L)$, of better technology entrepreneurs in the economy, i.e. $1 \geq \eta(L) \geq (\eta(L))^*$ such that the set of pricing functionals supporting the pooling equilibrium $\underline{j} = j^{pool}$ is:*

$$Q^{pool} = \left\{q_1(j) \mid E_s\left[D_j^1\right] \leq q_1(j) \leq q^{1*}(j)\right\}$$

Proof. The objective of this lemma is to derive the pricing functionals that support the pooling equilibrium in which all entrepreneurs issue security $\underline{j}$ only. Such functionals make technology "1" entrepreneurs - the worse technology - *at most* indifferent between issuing security $\underline{j} = j^{pool}$ and any other while technology "2" - the better one - entrepreneurs strictly prefer $\underline{j} = j^{pool}$.

Formally, the previous lemma has proved that $\underline{j} = j^{pool}$ is selected by entrepreneur with technology i if $\underline{j} \in \arg\min_{j \in \widehat{J}} \Phi\left[i,j\right]$, as defined in (28). Therefore, in order for a pricing functional to support the pooling equilibrium $\underline{j}$, it has to be the case:

$$\Phi\left[i,\underline{j}\right] \leq \Phi\left[i,j\right],\ \forall j \in \widehat{J}$$

The pricing functionals supporting the pooling equilibrium thus satisfy:

$$\begin{aligned} &\Phi\left[L,\underline{j}\right] < \Phi\left[L,j\right],\ \forall j \in \widehat{J} \quad \textit{better} \text{ technology entrepreneur} \\ &\Phi\left[H,\underline{j}\right] = \Phi\left[H,j\right],\ \forall j \in \widehat{J} \quad \textit{worse} \text{ technology entrepreneur} \end{aligned} \tag{29}$$

The first step of the argument is the characterization of the pricing functionals $q^{H*}(j)$ and $q^{L*}(j)$ that make, respectively, entrepreneurs "H" and "L" indifferent. Thus, $q^{i*}(j)$ is constructed to deliver $\Phi\left[i,\underline{j}\right] = \Phi\left[i,j\right]$, $\forall j \in \widehat{J}$:

$$\rho_i = \frac{\left[\widetilde{w}^i E_s\left[D_j^i(s)\right] + jC^i\right]}{q^{i*}(j)}; \rho_i \in R_+$$

where, for each entrepreneur i:

$$C^i = \varphi_1\left(\underline{j}, i\right)\left[q_1(\underline{j})E_s\left[g_i'\left(I_1^i; s\right)\right] - E_s\left[D_{\underline{j}}^i\right]\right]$$

Notice that $C^i = \left[\left(I_1^i - \widetilde{w}_1^i\right) E_s\left[g_i'\left(I_1^i; s\right)\right] - D_1^i\right]$ (keep in mind $\sum_{\varphi_1^i(j)} E_s\left[D_j^i(s)\right]\varphi_1^i(j) = D_1^i$) is computed taking into account that entrepreneur i is issuing only security $\underline{j}$. Since this is the equilibrium we are looking at, the choice is not arbitrary and C^i is treated as a constant when j changes.

Rephrasing (29), we can immediately observe that, in order to support the pooling equilibrium, it is *sufficient* to prove that the price increase that keeps technology "A" entrepreneur indifferent must make technology "L" entrepreneur strictly worse off if he chose security $j > \underline{j}$. The sufficiency comes from the fact that the starting point of the pricing functionals $q^{H*}(j)$ and $q^{L*}(j)$ is the same pooling equilibrium price:

$$q^{H*}(\underline{j}) = q^{L*}(\underline{j}) = q_1(\underline{j}) \tag{30}$$

where $q_1(\underline{j})$ is the pooling security price as determined by rational expectations given the measure of better $(0 \leq \eta(L) \leq 1)$ and worse $(\eta(H) = 1 - \eta(L))$ technology entrepreneurs in the given economy:

$$q_1(\underline{j}) = \frac{\eta(H)\varphi_1\left(\underline{j}, H\right) E_s\left[D^H_{\underline{j}}(s)\right] + \eta(L)\varphi_1\left(\underline{j}, L\right) E_s\left[D^L_{\underline{j}}(s)\right]}{\eta(H)\varphi_1\left(\underline{j}, H\right) + \eta(L)\varphi_1\left(\underline{j}, L\right)}$$

If $q_1(\underline{j})$ provides the "intercept" of $q^{H*}(j)$ and $q^{L*}(j)$, it is still necessary to determine their slopes. These slopes play a fundamental role in our argument. In fact, we have already observed that the support of the pooling equilibrium requires that $q^{L*}(j)$ lies above $q^{H*}(j)$ $\forall j \in \widehat{J}$. Graphically this results in the following:

Figure 6

I now turn to determine the explicit expressions for $q^{H*}(j)$ and $q^{L*}(j)$. For simplicity of exposition, I will divide the security space $\widehat{J}$ in three subsets:

$$\widehat{J}_1 = \left\{ j \in \Re_+ | \, \underline{j} = \tfrac{1}{1+\lambda_H} \leq j \leq \tfrac{1}{1+\lambda_L} \right\}$$
$$\widehat{J}_2 = \left\{ j \in \Re_+ | \, \tfrac{1}{1+\lambda_L} \leq j \leq \tfrac{1}{1-\lambda_L} \right\}$$
$$\widehat{J}_3 = \left\{ j \in \Re_+ | \, \tfrac{1}{1-\lambda_L} \leq j \leq \tfrac{1}{1-\lambda_H} = \overline{j} \right\}$$

This is convenient since $E_s\left[D_j^i(s)\right]$ changes slope depending on the interval considered:

$$\begin{array}{ll} i=H & \frac{\partial E_s\left[D_j^H(s)\right]}{\partial j}=(1-p)(1-\lambda_H),\ j\in\widehat{J} \\ i=L & \frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}=\begin{cases} 1+(2p-1)\lambda_L,\ j\in\widehat{J}_1 \\ (1-p)(1-\lambda_L),\ j\in\widehat{J}_2 \\ 0,\ j\in\widehat{J}_3 \end{cases} \end{array} \tag{31}$$

By definition, $q^{i*}(j)$ ensures that $\Phi\left[i,\underline{j}\right]=\Phi\left[i,j\right]$, $\forall j\in\widehat{J}$:

$$\frac{1}{\rho_i}=\frac{q^{i*}(j)}{\left[\widetilde{w}^i E_s\left[D_j^i(s)\right]+jC^i\right]};\rho_i\in R_+\forall j\in\widehat{J}$$

In particular:

$$\frac{q^{i*}(\underline{j})}{\left[\widetilde{w}^i E_s\left[D_{\underline{j}}^i(s)\right]+\underline{j}C^i\right]}=\frac{q^{i*}(j)}{\left[\widetilde{w}^i E_s\left[D_j^i(s)\right]+jC^i\right]} \tag{32}$$

The determination of $q^{H*}(j)$ is easier and so we give determine right away. It is possible to rewrite the RHS of (32) as follows, since we are restricting attention to linear pricing functionals:

$$\frac{q^{i*}(\underline{j})+\beta_i\left(j-\underline{j}\right)}{\left[\widetilde{w}^i E_s\left[D_{\underline{j}}^i(s)\right]+\underline{j}C^i\right]+\left(j-\underline{j}\right)\left[\widetilde{w}^i\left.\frac{\partial E_s\left[D_j^i(s)\right]}{\partial j}\right|_{\widehat{J}}+C^i\right]} \tag{33}$$

Rearranging we get the slope of $q^{i*}(j)$ in $\widehat{J}$:

$$\beta_i=q_1(\underline{j})\frac{\left[\widetilde{w}^i\left.\frac{\partial E_s\left[D_j^i(s)\right]}{\partial j}\right|_{\widehat{J}_1}+C^i\right]}{\left[\widetilde{w}^i E_s\left[D_{\underline{j}}^i(s)\right]+\underline{j}C^i\right]} \tag{34}$$

The slope of $q^{H*}(j)$ thus becomes:

$$\beta_H=q_1(\underline{j})\frac{\left[\widetilde{w}^H(1-p)(1-\lambda_H)+C^H\right]}{\underline{j}\left[\widetilde{w}^H(1+(2p-1)\lambda_H)+C^H\right]}<\frac{q_1(\underline{j})}{\underline{j}} \tag{35}$$

so that one can write:

$$\begin{array}{c} q^{1*}(j)=q_1(\underline{j})+\beta_H(j-\underline{j})= \\ =q_1(\underline{j})+q_1(\underline{j})\frac{\left[\widetilde{w}^H(1-p)(1-\lambda_H)+C^H\right]}{\underline{j}\left[\widetilde{w}^H(1+(2p-1)\lambda_H)+C^H\right]}(j-\underline{j}) \end{array} \tag{36}$$

Two observations can be made about $q^{H*}(j)$. Firstly, $\beta_H>(1-p)(1-\lambda_H)$, i.e. the pricing functional that keeps entrepreneur "H" indifferent increases faster than its respective expected delivery, $E_s\left[D_j^H\right]$, as we move toward securities with higher collateral level. If this were not the case, the entrepreneur would be choosing a security whose rate of repayment $R_1^H(j) = \frac{E_s\left[D_j^H(s)\right]}{q_1(j)}$ is smaller or equal than $\underline{j}$ and which costs strictly more in terms of collateral. Secondly - a consequence of the first observation - $q^{H*}(j)$ intersects the upper envelope of the price functionals, i.e. $E_s\left[D_j^H(s)\right]$, at collateral level $j_* < \frac{1}{1-\lambda_H}$.

The issue is slightly more involved for the pricing functional making entrepreneurs "L" indifferent, i.e. $q^{L*}(j)$ and, for a better understanding of the argument, the reader is reffered to Figure 5. We present $q^{L*}(j)$ construction splitting the argument according to the 3 relevant regions. The argument concludes that $q^{L*}(j)$ is continuous and strictly concave.

- Starting in $j \in \widehat{J}_1$, by (34) we get the slope of $q^{L*}(j)$ in $\widehat{J}_1$:

$$\beta_L = q_1(\underline{j}) \frac{\left[\widetilde{w}^L \left.\frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}\right|_{\widehat{J}_1} + C^L\right]}{\left[\widetilde{w}^L E_s\left[D_{\underline{j}}^L(s)\right] + \underline{j} C^L\right]} =$$
$$= q_1(\underline{j}) \frac{\left[\widetilde{w}^L(1+(2p-1)\lambda_L) + C^L\right]}{\underline{j}\left[\widetilde{w}^L(1+(2p-1)\lambda_L) + C^L\right]} = \frac{q_1(\underline{j})}{\underline{j}}$$

  This makes us conclude that $q^{L*}(j) > q^{1*}(j)$ in $\widehat{J}_1$ since the slope $q^{L*}(j)$ is larger than the slope of $q^{1*}(j)$ there. In fact, by (34) and (30), we have:

$$\beta_L > \beta_H$$
$$\Leftrightarrow$$
$$\frac{q_1(\underline{j})}{\underline{j}} > q_1(\underline{j}) \frac{\left[\widetilde{w}^H(1-p)(1-\lambda_H) + C^1\right]}{\underline{j}\left[\widetilde{w}^H(1+(2p-1)\lambda_H) + C^1\right]}$$

- In $j \in \widehat{J}_2$ $\frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}$ falls and the "intercept" of $q^{L*}(j)$ becomes $q^{L*}(\frac{1}{1+\lambda_L})$. I will show that the slope of $q^{L*}(j)$ falls from $\widehat{J}_1$ to $\widehat{J}_2$. Since we have:

$$\frac{q^{L*}\left(\frac{1}{1+\lambda_L}\right)}{\left[\widetilde{w}^L E_s\left[D^L_{\frac{1}{1+\lambda_L}}(s)\right] + \frac{1}{1+\lambda_L} C^L\right]} = \frac{q^{L*}\left(\frac{1}{1-\lambda_L}\right)}{\left[\widetilde{w}^L E_s\left[D^L_{\frac{1}{1-\lambda_L}}(s)\right] + \frac{1}{1-\lambda_L} C^L\right]}$$

  the RHS of the previous equation can be written:

$$\frac{q^{L*}(\frac{1}{1+\lambda_L}) + \beta_L'\left(\frac{1}{1-\lambda_L} - \frac{1}{1+\lambda_L}\right)}{\left[\widetilde{w}^L E_s\left[D^L_{\frac{1}{1+\lambda_L}}(s)\right] + \frac{1}{1+\lambda_L} C^L\right] + \left(\frac{1}{1-\lambda_L} - \frac{1}{1+\lambda_L}\right)\left[\widetilde{w}^L \left.\frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}\right|_{\widehat{J}_2} + C^L\right]}$$

  Rearranging we get the slope of $q^{L*}(j)$ in $\widehat{J}_2$:

$$\beta_L' = q^{L*}\left(\frac{1}{1+\lambda_L}\right) \frac{\left[\widetilde{w}^L \left.\frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}\right|_{\widehat{J}_2} + C^L\right]}{\left[\widetilde{w}^L E_s\left[D^L_{\frac{1}{1+\lambda_L}}(s)\right] + \frac{1}{1+\lambda_L} C^L\right]} =$$
$$= q^{L*}\left(\frac{1}{1+\lambda_L}\right) \frac{\left[\widetilde{w}^L(1-p)(1-\lambda_L) + C^L\right]}{\frac{1}{1+\lambda_L}\left[\widetilde{w}^L(1+(2p-1)\lambda_L) + C^L\right]}$$

Notice that, in order to keep $\Phi[L, j]$ constant, $\beta'_L < (1-p)(1-\lambda_L)$. Therefore, $\beta'_L < \beta_L$.

- Finally the slope of $q^{L*}(j)$ in $\widehat{J}_3$ can be easily found applying the same methodology as before Accounting for the fact that the relevant "intercept" is now $q^{L*}\left(\frac{1}{1-\lambda_L}\right)$, β''_L, i.e. the slope of $q^{L*}(j)$ in $\widehat{J}_3$, turns out to be:

$$\begin{aligned}\beta''_L &= q^{L*}\left(\tfrac{1}{1+\lambda_L}\right)\frac{\left[\widetilde{w}^L \frac{\partial E_s\left[D_j^L(s)\right]}{\partial j}\Big|_{\widehat{J}_3} + C^L\right]}{\left[\widetilde{w}^L E_s\left[D^L_{\frac{1}{1+\lambda_L}}(s)\right] + \frac{1}{1+\lambda_L}C^L\right]} = \\ &= q^{L*}\left(\tfrac{1}{1-\lambda_L}\right)\frac{C^L}{\frac{1}{1-\lambda_L}\left[\widetilde{w}^L(1+(2p-1)\lambda_L)+C^L\right]}\end{aligned}$$

which is strictly positive but $\beta''_L < \beta'_L$.

The fact that $\beta''_L < \beta'_L < \beta_L$ shows that $q^{L*}(j)$ is strictly concave. $q^{L*}(j)$'s continuity arises by construction. $q^{L*}(j)$'s concavity - together with $q^{H*}(j)$'s linearity over $\widehat{J}$ and the fact that $q^{H*}(j)$ intersects the upper envelope of the price functionals, i.e. $E_s\left[D_j^L(s)\right]$, at collateral level $j_* < \frac{1}{1-\lambda_H}$ - allows a simple methodology to determine the set of pooling equilibrium pricing functionals. To this purpose we make the following two observations.

Firstly, along $q^{H*}(j)$ entrepreneurs endowed with technology "H" are indifferent between issuing the pooling security $\underline{j}$ and any other security $j < j_*$. Moreover, by (35), the slope of $q^{H*}(j)$ is increasing in the pooling price $q_1(\underline{j})$ which, in turn, increases with the measure of entrepreneurs endowed with the better technology "2", i.e. $\eta(H)$, in the population.

Secondly, because of $q^{L*}(j)$'s concavity, the survival of the pooling equilibrium is linked to the fact that entrepreneurs endowed with technology "L" have no incentive to deviate, given prices, to securities with higher collateral. The absence of deviations is equivalent to the fact that better technology entrepreneurs get higher profits issuing $\underline{j} = j^{pool}$ instead of j_*. Formally:

$$Profits(L, \underline{j} = j^{pool}) \geq Profits(L, j_*)$$

Let us start observing that, when the population is made of good quality entrepreneurs only, i.e. $\eta(L) = 1$, we have:

$$\begin{gathered} Profits(L, \underline{j} = j^{pool}) > Profits(L, j_*) \\ \Longleftrightarrow \\ E\left[g_L\left(I_1^L\right)\right] - \varphi(\underline{j}, L)E\left[D^L_{\underline{j}}\right] > \\ > E\left[g_L\left(I_1^L\right)\right] - \varphi(j_*, L)E\left[D^L_{j_*}\right] \end{gathered} \tag{37}$$

In fact, since the collateralization constraint is binding, entrepreneurs sell all the securities they can:

$$\begin{gathered} I_1^L(j) = \widetilde{w}^L + \varphi(j, L)q(j) = \varphi(j, L)j \\ \Rightarrow \varphi(j, L) = \frac{\widetilde{w}^L}{j - q(j)} \end{gathered}$$

The inequality (37) above becomes:

$$E\left[g_L\left(\widetilde{w}^L+\widetilde{w}^L\frac{q_1(\underline{j})}{\underline{j}-q_1(\underline{j})}\right)\right]-\widetilde{w}^L\frac{E_s\left[D^L_{\underline{j}}\right]}{\underline{j}-q_1(\underline{j})}>$$
$$>E\left[g_L\left(\widetilde{w}^L+\widetilde{w}^L\frac{q_1(j_*)}{\underline{j}-q_1(j_*)}\right)\right]-\widetilde{w}^L\frac{E_s\left[D^L_{j_*}\right]}{j_*-q_1(j_*)}$$

since by construction we have that $q(j_*)=E_s\left[D^L_{j_*}\right]$ and $q(\underline{j})=E_s\left[D^L_{\underline{j}}\right]$, thus:

$$E\left[g_L\left(\widetilde{w}^L+\widetilde{w}^L\frac{E_s\left[D^L_{\underline{j}}\right]}{\underline{j}-E_s\left[D^L_{\underline{j}}\right]}\right)\right]-\widetilde{w}^L\frac{E\left[D^L_{\underline{j}}\right]}{\underline{j}-E\left[D^L_{\underline{j}}\right]}>$$
$$>E\left[g_L\left(\widetilde{w}^L+\widetilde{w}^L\frac{E_s\left[D^L_{j_*}\right]}{j_*-E_s\left[D^L_{j_*}\right]}\right)\right]-\widetilde{w}^L\frac{E\left[D^L_{j_*}\right]}{j_*-E\left[D^L_{j_*}\right]}$$

Observing:

$$\frac{d[Profits(L,j)]}{d\left[\frac{E\left[D^L_j\right]}{j-E\left[D^L_j\right]}\right]}>0$$
$$\Leftrightarrow$$
$$E\left[g'_L\left(I^L_1\right)\right]-1>0$$

because of (25) and

$$\frac{d\left(\frac{E\left[D^L_j\right]}{j-E\left[D^L_j\right]}\right)}{d\,(j)}<0$$

we have that the inequality in (37) is *strictly* satisfied. Thus the pooling is preserved. This is certainly not surprising since at $j=\underline{j}$ entrepreneurs with good quality technology "L" get a price equal to their expected delivery and issue the security entailing the least level of collateral.

We now conclude the argument testing whether the pooling equilibrium is preserved once the share of good quality entrepreneurs decreases, i.e. $\eta(L)$ falls below 1. The fall of $\eta(L)$ has two effects on the relation between $Profits(L,\underline{j}=j^{pool})$ and $Profits(L,j_*)$: on one side, it lowers the pooling price $q(\underline{j})$ and makes the pooling equilibrium less appealing since $Profits(L,\underline{j}=j^{pool})$ decreases; on the other side, it decreases the slope and intercept of $q^{H*}(j)$ increasing j_* and so lowering $Profits(L,j_*)$. But since the inequality in (37) is strict, we can argue that, by the continuity of the profit function, there is always a threshold share of good quality entrepreneurs $1>(\eta(L))^*\geq 0$ such that the pooling is preserved.

Therefore, given the productive technology, one can always find an open set of economies characterized by $1\geq\eta(L)\geq(\eta(L))^*$ such that the set of pricing functionals supporting the pooling equilibrium $\underline{j}=j^{pool}$ is:

$$Q^{pool}=\left\{q_1(j)\mid E_s\left[D^H_j\right]\leq q_1(j)\leq q^{H*}(j)\right\}$$

■

Lemma 3 *If* $1 \geq \eta(L) \geq (\eta(L))^*$*, the pooling equilibrium* $\underline{j} = j^{pool}$ *is robust in the sense of definition 2, i.e. it is the limit, for* $\varepsilon_j(n) \to 0$ *when* $n \longrightarrow +\infty$*, of a sequence of* ε*-economies in which the external agent issues* $\varepsilon_j(n)$ *shares paying* $E_s\left[D_j^{\bar{i}}(s)\right]$ *in each off-equilibrium security* $j > \underline{j} = j^{pool}$.

Proof. In order to prove the statement it is sufficient to focus on the case where only high quality (L) entrepreneurs are forced to issue securities that are not part of the equilibrium. I will present the argument in a very concise form since the reasoning is the same as Dubey and Geanakoplos (2003):

1. consider the average quality for security $\underline{j} = j^{pool}$ unaltered at the original level and force an "external agent" of measure $\varepsilon(n)$, indistinguishable to quality L entrepreneurs in terms of security payoffs, to issue off equilibrium securities, i.e. securities with collateral level $j > \underline{j}$;

2. after forcing the external agent to issue off equilibrium securities, the equilibrium of the economy is determined allowing all the remaining agents to optimize. Define this as the equilibrium of the $\varepsilon(n)$ economy;

3. finally take $\varepsilon(n) \to 0$, i.e. the measure of the external agent to zero. Clearly the limit $\varepsilon(\infty) = 0$ replicates the original economy. If the equilibrium of the $\varepsilon(n)$ economy converges to the pooling equilibrium $j^{pool} = \underline{j}$ of the original economy, then we say that the equilibrium survives the "external agent" perturbation. If the equilibrium survives this perturbation in which the external agent is equivalent to quality L entrepreneurs *only* issuing off equilibrium securities, then the equilibrium survives the perturbation in which the external agent is equivalent to any mix of good and bad quality entrepreneurs. Thus the equilibrium is robust.

In order to prove the argument it suffices to show that in the $\varepsilon(n)$ economy entrepreneurs with low quality technology (1) are indifferent between issuing $j^{pool} = \underline{j}$ and $j \neq \underline{j}$ while entrepreneurs with high quality technology (L) are at most indifferent, if not worse off. To this purpose consider the choice of an entrepreneur who is issuing $j^{pool} = \underline{j}$ and is now facing the introduced perturbation on security $j > \underline{j}$ in the $\varepsilon(n)$ economy. The previous lemma observed that if entrepreneur L deviates so does entrepreneur 1 but NOT viceversa. Therefore the equilibrium price on security $j > \underline{j}$ will reflect this asymmetry and will fall up to the point where entrepreneurs 1 are indifferent. But this is the condition necessary to support the pooling equilibrium: low quality technology holders are indifferent to issue different security while high quality technology holders are strictly worse off. In effect, the previous lemma guaranteed that when the share of good technology entrepreneurs is larger the required threshold, i.e. $\eta(L) \geq (\eta(L))^*$, the pooling equilibrium survives the perturbation consisting by forcing only good quality entrepreneurs on off equilibrium security.

In conclusion, since the deviations of low quality entrepreneurs is triggered by the perturbation consisting in forcing measure $\varepsilon(n)$ of entrepreneurs (L) to issue $j > \underline{j} = j^{pool}$, the measure of entrepreneurs 1 issuing securities different from $\underline{j} = j^{pool}$ converges to zero as $\varepsilon(n) \to 0$. Thus the pooling equilibrium survives any perturbation. ■

9.1.3 Baseline Model with Government Debt

h maximizes:

$$V^h\left[c_0^h, c_1^h, c_2^h = \left\{c_B^h, c_R^h\right\}\right] = E_{\pi_0}\left[c_0^h\right] + E_{\pi_1}\left[c_1^h\right] + E_s\left[c_2^h(s)\right]$$

s.t.

$$\begin{gathered}\left[B^h\left(q, w^h\right)\right]^D = \\ t=0: \\ c_0^h + (+I_0^h) \leq w_0 - \sum_i \sum_j q_0(j,i)\left(\theta_0^h(j,i) - \varphi_0^h(j,i)\right) - q_0^D D_0^h \\ t=1: \\ c_1^h + (+I_1^h) \leq w_1 + q_1^D(\gamma_D^h D_0^h - D_1^h) + \\ + \sum_j \left\{q_1^R(j)\left[\sum_i \left(\gamma_{ji}^h\right)\theta_0^h(j,i) - \left(\theta_1^h(j)\right)^R\right] - q_1(j)\left(\theta_1^h(j) - \varphi_1^h(j,i)\right)\right\} \\ t=2: \\ c_2^h(s) \leq \sum_i \sum_j \left\{D_j^i(s)\left[(1-\gamma_{ji}^h)\theta_0^h(j,i) - \varphi_0^h(j,i)\right]\right\} + \left(1-\gamma_D^h\right) D_0^h + D_1^h + \\ + \sum_j \left\{D_j^{i*}(s)\theta_1^h(j) - D_j^i \varphi_1^h(j,i) + \left[D_j^{i*}(s)\right]^R \left[\theta_1^h(j)\right]^R\right\} - T + \left(+g_i\left(I_t^h; s\right)\right)\end{gathered}$$

and

$$0 \leq \gamma_D^h \leq 1 \text{ iff } D_0^h > 0 \qquad \text{(GovRC)}$$

which is equivalent to the following, once one takes into account when h is a lender and whe he is not:

$\left[\Gamma^h\right]^D = \Gamma^h + \sum_{k=\pi_1,1-\pi_1}\left[\sum_j \sum_i \overline{\delta}_{ji,k}^h(1-\gamma_{ji,k}^h) + \sum_j \sum_i \underline{\delta}_{ji,k}^h \gamma_{ji,k}^h + \overline{\delta}_{D,k}^h(1-\gamma_{D,k}^h) + \underline{\delta}_{D,k}^h \gamma_{D,k}^h\right]$

where

$$\begin{aligned}\overline{\delta}_{D,k}^h &= GovRC \text{ multiplier upper bound for resale Government debt} \\ \underline{\delta}_{D,k}^h &= GovRC \text{ multiplier lower bound for resale Government debt}\end{aligned}$$

The problem is just a slight modification of the previous one. All the previous FOCs hold plus the followings:

$$q_0^D \mu_0^h(1-\pi_0) = \sum_{k=\pi_1,1-\pi_1} k\left(\mu_1^h(k)\gamma_D^h(k)q_1^D(j) + \sum_s \mu_{2s}^h(k)(1-\gamma_D^h(k))\right) \qquad (D_0^h)$$

$$q_1^D \mu_1^h(1-\pi_1) = \sum_s \mu_{2s}^h(1-\pi_1) \qquad (D_1^h)$$

$$\begin{gathered}q_1^D \mu_1^h(k) = \sum_s \mu_{2s}^h(k) + \overline{\delta}_{D,k}^h - \underline{\delta}_{D,k}^h \\ k = \pi_1, 1-\pi_1\end{gathered} \qquad (\gamma_D^h(k))$$

Observing that, by assumption, lenders (the one purchasing securities) have strictly positive consumption at $t = 0$, $t = 1$ and $t = 2$, one has:

$$q_1^D = 1$$

and, rearranging the FOCs with respect to γ_D^h, D_0^h and D_1^h:

$$q_0^D = (1 - \pi_1) + \pi_1 \left(\mu_1^h(k)\gamma_D^h(k) + (1 - \gamma_D^h(k)) \right) > 1$$