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Single and joint default in a structural model with purely discontinuous assets

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Abstract

Structural models of credit risk are known to present both vanishing spreads at very short maturities and a poor spread fit over longer maturities. The former shortcoming, which is due to the diffusive behavior assumed for asset values, can be circumvented by considering discontinuous assets. In this paper we resort to a pure jump process of the Variance-Gamma type.

First we calibrate the corresponding Merton type structural model to single-name data for the DJ CDX NA IG and CDX NA HY components. By so doing, we show that it circumvents also the diffusive structural models difficulties over longer horizons. In particular, it corrects for underprediction of low risk spreads and overprediction of high risk ones.

Then we extend the model to joint default, resorting to a recent formulation of the VG multivariate model and without superimposing a copula choice. We fit default correlation for a sample of CDX NA names, using equity correlation. The main advantage of our joint model with respect to the existing non diffusive ones is that it allows calibration without the equicorrelation assumption, but still in a parsimonious way. As an example of the default assessments which the calibrated model can provide, we price a FtD swap.

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In the credit risk literature the so-called structural form models, pioneered by Merton's 1974 contribution, play an important role, mainly because of the allure of endogenizing default arrival in an economic simple framework. As it is well known indeed, in the Merton (1974) model, default is triggered by the fact that the asset value at debt maturity is smaller than the debt one. Analogously, in the credit barrier models which have been inspired by Merton's original contribution, such as Black and Cox (1976), Longstaff and Schwartz (1995), Leland and Toft (1996), to mention a few, default can occur before maturity, if the asset value goes below an appropriate threshold.

In spite of this conceptual simplicity, the original Merton model, as well as its diffusion-based, threshold extensions, present two main weaknesses. On the theoretical side, they are unable to produce positive credit spreads in the very short run. Whenever the asset value follows a diffusion process indeed, default is not a totally unpredictable stopping time: as a consequence, the spread is null over close maturities. On the calibration side, their ability to explain actual spreads over Treasuries is under discussion. A number of papers, including Jones, Mason and Rosenfeld (1984) or, more recently, Lyden and Saraniti (2000), Demchuk and Gibson (2004), Eom, Helwege and Huang (2004) question the explanatory power of structural models, given that the percentage of the actual credit spread they are able to explain is modest. In particular, Eom et alii emphasize underprediction of the Merton model and overprediction of other structural models for high risk bounds, together with underprediction for low risk ones.

The theoretical shortcomings of the diffusion based models can be eliminated only by assuming an asset process with jumps, as in Zhou (2001) or Hilberink and Rogers (2002), or a pure jump asset process, as suggested in Madan (2000).The latter approach is intuitively quite convincing, since purely discontinuous processes can be interpreted as time changed Brownian motions. The time change in turn has been suggested by Clark (1973) and justified - both theoretically and on the empirical ground - by Geman, Madan and Yor (2000), Geman and Ané (1996), Geman (2005). Geman and Ané for instance report that the S&P 500, while not presenting normal returns under calendar time, shows them per unit trade. Geman et alii (2000) conclude that the jump component is not only present, but of such high activity that no continuous martingale component is necessary in order to represent financial asset dynamics.

This paper focuses on a particular pure jump Lévy process, a Variance Gamma (VG) one, in order to show that

- it circumvents also the calibration difficulties of diffusive structural models over longer horizons;
- it can be extended to multiple defaults without assuming equicorrelation;
- the multivariate extension can be calibrated in a parsimonious way using equity correlation.

The literature on credit risk models with pure jump asset values is still in its infancy. As concerns single defaults, Madan (2000) introduces a terminal default, Merton type model, with a Lévy process of the VG type for the log-asset value, while Cariboni and Schoutens (2004) provide its early-default version, together with an illustrative calibration to a small CDS sample. This paper intervenes in the single firm structural model discussion by providing a large scale calibration of the terminal default, Merton type model of Madan (2000). We examine the components of both the DJ.CDX.NA.HY and the CDX.NA.IG indices. We work on a sample of about 18700 single firm credit default swap (CDS) spreads, using firm specific market and accounting data for the leverage ratio and payout rate.

As concerns default correlation too the pure jump literature is quite thin. Schoutens (2006), as well as Moosbrucher (2006) and Baxter (2006), use a Gaussian copula, together with Lévy type margins, to infer from collateralized debt obligations (CDO) the implicit default correlation. By so doing, they impose equal pairwise correlation among all names in the basket. We depart from their approach in that we do not impose a copula on given VG margins. On the contrary, we move from a truly multivariate VG model, in which each single asset value is driven by a common and an idiosyncratic time change. This is made possible by the use of a novel version of the multivariate VG model, introduced in Semeraro (2006). We calibrate the pairwise default dependence of a sub-sample of the CDX pool to equity correlation, exactly as is done in traditional Merton type asset models. In such a way, we do not need the equicorrelation assumption and the large homogeneous portfolio hypothesis of the Gaussian copula.

The paper is structured as follows: section 1 recalls some basic properties of univariate pure jump processes of the VG type and introduces their multivariate version with common and idiosyncratic risk. Section 2 follows Madan (2000) in setting up a Merton type structural default model, i.e. a model with terminal default only, when the underlying asset follows a VG process. It then extends the structural model to multiple defaults. Section 3 computes the theoretical CDS spreads, which will be needed for the calibration. Section 4 presents the data and the single default probability calibration approach; section 5 comments on the results of the calibration. Section 6 calibrates default dependence and provides an example of its use, by pricing a First to Default (FtD) on two names in the pool. In section 7 the conclusions follow.

1 The VG asset model

We consider a structural model in which the (logarithm of the forward) firm asset value V_t, appropriately normalized so as to match the risk-neutral expectation property, follows a Lévy process of the Variance Gamma (VG) type. A symmetric version of this process has been introduced in the financial literature by Madan and Seneta (1990).

The asymmetric extension is due to Madan, Carr, Chang (1998) and Madan and Milne (1991), further generalized by Carr, Madan, Geman, Yor (2002).

We select the VG process since it is a very simple version of time changed Brownian motion, depending on three parameters, which induce asymmetry and kurtosis. Our multivariate version introduces an additional parameter for asset dependence.

1.1 Univariate version

A VG process is a real Lévy process $Y = \{Y(t), t \geq 0\}$ obtained as a Brownian motion with drift time-changed by a subordinator which is a gamma process.

A gamma process $\{G(t), t \geq 0\}$ with parameters (a, b) is a Lévy process so that the defining distribution of $Y(1)$ is gamma with parameters (a, b) (shortly $\mathcal{L}(Y(1)) = \Gamma(a, b)$). The parameters a and b are restricted to be positive.

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, $\{G(t), t \geq 0)$ be a gamma process with parameters $(\frac{1}{\alpha}, \frac{1}{\alpha})$, $\alpha > 0$, and let $\sigma > 0$, θ be real parameters; then the process $Y(t; \sigma, \alpha, \theta)$ -or simply $Y(t)$- is defined as

$$Y(t) = \theta G(t) + \sigma B(G(t)).$$

The characteristic function of the VG returns at time t is

$$\begin{aligned} E[\exp(\mathrm{i}uY(t))] &= \phi_{VG}(u; \sigma\sqrt{t}, \alpha/t, t\theta) \\ &= (\phi_{VG}(u; \sigma, \alpha, \theta))^t \\ &= (1 - \mathrm{i}u\theta\alpha + \sigma^2\alpha u^2/2)^{-t/\alpha}. \end{aligned}$$

A VG-process has infinitely many jumps in any finite time interval, no Brownian component and the following moments at time one:

mean	θ
variance	$\sigma^2 + \alpha\theta^2$
skewness	$\theta\alpha(3\sigma^2 + 2\alpha\theta^2)/(\sigma^2 + \alpha\theta^2)^{3/2}$
kurtosis	$3(1 + 2\alpha - \alpha\sigma^4(\sigma^2 + \alpha\theta^2)^{-2})$

The parameter θ is the instantaneous mean: negative values of θ give rise to negative skewness, so that θ is interpreted as a skewness indicator too. The other parameters, σ and α, control primarily the variance and kurtosis, as is evident from the case $\theta = 0$.

$Y(t; \sigma, \alpha, \theta)$ is assumed to represent asset returns, in excess of the risk-neutralizing component.

The firm asset value at time t, under the risk-neutral measure, is then

$$V_t = V_0 \exp\left[(r - q + w)\, t + Y(t; \sigma, \alpha, \theta)\right]$$

where r is the constant riskless rate, q is the dividend rate, w makes the risk neutral returns on V equal to $r - q$:

$$w = \alpha^{-1}\ln\left(1 - \theta\alpha - \frac{1}{2}\sigma^2\alpha\right).$$

The definition make sense provided that

$$\frac{1}{\alpha} \quad > \quad \theta + \frac{1}{2}\sigma^2 \tag{1.1}$$

or, equivalently, that

$$-\alpha(\theta + \frac{1}{2}\sigma^2) \quad := \quad k > 0. \tag{1.2}$$

The VG process has been extensively tested in the equity return domain. It has been shown to successfully describe stock indices behavior, since "it corrects strike and maturity biases in Black Scholes pricing (Madan, Carr, Chang, from now on MCC (1998))". Its estimates via options on stocks and stock indices, such as the S&P500, "show that the hypotheses of zero skewness and zero kurtosis can both be rejected (ibidem)".

1.2 Multivariate version

A first multivariate version of the previous model, in which a single time change applies to all the components of the process, is due to Madan and Seneta (1990). This version has two main drawbacks:

- independence cannot be captured;
- linear correlation cannot be fitted, since it is given, once the marginal parameters are fixed.

The multivariate extension studied here is due to Semeraro (2006) and further studied in Luciano and Semeraro (2007).

It is based on a multivariate time change, whose single components are the sum of an idiosyncratic and a common component.

Indeed, we define n subordinators as follows: let $a, \alpha_j, j = 1, ..., n$ be positive real parameters which satisfy

$$0 < a < \tfrac{1}{\alpha_j} \quad j = 1, ..., n. \tag{1.3}$$

Let X_j, $j = 1, ..., n$ and Z be independent gamma random variables with parameters respectively $(\frac{1}{\alpha_j} - a, \frac{1}{\alpha_j})$, $j = 1, ..., n$ and $(a, 1)$. Define the random vector $\boldsymbol{W}$ as the sum

$$\boldsymbol{W} = (W_1, W_2, ..., W_n)' = (X_1 + \alpha_1 Z, X_2 + \alpha_2 Z, ..., X_n + \alpha_n Z)', \tag{1.4}$$

where α_j, $j = 1, ..., n$ are real parameters.

Define $\boldsymbol{G} = \{\boldsymbol{G}(t), t \geq 0\}$ as the Lévy process which has the law $\mathcal{L}$ of $\boldsymbol{W}$ at time one:

$$\mathcal{L}(\boldsymbol{G}(1)) = \mathcal{L}(\boldsymbol{W}). \tag{1.5}$$

The process $\boldsymbol{G}$ is a multivariate subordinator with gamma margins given by

$$\mathcal{L}(G_j(t)) = \Gamma(t\frac{1}{\alpha_j}, \frac{1}{\alpha_j}), \;\; j = 1, ...n.$$

The multivariate version of the VG process is the subordination of a multivariate Brownian motion with independent components by the subordinator $\boldsymbol{G}$. Formally, let $B_j = \{B_j(t), t \geq 0\}$ $j = 1, ..., n$ be independent standard Brownian motions.

The $\mathbb{R}^n$ valued log price process $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t > 0\}$ is defined -analogously to the multivariate case- as:

$$\boldsymbol{Y}(t) = \begin{pmatrix} Y_1(t) \\ ... \\ Y_n(t) \end{pmatrix} = \begin{pmatrix} \theta_1 G_1(t) + \sigma_1 B_1(G_1(t)) \\ .... \\ \theta_n G_n(t) + \sigma_n B_n(G_n(t)) \end{pmatrix}, \tag{1.6}$$

where $\boldsymbol{G}$ is a multivariate subordinator defined by (1.5), independent from $\boldsymbol{B}$. The process $\boldsymbol{Y}$, as given by (1.6), is a Lévy process named $\boldsymbol{\alpha}$-VG, with characteristic function

$$\psi_{\boldsymbol{Y}(t)}(\boldsymbol{u}) = \prod_{j=1}^{n}(1 - \alpha_j(i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-t(\frac{1}{\alpha_j} - a)}(1 - \sum_{j=1}^{n} \alpha_n(i\theta_j u_j - \frac{1}{2}\sigma_j^2 u_j^2))^{-ta}. \tag{1.7}$$

The margins Y_j, $j = 1, ..., n$, of $\boldsymbol{Y}$ are VG processes with parameters $\theta_j, \sigma_j, \alpha_j$, so that each firm asset behavior is modelled by a VG process.

The $\boldsymbol{\alpha}$-VG process depends on the three marginal parameters $\theta_j, \sigma_j, \alpha_j$ and on an additional parameter a that will allow us to fit linear correlation.

The linear correlation coefficients between Y_j and Y_l, $l = 1, ...n$, $j = 1, ...n$ are time independent and equal to

$$\rho_{l,j} = \frac{\theta_l \theta_j \alpha_l \alpha_j a}{\sqrt{(\sigma_l^2 + \theta_l^2 \alpha_l)(\sigma_j^2 + \theta_j^2 \alpha_j)}}, \tag{1.8}$$

We can immediately observe that the correlation matrix $\boldsymbol{\rho} = [\rho_{l,j}]$, once the marginal parameters are fixed, is a function of a.

Under this model the $j-th$ firm asset value at time t, under the risk-neutral measure, is

$$V_j(t) = V_j(0) exp((r - q_j + \omega_j)t + Y_j(t)),$$

where for each $j = 1, ...n$, w_j is chosen as in the univariate model. Therefore

$$\omega_j = \alpha_j^{-1} \ln(1 - \alpha_j \theta_j - \frac{1}{2}\sigma_j^2 \alpha_j)), \tag{1.9}$$

and the parameters have to verify the constraints discussed in the univariate case. We stress that the linear correlation matrix of returns under the risk neutral measure remains the same that under the historical one.

2 The default triggering model

In the credit risk structural literature, the VG assumption has already been adopted by Madan (2000) and Cariboni and Schoutens (2004): the former built a model with terminal default only, the latter introduced the possibility of early default. Both have shown that the assumed dynamics allows for positive credit spreads over the short run, thus correcting the major theoretical drawback of diffusive structural models.

This section first reviews the structural model proposed by Madan (2000) and computes the corresponding debt value, recovery rate and equity value, as needed for calibration. Then it introduces multiple defaults.

2.1 Single name defaults

We start by assuming, as in the Merton's original approach, that the firm has a unique, zero-coupon debt issue with facial value F, maturity T. If default occurs, i.e. if $V(T)$ is smaller than F, a strict priority rule is assumed to apply: debt holders receive the asset value $V(T)$, while shareholders are deprived of any claim. If default does not occur, they maintain the right to F and $V(T) - F$ respectively. Therefore, bond holders have a claim of F and are short a European put on the firm value, with final payoff

$$\max(F - V(T), 0)$$

We can use well known results from option pricing under the VG assumption (see MCC(1998)) in order to compute

- the (risk-neutral) default probability, $\pi(T)$;
- the debt value, D_0;
- the corresponding recovery rate, R;
- the equity value, E_0.

As for the default probability at time T, $\pi(T)$, it coincides with the probability that $V(T) < F$, and it is the exercise probability of the above put option.

Taking the current date to be 0, and having defined the firm's quasi-leverage ratio as d :

$$d := \frac{F \exp(-(r-q)T)}{V_0} \tag{2.1}$$

it can be computed as

$$\pi(T) = 1 - \Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right) \tag{2.2}$$

where

$$k(d) := \frac{1}{s}\left[\ln\left(\frac{1}{d}\right) + \frac{T}{\alpha}\ln\left(\frac{1-c_1}{1-c_2}\right)\right]$$

$$c_1 := \frac{\alpha\left(\beta+s\right)^2}{2}$$

$$c_2 := \frac{\alpha\beta^2}{2}$$

$$s := \frac{\sigma}{\sqrt{1+\left(\frac{\theta}{\sigma}\right)^2\frac{\alpha}{2}}}$$

$$\beta := -\theta/\sigma^2$$

and the function Ψ can be obtained from the Hypergeometric function of two variables and the Bessel function of the second type[1], as in MCC (1998).

As for the current value of debt, D_0, it can then be obtained as the difference between the present value of F, computed at the riskless rate r, and the current value of the put option on $V(T)$ with strike F. Denote with $VGP\left(V_0, F, r, \sigma, \alpha, \theta\right)$ the VG European put price, with current value of the underlying V_0, strike F, riskless rate r. The debt value D_0 is then

$$D_0 = F\exp(-rT) - VGP\left(V_0, F, r, \sigma, \alpha, \theta\right) \tag{2.3}$$

Following MCC, its put component turns out to be:

$$VGP\left(V_0, F, r, \sigma, \alpha, \theta\right) =$$

$$\begin{aligned} & V_0\exp(-qT)\left[\Psi\left(k(d)\sqrt{\frac{1-c_1}{\alpha}}, (\beta+s)\sqrt{\frac{\alpha}{1-c_1}}, \frac{T}{\alpha}\right)-1\right]+ \\ & -F\exp(-rT)\left[\Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right)-1\right] \end{aligned} \tag{2.4}$$

and can be written in terms of the quasi-leverage ratio:

$$\frac{VGP\left(V_0, F, r, \sigma, \alpha, \theta\right)}{F\exp(-rT)} =$$

$$= VGP\left(\frac{1}{d}, 1, 0, \sigma, \alpha, \theta\right) = \frac{1}{d}VGP\left(1, d, 0, \sigma, \alpha, \theta\right) \tag{2.5}$$

[1]The solution (2.2) for the exercise probability is closed in the sense of being obtained by integration of elementary functions. It allows to perform some comparative statics, but has the main drawback of being computationally expensive, as MCC (1998) recognize. In the calibration of the model indeed the option price is found via partial integro differential equations (PIDEs).

It follows, as in Madan (2000), that debt equals

$$\begin{aligned} D_0 &= F\exp(-rT) - F\exp(-rT)VGP\left(\frac{1}{d}, 1, 0, \sigma, \alpha, \theta\right) = \\ &= F\exp(-rT) - V_0 VGP\left(1, d, 0, \sigma, \alpha, \theta\right) \end{aligned} \tag{2.6}$$

Combining (2.3) and (2.4) and simplifying according to (2.6) the debt value can be finally obtained in closed form as

$$\begin{aligned} D_0 &= F\exp(-rT) - V_0 \times \\ &\times \left\{ \begin{array}{l} \left[\Psi\left(k(d)\sqrt{\frac{1-c_1}{\alpha}}, (\beta+s)\sqrt{\frac{\alpha}{1-c_1}}, \frac{T}{\alpha}\right) - 1\right] \\ -d\left[\Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right) - 1\right] \end{array} \right\} \end{aligned}$$

As concerns the recovery rate R, i.e. the proportion of the face value which is recovered in case of default, it is endogenous, as in most structural models. It can be found by equating D_0 to the present value of its final expected payoff:

$$D_0 = [\pi(T)RF + (1-\pi(T))F]\exp(-rT)$$

Substituting for D_0 from (2.3), we get the recovery rate as

$$R = 1 - \frac{VGP\left(V_0, F, r, \sigma, \alpha, \theta\right)}{F\pi(T)\exp(-rT)} \tag{2.7}$$

Based on (2.5) and on the default probability assessment (2.2), the recovery rate can be explicitly quantified:

$$R = \frac{1}{d}\frac{\Psi\left(k(d)\sqrt{\frac{1-c_1}{\alpha}}, (\beta+s)\sqrt{\frac{\alpha}{1-c_1}}, \frac{T}{\alpha}\right) - 1}{\Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right) - 1} \tag{2.8}$$

As for equity, which is a call on V with strike F, from the option pricing results in MCC (1998) we have:

$$\begin{aligned} E_0 &= V_0\exp(-qT)\Psi\left(k(d)\sqrt{\frac{1-c_1}{\alpha}}, (\beta+s)\sqrt{\frac{\alpha}{1-c_1}}, \frac{T}{\alpha}\right) + \\ &\quad -F\exp(-rT)\Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right) \end{aligned} \tag{2.9}$$

The model then provides us with explicit formulation for all the relevant quantities: default probability, debt value, recovery rate and equity price.

2.2 Multiple defaults

The straightforward extension of the structural model studied so far to multiple defaults consists in considering that the n names will default at time T if and only if all of their asset values happen to be below the corresponding threshold at that time: the (risk neutral) joint default probability, π, is then

$$\bar{\pi}(T) = \Pr(V_1(T) \leq F_1, V_2(T) \leq F_2, ..., V_n(T) \leq F_n).$$

Similarly, all the mixed survival and default probabilities could be obtained from the joint distribution of asset values. Given the theoretical joint model of section 1.2 above, these probabilities are uniquely determined once the parameter of the common time change, a, is given. They cannot be computed in closed form, but can be obtained by Monte Carlo simulation or by numerical integration of their conditional values. Consider for simplicity the bivariate case and $T = 1$. Denote with S_i the risk neutral log returns on V_i, $S_i(t) = \ln(V_i(t)/V_{i0}) = m_i t + Y_i(t)$, where $m_i = r - q_i + \omega_i$. It follows that

$$\begin{aligned} \bar{\pi}(1) &= Pr(V_1(1) \leq F_1, V_2(1) \leq F_2) \\ &= P(S_1(1) \leq k_1, S_2(1) \leq k_2) = P(Y_1(1) \leq k_1 - m_1, Y_2(1) \leq k_2 - m_2), \end{aligned} \tag{2.10}$$

where $k_i = \ln(F_i/V_{i0})$. The random variable

$$\frac{S_i(1) - m_i - \theta_i(w_i + \alpha_i z_i)}{\sigma_i\sqrt{w_i + \alpha_i z_i}} = \frac{Y_i(1) - \theta_i(w_i + \alpha_i z_i)}{\sigma_i\sqrt{w_i + \alpha_i z_i}}, \ i = 1, 2$$

has conditional distribution, given both the idiosyncratic and the common time changes, that is unit normal. We can therefore compute the joint distribution function of $\boldsymbol{Y}(1)$ (see the Appendix for the derivation of the following equation)

$$\begin{aligned} F(x_1, x_2) &= P[Y_1(1) \leq x_1, Y_2(1) \leq x_2] \\ &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \Phi(\frac{x_1 - \theta_1(w_1 + \alpha_1 z)}{\sigma_1\sqrt{w_1 + \alpha_1 z}})\Phi(\frac{x_2 - \theta_2(w_2 + \alpha_2 z)}{\sigma_2\sqrt{w_2 + \alpha_2 z}}) \\ &\quad \cdot \frac{\frac{1}{\alpha_1}^{(\frac{1}{\alpha_1} - a)} e^{-\frac{1}{\alpha_1}(w_1)} w_1^{\frac{1}{\alpha_1} - a - 1}}{\Gamma(\frac{1}{\alpha_1} - a)} \frac{\frac{1}{\alpha_2}^{(\frac{1}{\alpha_2} - a)} e^{-\frac{1}{\alpha_2}(w_2)} w_2^{\frac{1}{\alpha_2} - a - 1}}{\Gamma(\frac{1}{\alpha_2} - a)} \frac{e^{-z} z^{a-1}}{\Gamma(a)} dw_1 dw_2 dz. \end{aligned} \tag{2.11}$$

From (2.2) it follows that the probability $\bar{\pi}(1)$ is

$$\bar{\pi}(1) = F(k_1 - m_1, k_2 - m_2). \tag{2.12}$$

As a subcase, i.e. when one of the arguments diverges, we can also obtain from the previous formula the marginal default probabilities:

$$\pi_1(1) = F(k_1 - m_1, +\infty) \tag{2.13}$$

$$\pi_2(1) = F(+\infty, k_2 - m_2). \tag{2.14}$$

which coincide with the closed form expression (2.2).

3 CDS spreads

This section studies the CDS spreads corresponding to the structural model just established. As it is known, a CDS with reference asset V is an OTC contract between two parties, the credit risk seller and buyer, by which the former pays a periodic fee against reimbursement by the latter of the loss given default on the underlying credit, or reference asset. The seller and buyer's streams of payments are called fee and default leg respectively.

Let us consider a CDS with maturity T and fee payments vF, proportional to the face debt value according to the constant v. Let the payment occur at the beginning of each time period $[t_{i-1}, t_i], i \geq 0$. For simplicity, let us assume annual fees ($t_i = i$). Let r_i be the yield to maturity i, and notice that the yield to the option maturity coincides with the rate r of section 2 ($r_T = r$). The fee leg value is the present value of the corresponding cash-flows, computed at the riskless rate:

$$\sum_{i=0}^{T-1} vF \exp(-ir_i)$$

Denoting the present value of an annuity as $\ddot{a}_T := \sum_{i=0}^{T-1} \exp(-ir_i)$, the fee leg value can be written as

$$vF\ddot{a}_T$$

As for the default leg, it consists of the loss given default, i.e. the difference between the facial and the recovery value of the reference asset, F and R respectively. In order to simplify the calibration procedure, let us assume that reimbursement takes place at maturity of the contract only, even if default occurred before, and that this maturity coincides with the debt maturity. The time-0 value of the loss given default, the so-called discounted expected loss, is therefore obtained using the risk neutral default probability at $T, \pi(T)$. In the option interpretation of structural models, if the CDS maturity is the same as the debt maturity, as required above, the discounted expected loss, and then the default leg, coincides with the put value, $VGP(V_0, F, r, \sigma, \alpha, \theta)$

By definition, the no-arbitrage CDS fee is the one which equates the two legs:

$$v = \frac{VGP(V_0, F, r, \sigma, \alpha, \theta)}{F\ddot{a}_T} \tag{3.1}$$

This is the spread for which we are going to collect data, and that will allow us, together with additional balance sheet and market data, to calibrate the VG parameters.

When the put property (2.5) applies, i.e. when $r_T = r$, the spread can be simplified into

$$v = \frac{\exp(-rT) VGP\left(\frac{1}{d},1,0,\sigma,\alpha,\theta\right)}{\ddot{a}_T} = \frac{\exp(-rT)}{\ddot{a}_T} \times$$
$$\times \left[1 - \Psi\left(k(d)\sqrt{\frac{1-c_2}{\alpha}},\ \beta\sqrt{\frac{\alpha}{1-c_2}}, \frac{T}{\alpha}\right) + \right.$$
$$\left. + \frac{\Psi\left(k(d)\sqrt{\frac{1-c_1}{\alpha}}, (\beta+s)\sqrt{\frac{\alpha}{1-c_1}}, \frac{T}{\alpha}\right) - 1}{d}\right] \tag{3.2}$$

For given interest rates, maturity, leverage ratio, it is then a function of the asset parameters α, θ, σ.

4 Data choice and marginal calibration

One of the major difficulties in the calibration of structural models is the fact that most corporate debt is not traded, and therefore, even for public firms, the asset value cannot be obtained equating it to the liabilities one, namely the sum of the current debt and equity values. As a response, traditional firm-specific calibrations of the Merton structural approach move from the relationship between the equity and asset value process on the one hand, and their volatilities on the other, to obtain the unobservable current value and volatility of the firm assets from the (observable) equity ones, for given debt facial value and maturity (see Crosbie and Bohn (2002)). This requires solving a non linear system of equations, in order to price the put in Merton model. Only after having solved the system they are able to compute in closed form the market value of debt, the default probability and credit spread. More recent firm-specific calibrations of structural models, such as Eom, Helwege, Huang (2004), from now on EHH, cope with the fact that most corporate debt is not traded, by assuming that its market value can be proxied by its book value. In turn, this assumption rests on the observation that most of the traded corporate debt is close to par. We will use this assumption too.

As for the other unobservable parameter, the instantaneous asset volatility, EHH (2004) proposes either to adjust the historical equity volatility for leverage or to use the bond implied volatility. In the first case, of the two relationships traditionally employed, only the relationship between the equity and asset value standard deviation is used. In particular, the knowledge of the derivative of the asset value with respect to the equity one is needed. As for the bond implied volatility, it is the one which matches previously observed bond prices with the theoretical values, in the same spirit of Black-Scholes implied volatility.

All the calibrations just mentioned use a diffusion model: with respect to them, we start from a much more flexible theoretical model, with asymmetry and kurtosis. This means also that we have two more parameters in addition to the volatility, respectively θ and α.

We decided to use an implied asset volatility, as well as implied asymmetry and kurtosis. These implied values will be obtained from CDS spreads instead of over Treasury spreads (or prices). We chose the former spreads instead of the latter for a number of reasons: CDSs are not subject to squeezes, are not in fixed supply, and have been shown to incorporate less liquidity premium than spreads over Treasury, independently of the definition of the riskless curve (see f.i. Longstaff, Mithal, Neis, 2004). Therefore, they seem to better isolate the credit risk of the reference asset. The choice of the data was as follows.

CDS spreads

We tried to collect a wide amount of observed spread data, $s|_{ob}$, in terms of representation of the universe of the US companies: to this end, we decided to consider the components of the Dow Jones investment grade cdx index, CDX.NA.IG.3 and the high yield index, CDX.NA.HY.3.

The first index, with its 125 names, is representative of the most liquid, investment grade names in the US. The second, with its 100 names, represents high yield names in the same market index. The ratings of the former, at the time of our data collection, were between AAA and BBB, with a particular concentration on BBB, which represented more than half of the index, immediately followed by A, which amounted to 38% of it. In the investment grade index 15 sectors were represented; the ones heavily represented - with a share of 10% or more - were basic industries, capital goods, consumer goods. The ratings of the high yield group instead were between split BBB and unrated, with more than 40% of the names in BB and more than 30% in B. As for sector, the high yield index covers 24 sectors: six sectors weight more than 7% (chemicals, energy, forest products, gaming and leisure, IT and utilities), while only IT is over 10%.

We considered the daily spreads along the observation period 9/21/04 –11/19/04: the initial date is indeed the one in which the investment grade index started to trade, while the high yield one had been introduced in July.

We looked at both the five and ten year maturity CDS, in order to have information on the term structure of the parameters. However, we observe that five year contracts are usually more liquid.

We had a total of approximately 18700 spreads referring to 224 names (for lack of data on one of them, Burlington Northern Santa Fe Corporation). As a total, 95% of the spreads were available, with at most 88 spreads for each name: with no missing data, we would have had 19800 of them. In particular, 93% of the ten year data and 97% of the five year ones were available.

Table 1 presents the CDS data statistics.

Table 1

Distribution of Average Spreads		
Whole Sample		
percentiles	5 year spreads	10 year spreads
1%	17.386	27.432
5%	23.273	37.227
10%	28.273	42.102
25%	39.205	56.227
50%	87.045	101.955
75%	240.284	233.477
90%	386.545	386.034
95%	556.293	625.205
99%	1986.250	1880.614
mean	.019	.020
st.deviation	.0310	.029
skewness	4.560	4.249
kurtosis	28.234	24.608

Seniority
The CDS of the two indices we are examining refer to senior debt. For each CDS but one we determined at least a corresponding deliverable bond: the missing entry was an investment grade name, MBIA insurance, which had to be eliminated from the sample, thus reducing it to 223 names. For all the five year spreads and all but three issuers among the ten year spread ones we had also a name-specific spread of the appropriate seniority. All these data were for unsecured bonds. Among the ten year ones, Celestica, Iron Mountain and Triton Pcs had only junior subordinated spreads available.

In the database, 54% of the spreads assume restructuring, the balance being non restructuring. This is a result of the fact that CDX.NA.IG.3 assumes no restructuring, in spite of the fact that generally IG names are modified restructuring. HY names instead generally trade with no restructuring.

Riskless rate
In order to extract from the CDS premium the implied put price, we considered as riskless rate $r_i, i = 1, ..10$, the LIBOR for the one year maturity and the US swap one for the two to ten year maturities. The riskless rate choice is, as well known, a crucial one, and most of the recent literature converges on suggesting the adoption of the swap curve instead of the Treasury one, because of the different liquidity between corporate and Treasuries. However, let us note that swap rates already include a counterparty risk premium, which is not included in the Government ones. We do not report here the riskless rates, which were taken from the Bloomberg database and updated daily both over the in sample and over the out of sample period.

Recovery rate
We selected the observed recovery rate, $R\ |_{ob}$, for each bond in the pool, adapting Macgilchrist (2004). Basically, we took into consideration for the recovery assignment the sector and the seniority (senior unsecured or junior subordinated) of the debt issue.

As for the sectors, they were defined based on the level 1 industry sector description provided by Bloomberg (API field "INDUSTRY_SECTOR"). This distinguishes the following ten sectors: Basic Materials, Communications, Consumer Cyclical, Consumer Non-cyclical, Diversified, Energy, Financial, Industrial, Technology, Utilities. Since the original data of Macgilchrist do not follow exactly the same classification, we grouped more detailed data when necessary.

As for seniority, the sector data of Macgilchrist referred to senior unsecured debt. As mentioned above, all our ten year and five year deliverable bonds were senior unsecured, with the exception of the ten year bonds for Celestica, Iron Mountain and Triton Pcs, which were junior subordinated. In order to reconstruct the recovery for the latter issues, we used the CMA data for recoveries by seniority, which aggregates all the sectors. We determined the relative ratios of recoveries for different seniorities and applied this ratios to the recovery rate found for each sector senior unsecured debt.

Table 2 presents the recovery data statistics for senior unsecured debt, which represents most of our sample:

Table 2

Recovery rates		
senior unsecured debt		
Sector	average recovery	standard deviation
basic materials	.6	.27
communication	.3	.2
consumer cyclical	.33	.24
consumer non-cyclical	.42	.21
diversified	.3	.25
energy	.4	.29
financial	.3	.29
industrial	.375	.27
technology	.3	.29
utilities	.42	.28

Leverage ratio

We took as an estimate of the market valued-leverage ratio $\frac{D_0}{V_0}\mid_{ob}$, the book ratio, $\frac{F}{F+E_0}$, as is done by most recent structural model calibrations. Instead of using median debt ratios, we collected appropriate firm specific data from StockVal. We define the debt ratio D_0/V_0 as

$$\frac{\text{(short term liabilities - account payable)} + \text{long term liabilities}}{\text{(short term liabilities - account payable)} + \text{long term liabilities} + \text{market cap}}$$

Since for some of the names in the pool the leverage ratio was not available, we dropped them from the sample: as a result, the number of observations reduced to 11400 approximately , of which 5900 referred to the five year horizon, the rest to the ten year. As for the number of names, depending on the observation date, we had from 133 to 136 firms at the five year level, from 122 to 129 at the ten year one.

Table 3 presents the summary statistics of the restricted sample : comparing it with table 1 above the reader can appreciate the fact that, in spite of reducing the number of data, we still have a sample representative of the two initial CDS indices, and therefore of the Dow Jones groups. The percentiles of the whole and restricted sample, as well as the other summary statistics, are indeed very close:

Table 3

Distribution of Average Spreads		
Restricted Sample		
percentiles	5 year spreads	10 year spreads
1%	17.023	30.773
5%	22.227	37.477
10%	26.318	41.568
25%	38.136	56.360
50%	83.159	119.705
75%	187.432	219.706
90%	349.682	348.273
95%	556.293	531.114
99%	2173.295	2052.386
mean	186.351	202.075
st.deviation	.034	.0314
skewness	4.836	4.531
kurtosis	29.016	25.707

Payout rate

The first step to determine the payout q was to match the cds names in the sample with corresponding tickers for which we could automatically get the average coupons paid. We in fact had only the CUSIP of a deliverable bond, but not the corresponding ticker. The match CUSIP - ticker was done taking into consideration that, even when the debt is issued by a subsidiary, the holding company is going to pay the dividends and the stock trading on the market is the holding company. Whenever we had to choose therefore, we selected the ticker of the holding company trading on the market.

Once the ticker assignment was complete, we took coupon rates from Bloomberg, using the debt distribution weighted average coupon of the individual securities of the ticker group. In the cases where this was not available, we used the debt distribution weighted average coupon of the individual securities of the issuer and its subsidiaries

As for dividend yields, we chose the sum of the gross dividends per share that had gone ex-dividend over the CDS observation period and approximatively the following six months, divided by the stock price. By so doing, we produced a proxy for the expected dividend, since we incorporated some (correct) new information, compared with the spreads. The data provider was again Bloomberg.

We then used the following formula to compute the payout rate for each name:

$$q = c \times \frac{D_0}{V_0}\mid_{ob} + m \times \left(1 - \frac{D_0}{V_0}\mid_{ob}\right)$$

where c is the average coupon, m is the dividend yield.

Table 4 shows statistics for the coupon, dividend and payout rate. From this we can see that for the names for which we have the debt ratio, the median average coupon is 6.7%, the median dividend yield is 1% and the median average payout is about 4%. The corresponding average values are close to the median ones. For the payout ratio, in particular, there is no particular evidence of skewness or kurtosis.

Table 4

Payout rate			
percentiles	average coupon	dividend yield, 12 months	payout rate
1%	428	0	72.093
5%	475	0	145.956
10%	539	0	195.401
25%	608	0	302.688
50%	673	.010	399.082
75%	761	.023	530.905
90%	811	.036	616.059
95%	870	.044	678.676
99%	976	.052	791.015
mean	.067	.013	.0409
largest st.deviation	.011	.014	.016
skewness	-.088	.904	.119
kurtosis	3.706	2.966	2.553

Calibration method

We divided the CDS data available in two time series of approximately equal size and we used the first half to calibrate the model, namely to select the parameters σ, α, θ, and the second half to discuss their reliability, by a sort of "out of sample test" of the results. The "in sample" choice for the parameters, which means having an implied vol, kurtosis and skewness, was done by quadratic error minimization, with an accurate study of multiple solutions. In the "out of sample" test, we considered the parameters obtained from the in sample calibration and we compared the corresponding CDS spreads with the actual ones. We did this in order to stress the robustness of our in sample parameter choice[2].

[2] We are aware of the fact that the number of data points is low with respect to a traditional in sample and out of sample study; however, the sense of the exercise is not that of a traditional in sample-out of sample one: it is more a robustness check in a truly cross sectional estimate, even tough we will maintain the vivid terminology "in and out of sample".

We did the calibration or in sample choice separately for the five and ten years spreads ($T = 5, 10$). For each maturity T we solved the following optimization problem:

$$\min_{\alpha,\sigma,\theta} \sum_{k=1}^{N} \left(s|_{ob}(k) - s|_{th}(k) \right)^2 \tag{4.1}$$

where N is the number of days for which we have in sample spreads, $s|_{ob}(k)$ and $s|_{th}(k)$ are respectively the observed and theoretical spread in date k.

The theoretical spread in turn is

$$s|_{th}(k) = \frac{VGP\left(V_0, F, r(k), \sigma, \alpha, \theta\right)}{\ddot{a}_T(k)}$$

where, with respect to the formula (3.2) given above, we have now signalled that both the riskless rate and the annuity values are updated daily, and therefore time (k) dependent. This makes the theoretical spread change over time too.

After having solved the minimization problem in (4.1), for each name we computed a number of out of sample pricing errors:

- the overall pricing error (OPE), defined as

  $$OPE = \sum_{k=1}^{M} \left(s|_{ob}(k) - s^*|_{th}(k) \right)^2$$

  where $s^*|_{th}(k)$ is the spread obtained using the optimal parameter values and M is the number of days for which we have out of sample spreads. Indeed, we had approximately 22 observations for the in sample piece, and an equal number for the out of sample check;

- the square root of the OPE ratio with respect to the number of observations, the so-called root mean square error or average daily error (ADE):

  $$ADE = \sqrt{\frac{\sum_{k=1}^{M} \left(s|_{ob}(k) - s^*|_{th}(k) \right)^2}{M}}$$

- the average of the percentage pricing error ($\%PE$), defined as

  $$\%PE = \frac{1}{M} \sum_{k=1}^{M} \frac{s^*|_{th}(k) - s|_{ob}(k)}{s|_{ob}(k)}$$

- and, last but not least, the average of the percentage error in absolute value ($\%APE$), namely

  $$\%APE = \frac{1}{M} \sum_{k=1}^{N} \frac{\mid s|_{ob}(k) - s^*|_{th}(k) \mid}{s|_{ob}(k)}$$

Based on the previous literature on stock pricing, we used the following constraints on the value of the variables: $0.003 < \sigma < 4.0,\ 0.05 < \alpha < 4.0,\ -4.0 < \theta < 4.0$.

5 Empirical Results for single name spreads

To start with, tables 5 to 7 below report the statistics of the calibration results, in terms of parameters for the asset value process, namely σ, for the volatility, α, for the kurtosis, θ, for the asymmetry. The parameters were obtained from the minimization procedure explained above, under the appropriate constraints, (4).

Table 5

Merton model calibrated parameters: σ		
	5 year horizon	10 year horizon
number of names excluded	3	3
percentiles		
1%	.037	.093
5%	.066	.118
10%	.077	.145
25%	.149	.224
50%	.224	.300
75%	.312	.310
90%	.384	.531
95%	.440	.686
99%	2.312	3.225
mean	.280	.412
st.deviation	.387	.533

Table 6

Merton model calibrated parameters: α		
	5 year horizon	10 year horizon
number of names excluded	3	3
percentiles		
1%	.057	.062
5%	.067	.073
10%	.901	.088
25%	.194	.112
50%	.268	.199
75%	.528	.496
90%	.649	.575
95%	.757	.652
99%	1.937	.891
mean	.389	.296
st.deviation	.398	.208

Table 7

Merton model calibrated parameters: θ		
	5 year horizon	10 year horizon
number of names excluded	3	3
percentiles		
1%	-.926	-.725
5%	-.673	-.547
10%	-.519	-.467
25%	-.398	-.309
50%	-.220	-.143
75%	-.100	.039
90%	.115	.394
95%	.375	.484
99%	.509	.545
mean	-.223	-.111
st.deviation	.290	.298

The reader must take into consideration that the minimization procedure slightly reduced the number of names, since for some of them either it did not converge, or it generated a numerical error in the out of sample check[3]. For this reason, the top of the table shows how many names have no meaningful solution: the reader can see that the number of cases so excluded is around 2%. The tables above show the distribution of the results obtained: the five year case is on the left, the ten year one on the right. We can see that for the 5 year spreads, the median σ, α, θ across all the names are respectively 22.4%, 26.8% and -22%. The corresponding average values are 28%, 39% and -22%, with a standard deviation smaller than 40% for the first two parameters, smaller than 30% for the last one. At the 10 year level, the median σ, α, θ are 30%, 19.9% and -14.3%, with average values 41%, 29% and -11%, and standard deviations equal respectively to 53%, 21% and 30%. Both in the five and ten year case, the variance parameter, σ, as well as the kurtosis, α, and the asymmetry one, θ, are slightly higher than the ones obtained in the previous literature for equities. In MCC (1998), for instance, they were 0.12, 0.17, -0.14 respectively. Their calibration however is realized on SPX listed options having shorter maturity than the CDS contracts considered here.
Having listed the features of the VG asset process, let us proceed to analyze the pricing errors of the model. Tables 8 and 9 are devoted to the statistics of the out of sample pricing errors. The ADE is in basis point.

[3]Whenever the minimization procedure gave more than one set of solutions, we chose the most appropriate one, in the sense of giving the least pricing error - out of sample.

Table 8

Pricing errors from 5-year CDS's			
percentiles	ADE	% PE	%APE
1%	0.609	.532	.008
5%	1.793	.397	.030
10%	2.696	.282	.051
25%	4.152	.148	.071
50%	11.331	.067	.100
75%	22.716	.002	.194
90%	41.33	-.112	.295
95%	66.758	-.183	.425
99%	468.886	.358	.916
mean	.002	.075	.150
st. deviation	.007	.190	.143

Table 9

Pricing errors from 10-year CDS's			
percentiles	ADE	% PE	%APE
1%	1.288	.439	.016
5%	2.738	.316	.031
10%	3.001	.239	.037
25%	5.219	.138	.069
50%	12.578	.073	.100
75%	25.309	-.004	.169
90%	48.403	-.082	.287
95%	75.925	-.163	.365
99%	1203.136	-.612	.625
mean	.005	.066	.139
st. deviation	.018	.171	.126

First of all, let us study the ADE, which gives an estimate of the average pricing error. We remark that the median value for the ADE - both for the 5 year and 10 year spread - is very low, slightly more than 10 basis points (bp). The mean of the ADE over 5 and 10 years is respectively 26 bp and 47 bp, while the standard deviation ranges from 0.7% to 1.8% bp. (The increase in the standard deviation over the longer horizon can be explained with the smaller liquidity inherent in the ten year data).

In order to assess the fit of the model, we also present the %PE and %APE of the model. The former is negative if on average the model underpredicts the actual spreads, and positive otherwise. The APE, on the contrary, gives an estimate of the pricing error, without compensating between negative and positive errors: therefore, it does not provide information about over or underpricing, but about the magnitude of the errors, independent of their signs.

Both the ADE and percentage errors are small in comparison with standard results in structural, diffusion based models. For the latter indeed errors are usually in the two digits order of magnitude[4].

In order to fully understand the advantages of the VG model with respect to diffusions, and to point out a better fit property not due to us using CDS data instead of spreads over Treasuries, let us show that we correct not only for the underestimation error, but also for the accuracy or bias. Let us recall indeed that, according for instance to the results in EHH, not only the Merton diffusion based model underpredicts spreads, but more sophisticated models, such as Leland and Toft (1996) or Longstaff-Schwartz (1995), severely overpredict spreads of high risk bonds and still underpredict safer bonds' ones. EHH concludes that the major challenge facing structural bond pricing modelers is to raise the average predicted spread for low risk bonds (typically short term investment grade bonds) and, at the same time, decreasing the spreads on risky bonds. In order to do this, we determined not only the overall percentage of under and overpredictions, through the %PE, but we separated the cases of overpricing from the ones of underpricing, and we divided HY from IG bonds.

Let us start from the amount of over and underpricing: with our model, over five years, on average, 29.43% of the spreads are underpriced by the jump model, with a standard deviation of 35.07%. Over ten years, the percentages become respectively 30.11 and 36.56.

As concerns the possibility of decreasing the underprediction on low risk bonds, without boosting the overprediction of high risk ones, let us present the %PE and APE pricing error results not only for the whole sample, which contains both IG and HY bonds, but also for the two classes separately Table 10 below contains the main statistics

[4]On their overall sample, EHH (2004) reports a mean %PE for the Merton model equal to -50.42%, while we have 7.52% over five years, 6.58% over ten. From the %PE change of sign and reduction in absolute value we infer that, considering the whole sample, underestimation of the Merton model is not only reduced with respect to EHH, but substituted by a small overestimation: this is accompanied by a strong reduction in the standard error, from 71.84% in EHH to 19% and 17.19% - respectively for 5 and 10 CDS's spreads - in our sample.

As for the %APE, EHH has 78.02%, while we have 15.05% over five years, 13.88% over ten: we still have a very strong reduction of the error. And also in the %APE case, there is an appreciable reduction in the standard deviation, from almost 39.96% to 14.35% and 12.65% over respectively 5 and 10 years.

EHH analyzes the performance not only of the Merton model, but also of Geske (1977), Leland and Toft (1996), Lonstaff and Schwartz (1995) and Collin-Dufresne and Goldstein (2001) models. Although some of these models overperform the Merton model, none of them has better statistics than the VG model we tested here, as the lowest %APE, obtained for the Geske model with face recovery, is 65.7% with a standard deviation of 28.34%.

of tables 8 and 9, for the two classes: for completeness, we also report the ADE results. Over 5 years the average %PE on HY bonds is 7.56%, the one on IG is 7.5%. Over 10 years the mean %PE for the HY is 5.4%, the one for IG is 7.46%. Over both horizons the two classes have errors of the same sign, so that the overall slight overpricing result does not arise from the compensation of underpricing of IG and severe overpricing of HY. The HY mispricing is even smaller than the other, in the ten year case, and very close to the other over five years. The same happens for the average APE over five years, which is respectively 12% and 17% for HY and IG. Over ten years, the difference in ADE is of two percentage points only, even tough in favor of IG bonds: we have 15% and 13% for HY and IG respectively.

Table 10

Pricing errors from 5-year IG CDS's			
percentiles	ADE	% PE	%APE
1%	0.231	-1.000	.010
5%	1.460	-.172	.040
10%	2.007	-.120	.059
25%	3.054	-.044	.078
50%	5.053	.065	.106
75%	12.700	.140	.194
90%	18.868	.342	.383
95%	22.716	.457	.465
99%	37.000	.916	1.000
mean	.001	.075	.171
st. deviation	.001	.227	.172

Table 11

Pricing errors from 5-year HY CDS's			
percentiles	ADE	% PE	%APE
1%	0.609	-.235	.001
5%	5.071	-.197	.022
10%	10.450	-.060	.042
25%	13.267	.014	.062
50%	23.340	.069	.094
75%	41.032	.150	.193
90%	72.811	.236	.244
95%	146.873	.282	.282
99%	644.226	.332	.332
mean	.005	.076	.122
st. deviation	.010	.121	.082

Table 12

Pricing errors from 10-year IG CDS's			
percentiles	ADE	% PE	%APE
1%	1.030	-.330	.016
5%	1.483	-.121	.033
10%	2.880	-.082	.040
25%	4.170	-.002	.069
50%	6.152	.070	.099
75%	12.398	.134	.145
90%	20.747	.232	.247
95%	24.706	.332	.336
99%	38.775	.764	.774
mean	.001	.075	.130
st. deviation	.001	.151	.114

Table 13

Pricing errors from 10-year HY CDS's			
percentiles	ADE	% PE	%APE
1%	2.781	-.625	.007
5%	5.385	-.601	.025
10%	10.183	-.067	.034
25%	16.217	-.006	.067
50%	29.140	.078	.104
75%	47.227	.170	.179
90%	86.107	.239	.308
95%	1202.504	.308	.601
99%	1255.854	.412	.625
mean	.010	.054	.150
st. deviation	.027	.197	.141

We can state therefore that a VG asset model, at least on the sample at hand, reduces both the underprediction for IG bonds and the overprediction for HY ones. Indeed, while on average the VG model is overestimating the credit spreads (positive %PE), this overestimation is not only slight, but also unbiased.

6 Default correlation calibration

For the multivariate process introduced in section 1.2 and applied to credit in 2.2 to be helpful in risk assessment, we need a calibration procedure for dependence. Diffusion based structural models are usually calibrated using equity correlation and assuming no correlation premium. Namely, the historical equity correlation matrix is used in order to infer risk neutral dependence. We will adopt the same device here[5]. In this section we will first illustrate how the calibration can proceed and then provide a numerical illustration, on a sub-sample of the CDX group.

[5]Indeed, one can show (see section 1.2) that the change of measure adopted here guarantees coincidence of the two matrixes. Numerical explorations by other authors found different estimates for the marginal parameters under the two measures, due to different time windows of the data and therefore to different information sets. One can argue that risk neutral correlations should be obtained with the same time window for data that we use for marginal calibration. We use a bigger time window for the correlation matrix in order to get accuracy. That is the reason why, in spite of the theoretical coincidence of the correlation coefficients, we introduce an explicit assumption.

6.1 Procedure

Assume that we have calibrated the parameters of the marginal distributions of returns, namely $\sigma_j, \theta_j, \alpha_j$. In order to calibrate the multivariate VG model we have also to determine a. This parameter is calibrated so as to fit the pairwise correlation between each couple of assets.

Assume indeed that we can also provide an estimate $\mathbf{r} = (r_{ij})$ of the (risk neutral and historical) linear correlation matrix $\boldsymbol{\rho} = [\rho_{ij}]$. As we can infer by (1.8) ρ_{ij}, for given margins, is a function of the parameter a only. Therefore the whole linear correlation matrix $\boldsymbol{\rho}$, given the marginal parameters, is a function of a.

In practice we find a by minimizing the distance between the estimate $\boldsymbol{r}$ of the linear correlation matrix and the theoretical $\boldsymbol{\rho}$. We minimize the root mean square error ($rmse$) between the estimated correlation coefficients and the model coefficients, which is given by

$$rmse = \sqrt{\sum_{i<j} \frac{(r_{ij} - \rho_{ij})^2}{(n^2 - n)/2}} \tag{6.1}$$

under the constraint[6] $a \leq 1/\alpha_j, j = 1, ...n$.

We end up having fitted both the marginal distributions and (in a minimum distance sense) the dependence structure. Thanks to this calibration possibility, the $\boldsymbol{\alpha}$ VG-process truly extends the multivariate Variance Gamma with a single subordinator used in the previous literature. In the latter case, one solves for the parameters of the margins, under the constraint

$$\alpha_j = \alpha_i, \;\; i, j = 1, ...n.$$

Given the previous constraint, the linear correlation is uniquely determined, and can also be different from the observed (risk neutral and/or historical) one. This drawback does not exist in the extended model provided in the present paper, since there is one more parameter, a, in order to take linear correlation into consideration.

6.2 Results

For the sake of simplicity, we considered a sub-sample of the CDX names, made by eighteen obligors. For privacy reasons, we do not report their exact name, but denote them with a number. We describe the implementation of the procedure on these names in several steps.

[6]Please note that the constraint is weakened with respect to its formulation in (1.3). The relaxed constraint allows us to include perfect correlation of the subordinator. This case was not included in section 1.2 since it requires some provisos, which are discussed in Semeraro (2006)

6.2.1 Step 1: marginal parameters

For each single name the marginal parameters have been estimated according to the procedure in section 5 above.

Company number	sigma	1/alpha	theta	k
1	0.096	0.693	-0.586	1.403
2	0.232	0.545	-0.822	1.434
3	0.069	0.716	-0.397	1.282
4	0.377	0.065	-1.254	1.077
5	0.114	0.650	-0.427	1.274
6	0.186	0.708	-0.402	1.272
7	0.160	0.233	-0.795	1.183
8	0.406	0.089	-0.271	1.017
9	0.200	0.257	-0.662	1.165
10	0.151	0.563	-0.418	1.229
11	0.066	0.431	-0.506	1.217
12	0.282	0.075	-0.192	1.011
13	0.384	0.079	-0.187	1.009
14	0.157	0.744	-0.400	1.289
15	0.225	0.647	-0.502	1.309
16	0.060	3.798	-0.519	2.963
17	0.134	0.575	-0.836	1.475
18	2.312	0.396	-0.220	0.028

As concerns the marginal data, let us stress that all of the corresponding standard deviations -computed from the parameters according to the formulas in section 1.1 - are smaller than 100%; all of the θ parameters are negative, signalling, as usual with stock prices, negative skewness; the values of all ratios $1/\alpha_j, j = 1, .., n$ are positive, as needed. The constraints (1.1) are satisfied, as shown by the number reported in the last column, k, defined in (1.2). This means that the conditions for well-posedness of the risk neutral measure are satisfied.

6.2.2 Step 2: correlation coefficients

According to the procedure illustrated above, we also need an estimate of the obligor's correlation coefficients, r_{ij}. This estimate was obtained considering the daily stock returns from 7/7/2003 to 7/7/2006. Having a total of 758 data, the standard error of the estimate was equal to 3.64%. We report below the corresponding correlation matrix.

	1	2	3	4	5	6	7	8	9	10	11	12	13	17	15	16	17	18
1	1																	
2	0.064	1																
3	0.025	0.116	1															
4	0.164	0.483	0.1743	1														
5	0.103	0.268	0.1639	0.271	1													
6	0.142	0.179	0.1116	0.166	0.225	1												
7	0.147	0.194	0.1626	0.292	0.211	0.195	1											
8	0.113	0.338	0.1318	0.32	0.272	0.137	0.189	1										
9	0.5	0.106	0.0776	0.162	0.139	0.147	0.177	0.146	1									
10	0.166	0.342	0.1249	0.37	0.321	0.214	0.27	0.319	0.196	1								
11	0.193	0.229	0.1741	0.281	0.236	0.308	0.332	0.194	0.248	0.252	1							
12	0.12	0.156	0.1709	0.171	0.188	0.171	0.202	0.145	0.169	0.243	0.218	1						
13	0.071	0.229	0.2139	0.278	0.223	0.228	0.251	0.237	0.141	0.258	0.256	0.236	1					
14	-0.01	0.087	0.6334	0.174	0.152	0.131	0.169	0.097	0.062	0.093	0.182	0.171	0.208					
15	0.104	0.093	0.1391	0.138	0.108	0.199	0.194	0.152	0.139	0.106	0.201	0.138	0.123	0.1544	1			
16	0.073	0.128	0.0725	0.186	0.202	0.124	0.151	0.138	0.091	0.19	0.183	0.135	0.136	0.066	0.08	1		
17	0.138	0.467	0.0831	0.445	0.253	0.116	0.265	0.318	0.173	0.325	0.279	0.22	0.266	0.2661	0.09	0.107	1	
18	0.193	0.244	-0.081	0.25	0.202	0.287	0.243	0.148	0.215	0.278	0.33	0.21	0.258	0.1698	0.17	0.17	0.239	1

6.2.3 Step 3: calibration of the parameter a

From the estimate of the correlation coefficient, together with the marginal parameters, we can obtain a value for the a parameter. The appropriate a is obtained using (6.1), under the constraint $a \leq \min(1/\alpha_j), j = 1, ...n$. The constraint in turn sums up the constraints on the parameters used in the process construction, (1.3), namely $a \leq 1/\alpha_j$ for every j. The estimated value for a is $a = .219$ and the corresponding $rmse$ is 0.184.

6.2.4 Step 4: a pricing application

Given the calibrated value of a, we can price - either by numerical integration or through a Monte Carlo approach - any credit derivative price written on the 18 names chosen as sub-sample. Suppose for instance that we want to price a first to default (FtD) on the first two names, with maturity one year and paying at expiration, not at the time of default. It is known that the forward value of a first to default is the complement to one of the (risk neutral) survival probability. In the structural model of section (2.2), this probability in turn is

$$\Pr(V_1(T) > F_1, V_2(T) > F_2) = 1 - \pi_1(1) - \pi_2(1) + \bar{\pi}(1),$$

where $\pi_j(1)$, $j = 1, 2$, is the marginal default probability of firm i, given by (2.2) or -equivalently - by (2.13), while $\bar{\pi}(1)$ is the joint one, given by ((2.12)).

The FtD price is then

$$\exp(-r)(\pi_1(1) + \pi_2(1) - \bar{\pi}(1)).$$

Consider that, on the last observation day, the names 1 and 2 had a leverage ratio F_i/V_{i0} respectively of 22.67% and 28.89%. Assume that they were going to pay no dividends in the incoming year and consider that the observed one year riskless rate (US Treasury bills) on the same day was 2.579%. The marginal parameter values in the previous table, together with these market data, gave respectively $m_1 = 44.8\%, m_2 = 51.6\%$.

As a whole, the corresponding marginal and joint default probability were very modest, and the fair price of a FtD on those two names, at the closing date of the observation period, was close to zero. However, if we consider an increase in the leverage ratio, for instance if we increase it by one, the FtD no arbitrage price becomes positive. More precisely, the corresponding default probabilities, in basis points, become

$$\begin{aligned} \pi_1(1) &= 2.544 \times 10^{-9} \\ \pi_2(1) &= 2.325 \\ \bar{\pi}(1) &= 1.93 \times 10^{-11} \end{aligned}$$

and the FtD price is 2.266 basis points.

7 Conclusions

This paper presents a multivariate extension of a terminal default model à la Merton with a VG asset value. It provides an empirical application of it, based on an extensive single name univariate calibration of the CDX NA HY and IG components.

At the univariate level, our analysis is based on the comparison between predicted and actual CDS spreads of both the DJ CDX NA IG and CDX NA HY components. We show that VG jumps in asset values are able to give small prediction errors and biases. Indeed, the VG Merton model seems to address appropriately the main problems left unsolved by diffusion based structural models, namely the understatement of credit spreads of the basic Merton case and the overstatement of the other diffusion models. The unpredictability of default which is a result of a pure jump asset value - such as the VG - seems therefore to be important not only at the theoretical, but also at the calibration level.

Based on the univariate credit risk calibration, we build the multivariate one. At the joint default level we are able to fit default dependence from equity correlation, without imposing a specific, exogenously given copula. At the opposite, we fit dependence consistently with the existence of a dynamic asset process which drives default. Opposite to the existing multivariate models with VG asset values, we do not need to resort to equicorrelation. Our procedure can be used as a first step for multiple default derivative pricing, such as first to default or k to default. We present an example in this sense.

At the multivariate level we consider the possibility of fitting default correlation without equicorrelation, when asset values are pure jump processes, as the main contribution of the paper.

Appendix

Derivation of equation (2.11). Let $\tilde{B}_i(t) = \theta_i t + \sigma_i B(t)$, then

$$
\begin{aligned}
F_T(x_1, x_2) &:= P[Y_1(T) \le x_1, Y_2(T) \le x_2] \\
&= P[\tilde{B}_1(X_1(T) + \alpha_1 Z(T)) \le x_1, \tilde{B}_2(X_2(T) + \alpha_2 Z(T)) \le x_2] \\
&= \int_0^{+\infty} P[\tilde{B}_1(X_1(T) + \alpha_1 z) \le x_1, \tilde{B}_2(X_2(T) + \alpha_2 z) \le x_2 | Z(T) = z] f_{Z(T)}(z) dz \\
&= \int_0^{+\infty} P[\tilde{B}_1(X_1(T) + \alpha_1 z) \le x_1] P[\tilde{B}_2(X_2(T) + \alpha_2 z) \le x_2] f_{Z(T)}(z) dz \\
&= \int_0^{+\infty} \{ \int_0^{+\infty} P[\tilde{B}_1(w_1 + \alpha z) \le x_1 | X_1(T) = w_1] f_{X_1(T)}(w_1) dw_1 \\
&\cdot \int P[\tilde{B}_2(w_2 + \alpha_2 z) \le x_2 | X_2(T) = w_2] f_{X_2(T)}(w_2) dw_2 \} f_{Z(T)}(z) dz \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} P[\tilde{B}_1(w_1 + \alpha_1 z) \le x_1] P[\tilde{B}_2(w_2 + \alpha_2 z) \le x_2] \\
&\qquad \cdot f_{X_1(T)}(w_1) f_{X_2(T)}(w_2) f_{Z(T)}(z) dw_1 dw_2 dz \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \Phi(\frac{x_1 - \theta_1(w_1 + \alpha_1 z)}{\sigma_1 \sqrt{w_1 + \alpha_2 z}}) \Phi(\frac{x_2 - \theta_2(w_2 + \alpha_2 z)}{\sigma_2 \sqrt{w_2 + \alpha_2 z}}) \\
&\qquad \cdot f_{X_1(T)}(w_1) f_{X_2(T)}(w_2) f_{Z(T)}(z) dw_1 dw_2 dz.
\end{aligned}
$$

In case $T = 1$:

$$
\begin{aligned}
F(x_1, x_2) &:= P[Y_1(1) \le x_1, Y_2(1) \le x_2] \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \Phi(\frac{x_1 - \theta_1(w_1 + \alpha_1 z)}{\sigma_1 \sqrt{w_1 + \alpha_1 z}}) \Phi(\frac{x_2 - \theta_2(w_2 + \alpha_2 z)}{\sigma_2 \sqrt{w_2 + \alpha_2 z}}) \\
&\qquad \cdot f_{X_1}(w_1) f_{X_2}(w_2) f_Z(z) dw_1 dw_2 dz \\
&= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \Phi(\frac{x_1 - \theta_1(w_1 + \alpha_1 z)}{\sigma_1 \sqrt{w_1 + \alpha_1 z}}) \Phi(\frac{x_2 - \theta_2(w_2 + \alpha_2 z)}{\sigma_2 \sqrt{w_2 + \alpha_2 z}}) \\
&\qquad \cdot \frac{\frac{1}{\alpha_1}^{(\frac{1}{\alpha_1} - a)} e^{-\frac{1}{\alpha_1}(w_1)} w_1^{\frac{1}{\alpha_1} - a - 1}}{\Gamma(\frac{1}{\alpha_1} - a)} \frac{\frac{1}{\alpha_2}^{(\frac{1}{\alpha_2} - a)} e^{-\frac{1}{\alpha_2}(w_2)} w_2^{\frac{1}{\alpha_2} - a - 1}}{\Gamma(\frac{1}{\alpha_2} - a)} \frac{e^{-z} z^{a-1}}{\Gamma(a)} dw_1 dw_2 dz.
\end{aligned}
$$

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