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## On the impossibility of protecting risk-takers

Toomas Hinnosaar

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# On the impossibility of protecting risk-takers\*

Toomas Hinnosaar<sup>†</sup>

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## Abstract

Risk-neutral sellers can extract high profits from risk-loving buyers by offering them lotteries. To limit risk-taking, gambling is heavily regulated in most countries. I show that protecting risk-loving buyers is essentially impossible.

Even if buyers are risk-loving only asymptotically, the seller can construct a non-random winner-pays auction that ensures unbounded profits. Buyers are asymptotically risk-loving, for example, when their preferences satisfy Savage's axioms or they have prospect theory preferences. The profits are unbounded even if the seller cannot use any mechanism that resembles a lottery.

*JEL:* D82, D44, C72, D81

*Keywords:* auctions, gambling, risk preferences, prospect theory

## 1 Introduction

Gambling is either illegal or heavily regulated in most countries. The primary economic reason for gambling restrictions is protecting agents against decisions that are considered harmful to themselves. Namely, when selling lotteries to risk-loving agents, a risk-neutral seller can raise large profits.<sup>1</sup>

In practice, the line between risky but legal sales mechanisms and gambling is unclear. Selling goods at posted prices is not considered harmful even when only a random fraction of buyers gets to purchase the goods.<sup>2</sup> Similarly, most auctions are not considered gambling, although auction outcomes are random from an individual bidder's perspective.

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<sup>†</sup>Collegio Carlo Alberto, [toomas@hinnosaar.net](mailto:toomas@hinnosaar.net).

<sup>1</sup>Azevedo and Gottlieb (2012) showed that a risk-neutral monopolist can raise infinite profits when selling lotteries to agents with prospect theory preferences.

<sup>2</sup>For example, when Led Zeppelin gave a concert in 2007, about 20 million people requested tickets priced at £125, but there were only 16,000 tickets available. According to an online ticket reseller, the market value of a ticket was £7,425. <http://www.blabbermouth.net/news/average-led-zeppelin-ticket-price-for-reunion-show/>

There are other selling mechanisms that are closer to lotteries, such as selling random goods. For example, at [priceline.com](http://priceline.com), buyers learn which particular products they get only after the purchase.<sup>3</sup> As another example, sellers often give out surprise gifts (sweepstakes) to a random subset of buyers. At least so far, [priceline.com](http://priceline.com)'s model has not been called gambling, whereas in the US, sweepstakes are considered a loophole for lotteries, so the FTC has enforced a "no purchase or payment necessary" rule.<sup>4</sup> Finally, some sellers use auction formats that seem to be close to lotteries in terms of outcomes and are highly profitable (for example, penny auctions and lowest unique bid auctions).<sup>5</sup>

This paper is motivated by the perspective of a regulator who would be interested in the types of mechanisms that should be regulated by gambling laws to avoid the possibility of extracting large profits from risk-loving buyers. To do so, the regulator may want to prohibit randomized payment rules and monetary payments from the seller to the buyers because these instruments provide a way to create lotteries. On the other hand, prohibiting everything except the standard posted price mechanism is going too far because doing so would exclude mechanisms that are Pareto-improvements when buyers are risk-neutral or risk-averse.

The main result of the paper is an impossibility result. For a very general class of preferences, even with the strongest reasonable restrictions on the selling mechanisms, the seller can extract unboundedly large profits. The result only requires that buyers are asymptotically risk-loving, which means that their marginal utility converges to zero at negative infinity. This is a very mild condition, which is satisfied, for example, when agents have prospect theory preferences<sup>6</sup> or have bounded utility function.

The class of mechanisms that allows extracting infinite profits is non-random winner-pays auctions, where the highest bidder gets the object and pays a transfer that is a deterministic function of bids and the other bidders pay nothing. This class includes first- and second-price auctions (with or without reserve price), but excludes all-pay auctions and anything reminiscent of lotteries.

To see how the seller can obtain unbounded profits with auctions of this type, consider the following modification of the second-price auction: Fix a large transfer  $T > 0$ . Let the highest (winning) bid be  $b$ . The winner pays  $T$  if the second highest bid is in the interval  $[0, \gamma(b)]$  and 0 otherwise, where  $\gamma(b)$ , is chosen so that bidding one's true value is optimal for buyer of type  $b$ . If buyers were risk-neutral, the mechanism would be equivalent to the second-price auction for any  $T$ . Increase in  $T$  and corresponding decrease of  $\gamma(b)$  would

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<sup>3</sup>When purchasing a hotel room or flight from [priceline.com](http://priceline.com), consumers often learn the exact itinerary or hotel location only after the purchase, which is non-refundable.

<sup>4</sup>If the seller offers a prize, then it must disclose instructions on how to participate in the draw without paying any money or purchasing any goods or services. <https://www.ftc.gov/tips-advice/business-center/guidance/complying-telemarketing-sales-rule>

<sup>5</sup>For example, penny auctions are reminiscent of English auctions but include a large bid fee and are, therefore, highly random in equilibrium (Hinnosaar, 2014). The Gambling Commission in the UK said that it "was not convinced that penny auctions amounted to gambling." <http://news.bbc.co.uk/2/hi/business/7793054.stm> Similarly, the European Commission spokesperson commented that penny auctions are governed by rules on auctions (rather than on gambling). <http://www.technologizer.com/2008/10/17/swoopo-seems-safe-from-legal-action-in-eu/>

<sup>6</sup>When agents have nonlinear probability weighting function, the condition on utility function needs to be adjusted with the weighting function. The combined sufficient condition is satisfied essentially by all functional forms used in the literature.

not affect neither expected transfers nor profits.

However, large  $T$  makes the auction risky from bidders' perspective, who essentially face a gamble where they may get the object for free with high probability or pay a large transfer with low probability. When buyers are asymptotically risk-loving, the expected payment increases fast enough that profits can be made arbitrarily large. For example, if utility function is bounded below, then as  $T$  gets large enough, its increase changes utility very little and therefore the seller can raise the transfer almost without reducing the probability with which agents pay it.

Because infinite profits can be achieved selling just one object to two buyers who are asymptotically risk-loving, the result extends to many other situations. When buyers are heterogeneous in risk attitudes, it suffices to have two buyers whose preferences satisfy the sufficient condition (i.e., they are asymptotically risk-loving). Furthermore, if the seller is able to use asymmetric mechanisms, it suffices to have just one such buyer. When there is more than one object, the result extends trivially because the seller can focus only on a single object.

The paper contributes to two branches of literature. First, mechanism design literature has studied profit-maximizing mechanisms under various preferences. Optimal auctions with risk-neutral agents were characterized by Myerson (1981) and Riley and Samuelson (1981). Matthews (1983) and Maskin and Riley (1984) characterized optimal mechanisms with risk-averse buyers. In recent years, a new stream of literature has started to analyze optimal mechanisms with non-standard preferences. For example, Di Tillio, Kos, and Messner (2012) and Bose and Renou (2014) characterized the optimal mechanisms when bidders are ambiguity averse, and Carbajal and Ely (2015) when bidders are loss averse (with a piecewise linear function as in Kőszegi and Rabin (2006)). The optimal mechanisms in all these papers ensure finite profit.

All these papers assume that the utility function is piecewise linear or concave, so marginal utility at negative infinity is bounded away from zero. I show that results change drastically when the agents are asymptotically risk-loving; profits from the optimal mechanism are unboundedly large. Modeling agents as asymptotically risk-loving covers several important classes of preferences that are so far uncovered by mechanism design literature: risk-loving preferences, prospect theory preferences, and bounded utility.

Second, the paper also contributes to the discussion on the implications of prospect theory. Prospect theory, formulated by Kahneman and Tversky (1979; 1992), is widely supported by experimental evidence. However, in market settings, it has some undesirable features. Azevedo and Gottlieb (2012) showed that when sellers are able to offer arbitrary lotteries, profits are unbounded and Pareto-efficiency is not well defined.<sup>7</sup> Ebert and Strack (2015) proved that due to skewed preferences over gains and losses and the probability weighting function, an agent with prospect theory preferences in a dynamic context can potentially continue taking gambles until all the agent's resources are exhausted. De Giorgi, Hens, and Rieger (2010) showed that in an economy where agents have prospect preferences, financial market equilibria may not exist due to an infinite short-selling problem. The current paper adds another feature by showing that for un-

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<sup>7</sup>Rieger and Wang (2006) pointed out that with (standard) prospect theory preferences lotteries with finite expected value may lead to infinite utility. They proposed alternative specifications for value and probability weighting functions, but these specifications do not solve the problem highlighted here.

bounded profits, it is not necessary to have arbitrary lotteries or infinite short-selling opportunities, but it suffices that the seller can offer arbitrary auction formats.

## 2 Main result

A monopolistic seller sells a single object at zero cost. There are  $n \geq 2$  buyers with private types, such that buyer  $i$ 's type is  $\theta_i \in [0, \bar{\theta}]$ . Types  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  are drawn from a symmetric joint distribution  $N$  with the following property. Let  $\theta_{-i} = \max_{j \neq i} \{\theta_j\}$ , let  $G$  be the cumulative distribution function of  $\theta_{-i}$ , and  $g$  the probability density function. I assume that  $\theta_{-i}$  has full support, i.e.  $g(\theta_{-i}) > 0$  for all  $\theta_{-i} \in [0, \bar{\theta}]$ .<sup>8</sup>

Buyer of type  $\theta$  who gets the object and pays  $t$  gets utility  $u(\theta, t)$ , and the buyer who does not get the object and does not pay anything<sup>9</sup> gets utility 0. I assume that the function  $u$  is common for all buyers, continuously differentiable,  $u(0, 0) = 0$ ,  $\lim_{t \rightarrow \infty} u(\theta, t) < 0$ ,  $u_1(\theta, t) = \frac{\partial u(\theta, t)}{\partial \theta} > 0$ , and  $u_2(\theta, t) = \frac{\partial u(\theta, t)}{\partial t} < 0$ .<sup>10</sup>

There are two assumptions that are crucial for the result: buyers are *asymptotically risk-loving*, and the seller can use *non-random winner-pays auctions*.

**Definition 1.** *An expected-utility maximizing agent with type  $\theta$  is asymptotically risk-loving if  $\lim_{t \rightarrow \infty} u_2(\theta, t) = 0$ .*

As Lemma 1 shows, Definition 1 is satisfied in a wide range of situations. When utility is bounded from below, the agent is trivially asymptotically risk-loving because the utility converges to a constant. Bounded utility may be a property of preferences<sup>11</sup> or arise in situations where agents can declare bankruptcy after incurring losses that are too large. Agents are also asymptotically risk-loving whenever they become in the limit a risk-lovers according to the Arrow-Pratt relative or absolute risk measures. For example, any agent who is risk-loving globally or at least in the in losses is also asymptotically risk-loving.

**Lemma 1.** *An agent with type  $\theta$  is asymptotically risk-loving,*

1. *if and only if  $\lim_{t \rightarrow \infty} \frac{u(\theta, t)}{t} = 0$  and  $\lim_{t \rightarrow \infty} u_2(\theta, t)$  exists, or*
2. *if the utility function is bounded from below, or*
3. *if the second derivative with respect to transfer  $u_{22}$  exists and either  $\lim_{t \rightarrow \infty} \frac{tu_{22}(\theta, t)}{u_2(\theta, t)} = \underline{r}^* > 0$  or  $\lim_{t \rightarrow \infty} \frac{u_{22}(\theta, t)}{u_2(\theta, t)} = \underline{r} < 0$  (i.e., asymptotically risk-loving according to the Arrow-Pratt relative or absolute risk measure).*

<sup>8</sup>For example, in the independent values case, if each  $\theta_i$  is an independent draw from the same distribution  $F$ , then  $G(\theta_{-i}) = F(\theta_{-i})^n$ , which has full support if type distribution  $F$  has full support. The seller can obtain unbounded profits even if the values are correlated more (for example with common values), using mechanisms that induce mixed strategy equilibria, but this requires relaxing truth-telling requirement.

<sup>9</sup>Because I am only considering mechanisms where only the winner pays, there is no need to define preferences over money for agents who do not receive the object.

<sup>10</sup>These assumptions make the model tractable and more general than standard assumptions in the literature, but several assumptions can be relaxed even further without changing the result: there could be more than one object, opportunity costs, and the utility function does not have to be differentiable.

<sup>11</sup>For example, Savage's axioms imply bounded utility (Fishburn, 1970).

The proof of this lemma is in Appendix A.

**Definition 2.** *A non-random winner-pays auction satisfies the following rules:*

1. *Bidders simultaneously submit bids.*
2. *The highest bid gets the object if the bid is higher than some minimal level.*
3. *All other bidders do not get the object and do not pay anything.*
4. *The highest bidder pays an amount that is a deterministic function of bids.*

For example, first-price auctions and second-price auctions with or without reserve prices satisfy these rules, whereas the rules exclude lotteries (where transfers are random) and all-pay auctions (where losers pay positive amounts). Essentially, the restrictions exclude all mechanisms that resemble lotteries and other risky selling mechanisms used in practice, but they still allow for efficient and profit-maximizing mechanisms when agents are risk-neutral. This class of mechanisms is as restrictive as possible without specifying the particular transfer functions that the seller can use. To specify transfer functions, regulators would need information they do not have, or they would need to exclude some mechanisms that are necessary for ensuring efficiency with risk-neutral or risk-averse bidders.

**Proposition 1.** *Suppose that buyers are asymptotically risk-loving. There exists a non-random winner-pays auction, which*

1. *allocates the object to the buyer with the highest type,*
2. *has all buyers with positive probability of getting the object pay unboundedly large transfers in expectation,*
3. *ensures unboundedly large profits for the seller.*

The proof in Appendix A is constructive. To achieve unbounded revenue with agents who are asymptotically risk-loving, I construct a modification of a second-price auction that can be made arbitrarily risky in losses. The idea is the following: Fix a (large) transfer  $T > 0$ . Suppose that  $i$  is the highest bidder with bid  $b_i$ , and the second highest bid is  $b_{-i}$ . Then  $i$  gets the object without any payment if  $b_{-i} > \gamma(b_i)$  and has to pay  $T$  otherwise. For each  $T$ , the function  $\gamma(b_i)$  is chosen so that each type prefers to bid one's own value.

From the winner's perspective, the transfer is  $T$  with probability  $G(\gamma(\theta_i))$  and zero otherwise. As  $T$  increases, the probability of paying  $T$  decreases, and the lottery becomes more risky. If agents were risk-neutral, increasing  $T$  would simply shift realized payments in a way that the expected transfer remains constant. However, with asymptotically risk-loving agents, the transfer eventually increases more quickly than the probability of paying it decreases, so the expected profit can be made arbitrarily large.

**Example** Suppose  $n = 2$ , types are distributed independently uniformly in  $[0, 1]$ , bidders are expected-utility maximizers, and a bidder who receives the object and pays  $t_i \geq 0$  gets utility  $u(\theta_i, t_i) = \theta_i - t_i^\alpha$  where  $\alpha > 0$ .<sup>12</sup>

Then for all  $T > 1$ , probability  $\gamma(\theta_i) = G(\gamma(\theta_i))$  of paying  $T$  is defined by<sup>13</sup>

$$\gamma(\theta_i) = \int_0^{\theta_i} \frac{u(\theta'_i, 0)}{u(\theta'_i, 0) - u(\theta'_i, T)} d\theta'_i = \frac{\theta_i^2}{2T^\alpha}. \quad (1)$$

The expected transfer from buyer with type  $\theta_i > 0$  is  $t(\theta_i) = T\gamma(\theta_i) = \frac{\theta_i^2}{2}T^{1-\alpha}$ . Therefore  $\lim_{T \rightarrow \infty} t(\theta_i) = \infty$  and thus also the profits are unbounded whenever  $\alpha < 1$ . Notice that with these preferences, the agents are risk-loving in money in the Arrow-Pratt relative risk measure if and only if  $\alpha < 1$ .

**Prospect-theory preferences** In the case of prospect theory preferences, the agents may have non-linear probability weighting function to compute the utility from random outcomes, so that the definition of asymptotically risk-loving agent must be adjusted. Proposition 1 holds under the following definition of asymptotically risk-loving preferences.

**Definition 3.** *A prospect-theory agent with type  $\theta$  is asymptotically risk-loving if for all  $c > 0$ ,  $\lim_{t \rightarrow \infty} w^-\left(\frac{c}{t}\right) u(\theta, t) = 0$ .*

With expected utility weighting function,  $w^-(p) = p$ , Definition 3 is equivalent to the definition Definition 1 above. Prospect theory literature assumes that weighting function is concave in small probabilities. In this case, Definition 3 is stronger than Definition 1.<sup>14</sup>

Lemma 2 gives sufficient conditions that are easier to verify than the condition in Definition 3. Again, any agent with bounded utility function is asymptotically risk-loving. The second and the third part of the lemma show that if either  $\lim_{p \rightarrow 0} \frac{w^-(cp)}{w^-(p)}$  or  $\lim_{t \rightarrow \infty} \frac{u(\theta, ct)}{u(\theta, t)}$  are finite for all  $c > 0$ , then it suffices to have  $\lim_{p \rightarrow 0} w^-(p)u\left(\theta, \frac{1}{p}\right) = 0$ . This condition was used by Azevedo and Gottlieb (2012), who verified that this condition is satisfied by virtually all models proposed in prospect theory literature. Since essentially all models proposed the prospect theory literature<sup>15</sup> use either bounded or power utility function (which has  $\lim_{t \rightarrow \infty} \frac{u(\theta, ct)}{u(\theta, t)} \in \mathbb{R}$ ), we can conclude that with the functional forms used in the literature, prospect-theory agents are asymptotically risk-loving.

**Lemma 2.** *A prospect-theory agent with type  $\theta$  is asymptotically risk-loving, if either*

1.  $u(\theta, t) \geq \underline{u} > -\infty$  for all  $t \in \mathbb{R}$ , or
2.  $\lim_{p \rightarrow 0} w^-(p)u\left(\theta, \frac{1}{p}\right) = 0$  and  $\lim_{p \rightarrow 0} \frac{w^-(cp)}{w^-(p)} \in \mathbb{R}$  for all  $c > 0$ , or
3.  $\lim_{p \rightarrow 0} w^-(p)u\left(\theta, \frac{1}{p}\right) = 0$  and  $\lim_{t \rightarrow \infty} \frac{u(\theta, ct)}{u(\theta, t)} \in \mathbb{R}$  for all  $c > 0$ .

<sup>12</sup>For example, if utility from negative transfers is  $u(\theta_i, t_i) = \theta_i + \frac{1}{\lambda}(-t_i)^\alpha$ , then the preferences are quasilinear over the value of the object and Tversky and Kahneman (1992) preferences over money.

<sup>13</sup>The function  $\gamma$  is determined by the condition that ensures that each buyer is willing to report the type truthfully (see Equation (11) in Appendix A).

<sup>14</sup>Definition 3 can also be weaker than Definition 1; for example, if utility function is linear in money, but the weighting function is convex in small probabilities.

<sup>15</sup>Including for example Tversky and Kahneman (1992), Prelec (1998), and Rieger and Wang (2006).

### 3 Discussion

Prohibiting all known types of gambling and selling mechanisms that resemble lotteries is not sufficient to deter the seller from extracting unbounded profits from asymptotically risk-loving buyers.<sup>16</sup> In addition, a regulator would have to limit specific functional forms that the payments can take. Doing so would either require situation-specific information, which would mean regulating all non-standard auctions with gambling laws or excluding some mechanisms that are necessary to achieve efficiency with risk-neutral and risk-averse agents. In particular, it suffices to set an upper bound for the payments, but setting an upper bound would either create inefficiencies (e.g., making it impossible to sell valuable objects) or require more information than a regulator would have. After all, auctions are used precisely because the seller does not know the values of the buyers.

A practical way to avoid infinite profits is to require the seller to offer a return policy, where any buyer can opt out from the mechanism after learning the outcomes. This strong restriction eliminates all mechanisms with uncertainty as well as auctions that are not dominant-strategy incentive compatible.

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<sup>16</sup>There are other reasons why the regulator may still want to regulate this type of mechanisms, for example to protect gambling addicts and to ensure that gambling institutions do not cheat their clients.

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# Appendices

## A Proofs

**Lemma 1.** *An agent with type  $\theta$  is asymptotically risk-loving,*

1. *if and only if  $\lim_{t \rightarrow \infty} \frac{u(\theta, t)}{t} = 0$  and  $\lim_{t \rightarrow \infty} u_2(\theta, t)$  exists, or*
2. *if the utility function is bounded from below, or*
3. *if the second derivative with respect to transfer  $u_{22}$  exists and either  $\lim_{t \rightarrow \infty} \frac{tu_{22}(\theta, t)}{u_2(\theta, t)} = \underline{r}^* > 0$  or  $\lim_{t \rightarrow \infty} \frac{u_{22}(\theta, t)}{u_2(\theta, t)} = \underline{r} < 0$  (i.e., asymptotically risk-loving according to the Arrow-Pratt relative or absolute risk measure).*

### Proof

1.  $\Leftarrow$  If  $\lim_{t \rightarrow \infty} u(\theta, t) = \underline{u} > -\infty$ , then the claim holds trivially. If  $\lim_{t \rightarrow \infty} u(\theta, t) = -\infty$ , then the claim follows from the L'Hôpital's rule.  
 $\Rightarrow$  By definition, for any  $\varepsilon > 0$ , there exists  $\hat{t} > 0$  such that  $|u_2(\theta, t)| < \varepsilon$  for all  $t > \hat{t}$ . Therefore, for all  $t > \hat{t}$ ,  $-\varepsilon(t - \hat{t}) < u(\theta, t) - u(\theta, \hat{t}) < \varepsilon(t - \hat{t})$ ; thus,

$$\frac{u(\theta, \hat{t}) + \varepsilon \hat{t}}{t} - \varepsilon < \frac{u(\theta, t)}{t} < \frac{u(\theta, \hat{t}) - \varepsilon \hat{t}}{t} + \varepsilon \quad (2)$$

so that  $\lim_{t \rightarrow \infty} \frac{u(\theta, t)}{t} \in (-\varepsilon, \varepsilon)$ . By taking  $\varepsilon \rightarrow 0$ ,  $\lim_{t \rightarrow \infty} \frac{u(\theta, t)}{t} = 0$ .

2. Trivial.

3. Fix  $\alpha \in (0, \underline{r}^*)$ . By Lemma 3, there exist  $\hat{t} > 0$ ,  $c_1 > 0$ , and  $c_2 \in \mathbb{R}$  such that for  $t > \hat{t}$ ,  $\frac{u(\theta, t)}{t} > c_1 t^{-\alpha} + \frac{c_2}{t}$ , and  $\lim_{t \rightarrow \infty} u(\theta, t) < 0$  by assumption. Thus,  $\lim_{t \rightarrow \infty} \frac{u(\theta, t)}{t} = 0$ .

Finally,  $\lim_{t \rightarrow \infty} -\frac{u_{22}(\theta, t)}{u_2(\theta, t)} = \underline{r} < 0$  implies  $\lim_{t \rightarrow \infty} -\frac{tu_{22}(\theta, t)}{u_2(\theta, t)} = \lim_{t \rightarrow \infty} t\underline{r} = \infty > 0$ .

□

**Lemma 3.** Suppose  $\underline{r}^* = \lim_{t \rightarrow \infty} -\frac{tu_{22}(\theta, t)}{u_2(\theta, t)}$ , and  $\theta \in [0, \bar{\theta}]$ .

1. For any  $\alpha < \underline{r}^* \leq 1$ , there exist  $\hat{t} > 0$ ,  $c_1 > 0$ , and  $c_2 \in \mathbb{R}$  such that  $u(\theta, t) > c_1 t^{1-\alpha} + c_2$  for all  $t > \hat{t}$ .
2. For any  $1 > \alpha > \underline{r}^*$ , there exist  $\hat{t} > 0$ ,  $c_1 > 0$ , and  $c_2 \in \mathbb{R}$  such that  $u(\theta, t) < c_1 t^{1-\alpha} + c_2$  for all  $t > \hat{t}$ .

**Proof** Suppose  $\underline{r}^* > \alpha$ . Following from the definition of  $\underline{r}^*$ , there exist  $\hat{t}$  such that  $-\frac{tu_{22}(\theta, t)}{u_2(\theta, t)} > \alpha$  or equivalently  $\frac{u_{22}(\theta, t)}{u_2(\theta, t)} < -\frac{\alpha}{t}$  for all  $t > \hat{t}$ . Therefore,

$$\ln \frac{u_2(\theta, t)}{u_2(\theta, \hat{t})} = \int_{\hat{t}}^t \frac{u_{22}(\theta, x)}{u_2(\theta, x)} dx < -\alpha \ln \frac{t}{\hat{t}} \Rightarrow u_2(\theta, t) > c_1 (1 - \alpha) t^{-\alpha}, \quad (3)$$

where  $c_1 = \frac{u_2(\theta, \hat{t})}{(1-\alpha)\hat{t}^{-\alpha}} > 0$ . Now for all  $t > \hat{t}$ ,

$$u(\theta, t) - u(\theta, \hat{t}) > c_1 t^{1-\alpha} - c_1 \hat{t}^{1-\alpha} \Rightarrow u(\theta, t) > c_1 t^{1-\alpha} + c_2, \quad (4)$$

where  $c_2 = u(\theta, \hat{t}) - c_1 \hat{t}^{1-\alpha}$ . Proof for  $\alpha > \underline{r}^*$  is analogous.

□

**Proposition 1.** Suppose that buyers are asymptotically risk-loving. There exists a non-random winner-pays auction, which

1. allocates the object to the buyer with the highest type,
2. has all buyers with positive probability of getting the object pay unboundedly large transfers in expectation,
3. ensures unboundedly large profits for the seller.

**Proof** Fix a large transfer  $T > 0$  and a critical type  $\theta^* \in (0, \bar{\theta})$ . Consider the following non-random winner-pays mechanism, which I will refer below briefly to as  $T$ -mechanism: If buyer  $i$  has the highest type (so that  $\theta_i > \max_{j \neq i} \theta_j = \theta_{-i}$ ), then  $i$  gets the object if the value is above critical type  $\theta_i \geq \theta^*$ . Other bidders do not get the object or pay anything, and buyer  $i$  pays

$$\bar{t}(\boldsymbol{\theta}) = \begin{cases} 0 & \text{if } \theta_{-i} \geq \gamma(\theta_i), \\ T & \text{if } \theta_{-i} < \gamma(\theta_i), \end{cases} \quad (5)$$

where  $\gamma : [0, \bar{\theta}] \rightarrow [0, 1]$  is a function that guarantees that bidding one's true value is optimal for each type  $\theta_i \in [0, \bar{\theta}]$ .

I divide the proof itself into two lemmas: First, Lemma 4 shows that for any  $T$ -mechanism, I can construct the function  $\gamma$  that makes truth-telling optimal for all types. Second, Lemma 5 shows that when buyers are asymptotically risk-loving, then with sufficiently large  $T$ , the  $T$ -mechanism ensures arbitrarily large expected transfers from each types and therefore unbounded profits.

Remark: I will prove these lemmas allowing arbitrary weighting functions for gains and losses. If agents are expected utility maximizers, then  $w^+(p) = w^-(p) = p$ .  $\square$

**Lemma 4.** *For a big  $T$ , there exists function  $\gamma$ , such that bidding one's own value is an equilibrium in  $T$ -mechanism.*

**Proof** Fix arbitrary  $T$ -mechanism. The expected utility for bidder with value  $\theta_i$  who bids  $\theta'_i$  is<sup>17</sup>

$$U(\theta'_i|\theta_i) = [w^+(G(\theta'_i)) - w^+(G(\gamma(\theta'_i)))]u(\theta_i, 0) + w^-(G(\gamma(\theta'_i)))u(\theta_i, T), \quad (6)$$

The condition for the optimality of bidding one's own type is  $\left. \frac{dU(\theta'_i|\theta_i)}{d\theta'_i} \right|_{\theta'_i=\theta_i} = 0$ , which gives the condition

$$\frac{dw^+(G(\theta_i)) - w^+(G(\gamma(\theta_i)))}{d\theta_i} u(\theta_i, 0) = \frac{dw^-(G(\gamma(\theta_i)))}{d\theta_i} [-u(\theta_i, T)], \quad \forall \theta_i \in [0, \bar{\theta}]. \quad (7)$$

When  $T$  is sufficiently large, the necessary condition Equation (7) is also the sufficient condition under which reporting one's own value is the unique maximizer of expected utility because if  $\gamma$  satisfies Equation (7), then

$$\frac{d^2U(\theta'_i|\theta_i)}{d\theta_i d\theta'_i} = \frac{dw^-(G(\gamma(\theta'_i)))}{d\theta'_i} \left[ \frac{-u(\theta'_i, T)}{u(\theta'_i, 0)} u_1(\theta_i, 0) + u_1(\theta_i, T) \right] > 0, \quad \forall \theta_i, \theta'_i, \quad (8)$$

as long as  $T$  is large enough so that  $u(\theta'_i, T) < 0$ .

Next, I define a new probability weighting function  $\bar{w}(\cdot, \theta_i)$  as

$$\bar{w}(G(p, \theta_i)) = w^-(G(x, \theta_i)) \frac{-u(\theta_i, T)}{u(\theta_i, 0) - u(\theta_i, T)} + w^+(G(x, \theta_i)) \frac{u(\theta_i, 0)}{u(\theta_i, 0) - u(\theta_i, T)}. \quad (9)$$

For each  $\theta_i$  it is simply a weighted average of weighting functions  $w^+$  and  $w^-$ , therefore it is also a weighting function with the same properties: in particular, it is continuous, strictly increasing, and  $\bar{w}(0) = 0$ .

Using  $\bar{w}$ , I can rewrite Equation (7) as

$$\frac{d\bar{w}(G(\gamma(\theta_i)))}{d\theta_i} = \frac{dw^+(G(\theta_i))}{d\theta_i} \frac{u(\theta_i, 0)}{u(\theta_i, 0) - u(\theta_i, T)}, \quad \forall \theta_i \in [0, \bar{\theta}]. \quad (10)$$

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<sup>17</sup>Remember that  $G$  is the distribution of  $\theta_{-i} = \max_{j \neq i} \theta_j$ , i.e. the highest valuation among the other bidders, and  $w^+$  and  $w^-$  are the probability weighting functions for gains and losses respectively. I assume here that  $T$  is sufficiently big, so that paying  $T$  for the object falls into the losses domain.

Integrating both sides from 0 to  $\theta_i$  gives

$$\bar{w}(G(\gamma(\theta_i))) = \int_0^{\theta_i} \frac{dw^+(G(\theta'_i))}{d\theta'_i} \frac{u(\theta'_i, 0)}{u(\theta'_i, 0) - u(\theta'_i, T)} d\theta'_i, \quad \forall \theta_i \in [0, \bar{\theta}]. \quad (11)$$

The right-hand side of Equation (11) is an integral of a continuous function that is strictly positive for each  $\theta'_i$ , which means that it is a continuous strictly increasing function of  $\theta_i$ . Notice that on the left-hand side, function  $\bar{w}(G(\cdot))$  is strictly increasing and continuous, therefore the equation defines unique  $\gamma(\theta_i)$  for each  $\theta_i$ . Moreover, it is continuous and strictly increasing.

Therefore for any  $T$ -mechanism with sufficiently large  $T$ , I have constructed (a unique) function  $\gamma$  with which truth-telling is optimal for each buyer, assuming that other buyers bid their types. □

**Lemma 5.** *If  $\lim_{T \rightarrow \infty} w^-(\frac{c}{T})u(\theta^*, T)$  for each  $c > 0$ , then using  $T$ -mechanisms, the seller can ensure*

1. *unboundedly large expected transfer from each type who receives the object with positive probability,*
2. *unbounded expected profits.*

**Proof** We need to show that the expected transfer can be made arbitrarily large from all types  $\theta_i \geq \theta^*$ . The expected transfer from type  $\theta_i$ , denoted by  $t(\theta_i)$ , is

$$t(\theta_i) = \int_0^{\gamma(\theta_i)} T dG(\theta_{-i}) = TG(\gamma(\theta_i)), \quad (12)$$

where the function  $\gamma$  is characterized by Equation (7) in the proof of Lemma 4. Suppose that the claim does not hold for some type  $\theta_i$ , i.e.  $\lim_{T \rightarrow \infty} t(\theta_i) = c < \infty$ . Then  $\lim_{T \rightarrow \infty} G(\gamma(\theta_i)) = \frac{c}{T}$ .

Notice that by monotonicity,  $u(\theta^*, 0) \leq u(\theta_i, 0)$  and  $-u(\theta_i, T) \leq -u(\theta^*, T)$ . Integrating Equation (7) on both sides from 0 to  $\theta_i$  and using these inequalities gives

$$[w^+(G(\theta_i)) - w^+(G(\gamma(\theta_i)))]u(\theta^*, 0) \leq w^-(G(\gamma(\theta_i)))[-u(\theta^*, T)]. \quad (13)$$

Taking the limit from both sides of Equation (13) gives

$$\lim_{T \rightarrow \infty} [w^+(G(\theta_i)) - w^+(G(\gamma(\theta_i)))]u(\theta^*, 0) \leq \lim_{T \rightarrow \infty} w^-(\frac{c}{T})[-u(\theta^*, T)] = 0. \quad (14)$$

This leads to a contradiction, because the left-hand side is strictly positive whenever  $\lim_{T \rightarrow \infty} G(\gamma(\theta_i)) < G(\theta_i)$ . But if this inequality is reversed, then  $c \geq \lim_{T \rightarrow \infty} TG(\gamma_i) \geq \lim_T TG(\theta_i) = \infty$  leads to a contradiction. □