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Abstract

This paper presents a model in which homogeneous rational agents choose between two competing technologies. Agents observe a private signal and a sample of other agents' previous choices. The signal has both an idiosyncratic and an aggregate component of uncertainty. I derive the optimal decision rule when each agent observes the decision of exactly two agents. Due to aggregate uncertainty, aggregate behavior does not necessarily reflect the true state of nature. Nonetheless, agents still find others' choices a good source of information, and they base their decisions partly on the behavior of others. Consequently, bad choices can be perpetuated in this environment: I show that aggregate uncertainty can lead to agents herding on the inferior technology with positive probability. I also present examples in which herding occurs for arbitrarily large sample sizes.

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1. Introduction

In many circumstances, individuals have heterogeneous private information about the environment they face. Consequently, as long as there is some correlation in players' valuations, the behavior of other agents provides useful information. In these situations, rational agents will base their decisions at least partially on the decisions of others. This kind of behavior is known as *word-of-mouth* learning.

Word-of-mouth learning is present in many economic situations. When a financial institution must decide whether or not to lend money to a firm, it can perform a costly analysis or wait and see how other financial institutions behave. Similarly, when a new product is introduced in a market, agents often base their buying decision on reports from other agents. In the same way, when deciding where to have dinner, consumers often take into account other consumers' decisions: when choosing between two restaurants it is common to take the number of people currently patronizing each of them as an indicator of quality. Similar stories apply to exploration for natural resources and research and development decisions.

Since agents rely on other agents' choices when making decisions in various activities, it is worthwhile to understand the outcomes of this sort of behavior. In this paper I analyze possible outcomes of word-of-mouth learning. I focus on the following questions: In which situations will all agents end up making the same decision? If two competing new technologies are introduced in a market, will only one be adopted by consumers? And if so, will the best one be adopted? In particular, I focus on the concept of herds. In this paper, I will say that a herd occurs when: (1) agents optimally choose to disregard some of their own information and follow the behavior of others, and (2) this behavior leads to the inferior technology being chosen in the long run.

The literature on herding started with Banerjee [1992] and Bikhchandani, Hirshleifer, and Welch [1992]. In their papers, a set of rational agents choose sequentially between two technologies. The payoff from this decision is common but unknown to all. In each period, one agent receives a signal and observes what all agents before him have chosen. Afterwards, this agent makes a once-and-for-all decision between the two technologies. Given that each agent knows that the signal he has received is no better than the signal other players have received, agents follow the behavior of others. As a result, herds occur. Consequently, Banerjee and Bikhchandani et al. show that the optimal behavior of rational agents can impede social learning.

Ellison and Fudenberg [1993, 1995] analyze social learning in two papers that share the following setup. A continuum of agents choose between two competing technologies. Each period, some fraction of the agents reevaluate their choice. They observe the behavior and payoffs of the previous period's agents. Payoffs suffer i.i.d. aggregate shocks in every period. Individuals follow an exogenously given rule of behavior. In Ellison and Fudenberg [1993], agents observe the aggregate behavior and the average payoffs in the population, whereas in Ellison and Fudenberg [1995], agents observe a finite sample of behavior and payoffs.[1] Ellison and Fudenberg [1993] propose a rule of thumb that takes into account both previous-period popularity and payoffs from choosing each technology. If agents place a specific weight on popularity in the decision rule, then the superior technology is chosen with probability one in the long run. If other weights are chosen, herds can occur. Ellison and Fudenberg [1995] assume the following rule of behavior: choose the technology with the highest observed average payoff. In that setting, herds can occur when agents' samples of others' behavior are small. These models by Ellison and Fudenberg differ from those of Banerjee and Bikhchandani et al. in the following ways. First, there is a continuum of agents making choices in each period, instead of only one agent. Second, agents observe payoffs but receive no signal. Finally, agents follow exogenously given rules of behavior instead of solving optimization problems. Despite these differences, herds also occur in Ellison and Fudenberg's models.

In a recent paper, Banerjee and Fudenberg [2004] present a model of social learning by rational agents. The setup includes elements of Banerjee [1992], Bikhchandani et al. [1992] and Ellison and Fudenberg [1993, 1995]. A continuum of agents choose between two competing technologies. During each period, a fraction of the agents is replaced. The newcomers make once-and-for-all decisions after observing a finite sample of previous-period agents' choices and receiving an informative signal.[2] Banerjee and Fudenberg [2004] show that if the signal is sufficiently informative and agents sample more than one agent's choice, then agents choose the superior technology in the long run. Consequently, in contrast to the other models discussed above, herds do not occur in Banerjee and Fudenberg [2004].

In terms of the technology adoption example, the result in Banerjee and Fudenberg [2004] can be explained as follows. Two new competing software packages are released: one from a well established firm and the other from a new company. The quality of the software from the well established firm is well-known. The quality of the software from the new firm is uncertain. Assume, for simplicity, that tastes are homogeneous. Moreover, assume that the quality of the software from the new firm is actually higher (although consumers do not know this). Agents receive a noisy signal of the quality of the new technology and base their decision both on the signal and on the behavior of previous-period agents. If there is a large number of individuals, the distribution of the signals will resemble the true quality

[1] Ellison and Fudenberg [1993] present two models. In the first, one technology is better for all agents. In the second, agents are heterogeneous in their preferences. In particular, agents' preferences are similar to those of other agents close to them. Ellison and Fudenberg [1995] also study two models. Both technologies have equal payoffs on average in the first. In the second, one technology is better for all agents. In the model I present here, preferences are homogeneous and there exists a superior technology. My model is closer to Ellison and Fudenberg [1995] because the sample observed by agents is finite.

[2] In Banerjee and Fudenberg's paper, the signal may be correlated with the sample. In the present paper, conditional on the state of the world, the signal is independent of the sample.

distribution of the new firm's software. Consequently, it is more likely that individuals will choose the new software. This will increase the likelihood that next period's agents will observe a high number of individuals choosing the right technology. As a result, word-of-mouth will spread and most agents will end up choosing the software from the new firm.

In this paper, I incorporate aggregate uncertainty into a word-of-mouth learning model and show how this alters some of the model's fundamental results. I argue that aggregate uncertainty is a realistic feature in word-of-mouth settings. Consider once more the technology example. What would happen if the first version of the new company's software contained a bug that seemed problematic but was in actuality not disruptive? In that case, a high fraction of the first potential buyers will receive low-quality signals for this software. In other words, aggregate uncertainty implies that the realized distribution of agents' signals does not reflect the true quality distribution of the new technology. This causes a high fraction of early buyers to choose the technology from the established company. Later potential buyers observe this. Consequently, even if the bug is rapidly fixed, word-of-mouth will spread that the software is not very good. As a result, individuals will tend to buy the lower-quality software instead of the higher-quality software from the new company. Therefore, bad shocks in early periods can be perpetuated, leading to herds.

I present a model of technology adoption with a setup similar to that in Banerjee and Fudenberg [2004]. Agents learn about technologies' qualities from idiosyncratic signals and the behavior of others. In the present model, an aggregate shock affects the distribution of signals in each period. This may lead to a lack of correspondence between the true state of the world and aggregate behavior. As I describe later, my model differs from both of Ellison and Fudenberg's papers in the following ways. First, I assume that agents observe a private signal instead of previous-period payoffs.[3] Second, agents are rational in this model; hence they solve an optimization problem.[4]

I derive the optimal decision rule for the case in which each individual observes the behavior of exactly two agents from the previous period. In spite of the existence of aggregate uncertainty, agents find the observation of others' behavior a good source of information and thus base their decisions partly on others' choices. Consequently, agents may perpetuate a bad choice by behaving optimally. In fact, I show that herds can arise due to aggregate uncertainty. I present necessary and sufficient conditions for herds to occur when exactly two agents are observed.

I do not provide a generic characterization of the optimal decision rule for cases in which a sample

[3] I introduce the aggregate shock in the distribution of signals instead of introducing it in the distribution of payoffs to simplify the analysis of aggregate uncertainty.

[4] Ellison and Fudenberg [1993, 1995] allow for periodic reevaluation of the choice, whereas the decision presented in this paper is once and for all. However, this distinction has no practical consequences, since in Ellison and Fudenberg [1993, 1995], past decisions and information do not affect current decisions.

of more than two agents is observed. I show that in those cases the decision rule may change over time: agents facing the same sample and signal may react differently in distinct periods. This makes agents' decisions difficult to derive. In spite of this limitation, I show that herds can occur for all odd sample sizes greater than two.

Several papers analyze other elements of social learning. Schlag [1998] presents a setting in which a finite number of agents face the same multi-armed bandit. Each period, some agents are replaced by new ones. Each agent in the population observes the behavior and payoff of exactly one other individual. Agents are boundedly rational in that they have short term memory and are myopic. Under some further restrictions, Schlag [1998] shows that the optimal behavior rule involves imitating the sampled behavior with positive probability, even if the sampled payoff is lower than the own past period payoff. Switching to an arm that did better is more likely than switching to an arm that did worse.[5] From a different perspective, Bolton and Harris [1999] present an N-player game with perfect information in which all agents face an identical two-armed bandit. Both the behavior and payoffs of *every individual* are observed. The authors show that individuals engage in a socially suboptimal level of experimentation.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 discusses the difficulties of providing a full characterization for any sample size and studies the case in which agents only observe one agent. Section 3 also characterizes the optimal behavior of agents if they observe the behavior of two agents, showing that herds can arise in that case. Section 4 concludes.

2. The Model

There is a unit-mass continuum of agents. Each agent lives for one period. Once they die, they are replaced by another continuum of agents of mass one.[6] Time is discrete, and periods are indexed by $t = 1, 2, \ldots$. During each period, each agent chooses one of two different technologies: a and b. There are two states of the world: $\theta \in \Theta = \{A, B\}$. Technology a is the optimal choice in state A and technology b is the optimal choice in state B.

The timing of this game is pictured in Figure 1. First, nature chooses $\theta \in \Theta$. Afterwards, period-one agents are born. The true state of the world is not revealed directly to agents. Instead, each agent receives a noisy signal about the true state of the world. This signal has both an idiosyncratic and an aggregate component of uncertainty. Next, each agent chooses between a and b without observing other agents' decisions. Payoffs are collected and agents die. In this way, each agent's decision is once-and-

[5] Schlag [1998] shows that the difference in the probabilities of switching from arm i to arm j and switching from arm j to arm i should be proportional to the difference in payoffs between arms j and i.

[6] The assumption that all agents are replaced in every period is not crucial. If a fraction of agents remain in every period, versions of my results still hold.

for-all. Then, period $t = 2$ starts. This period is different from period one since now there are past decisions available to observe. A new group of agents is born. Each of these period-two agents observes a signal s and an independent random sample of the decisions of period-one agents. As I describe later, the sample is characterized by ζ, the number of individuals in the sample choosing technology a, and N, the sample size. With these two sources of information, each agent decides between a and b, collects payoffs and dies. The timing of each subsequent period is identical to that of period $t = 2$.

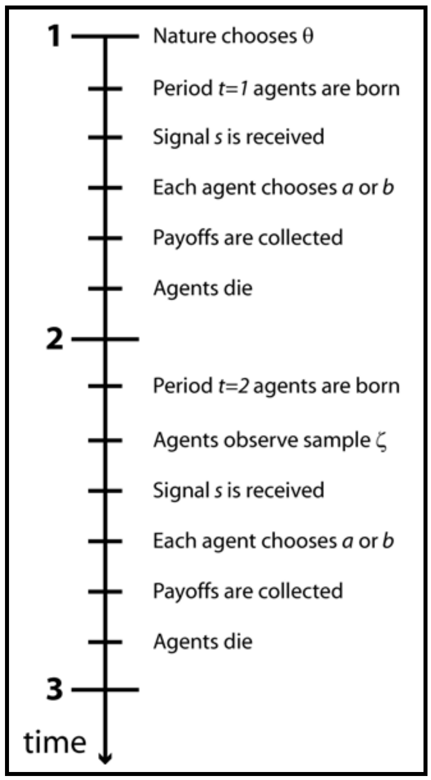

Figure 1: Timing of the Game

2.1 The Payoffs

Table 1 presents the structure of payoffs. In the present model, I assume $u(a, A) > u(b, A)$ and $u(b, B) > u(a, B)$. Moreover, I assume that both technologies are equally attractive ex-ante:

$$E[u(a)] = E[u(b)] \text{ or equivalently,}$$
$$\Pr(\theta = A)\, u(a, A) + \Pr(\theta = B)\, u(a, B) = \Pr(\theta = A)\, u(b, A) + \Pr(\theta = B)\, u(b, B) \tag{1}$$

Under symmetry assumption (1) agents are indifferent between technologies unless they receive fur-

		True State (θ)	
		A	B
Technology	a	$u(a, A)$	$u(a, B)$
	b	$u(b, A)$	$u(b, B)$

Table 1: General Payoff Function

ther information. This assumption substantially simplifies the analysis of the model. In fact, if equation (1) holds, agents behave as if $\Pr(\theta = A) = \Pr(\theta = B) = \frac{1}{2}$ and payoffs are as presented in Table 2.[7] Consequently, from now on I assume payoffs are as specified in Table 2 and that both technologies are ex-ante equally likely to be superior.

		True State (θ)	
		A	B
Technology	a	1	0
	b	0	1

Table 2: Payoff Function

2.2 The Signal

Agents receive noisy information about the true state of the world via a signal $s \in \{\alpha, \beta\}$. [8] Each agent receives an independent draw of s. In state A, the signal takes the value α with probability q. Symmetrically, in state B, the signal takes the value β with probability q. Table 3 summarizes the likelihood of signals given the state of the world.[9]

		$\Pr(s \mid A)$	$\Pr(s \mid B)$
Signal (s)	α	q	$1-q$
	β	$1-q$	q

Table 3: Likelihood of Signals

In each period some agents receive signal α while others receive signal β. Therein lies the *idiosyncratic* nature of the signal. To introduce *aggregate uncertainty*, I let the fraction q of individuals that receive the correct signal be random. Agents do not observe the value of q each period; they just observe either $s = \alpha$ or $s = \beta$. Let the aggregate shock q take only two values: $q \in Q = \{l, h\}$ with $0 < l < h < 1$. The draws of q are i.i.d. across periods with $p \equiv \Pr(q = h)$. I assume the signal is informative. By this I mean

[7]This equivalence is explained in detail in Appendix A.1.

[8]The results I present in this paper do not depend on the discreteness of the signal. If agents observe a signal $\tilde{s} = s + \epsilon_i$ where $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ and σ_ϵ is small, my results still hold.

[9]Note that conditional on the the state of the world, the signal is independent of the sample.

that $E[q] = ph + (1-p)l > \frac{1}{2}$, so that, on average, agents receive the right signal more often than not. To illustrate the restriction $E[q] > \frac{1}{2}$, the shaded area in Figure 2 shows all possible values for l and h for the case $p = \frac{1}{2}$.

Assume the true state of the world is A. At the start of each period, $q_t = h$ with probability p. Similarly, $q_t = l$ with probability $1-p$. Then, if $q_t = h$, each agent independently has probability h of receiving signal α and probability $1-h$ of receiving signal β. In the same way, if $q_t = l$, each agent independently has probability l of receiving signal α and probability $1-l$ of receiving signal β.

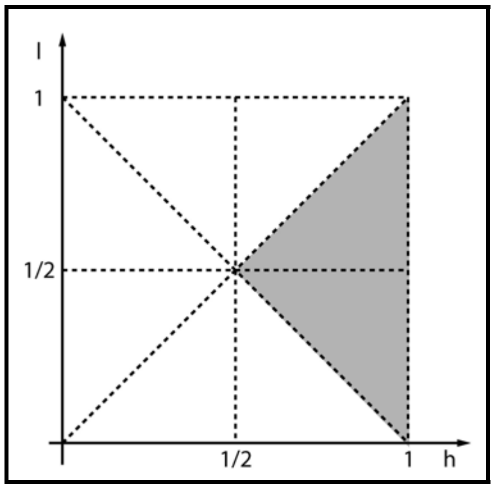

Figure 2: Possible Values for l and h when $p = \frac{1}{2}$

If the agent had no information other than the signal, his belief about the state would be summarized by $\Pr(A \mid \alpha) = \Pr(B \mid \beta) = E[q]$. Since agents combine information from the sample and the signal, it is useful to define the signal likelihood ratio $\psi(s)$ by

$$\psi(s) \equiv \frac{\Pr(s \mid A)}{\Pr(s \mid B)} = \begin{cases} E[q] / (1 - E[q]) & \text{if } s = \alpha \\ (1 - E[q]) / E[q] & \text{if } s = \beta \end{cases}$$

2.3 Sampling

Let x denote the fraction of agents choosing the superior technology. For example, if A is the true state of the world, x is the fraction of agents choosing a. At the beginning of his lifetime, each individual observes a sample of N individuals' choices in the previous period. I define ζ to be the number of those individuals that chose a. Sampling is assumed to be proportional. As a result, the likelihood of each sample is given by:

$$\Pr(\zeta \mid x, \theta) = \begin{cases} \binom{N}{\zeta} x^{\zeta} (1-x)^{N-\zeta} & \text{if } \theta = A \\ \binom{N}{\zeta} (1-x)^{\zeta} x^{N-\zeta} & \text{if } \theta = B \end{cases}$$

For notational simplicity, I define $\Pr(\zeta \mid x) \equiv \Pr(\zeta \mid x, A)$. As a result, $\Pr(\zeta \mid x, B) = \Pr(\zeta \mid 1-x)$, which I use later.

2.4 The Position of the System

Since there is a continuum of agents, the fraction of individuals that observes sample ζ is exactly $\Pr(\zeta \mid x, \theta)$. In the same way, the fraction of agents in the population who receive signal s is given exactly by $\Pr(s \mid \theta)$. Define $\mathbf{q}^t = (q_1, q_2, \ldots, q_t)$ to be the list of the signals' qualities (i.e., the realization of the aggregate shocks) from the beginning of time. Thus, in every period t, the position of the system is completely determined by the state variable $\mathbf{q}^t$. Consequently, I define $x_t : Q^t \to [0,1]$ to be the map from the state at time t to the fraction of agents choosing the superior technology. In period t, there are 2^t possible values for $\mathbf{q}^t$. Consequently, x_t can take 2^t possible values in that period.

2.5 The Decision Rule

Each agent makes a decision based on the information provided by the sample ζ and the signal s. Consequently, an agent born in period $t+1$ considers the following likelihood ratio when choosing a technology:[10]

$$\frac{\Pr(A \mid \zeta, s)}{\Pr(B \mid \zeta, s)} = \frac{\Pr(\zeta, s \mid A)}{\Pr(\zeta, s \mid B)} \tag{2}$$

The agent does not know the identity of the superior technology. In the same way, he knows neither the fraction x_t choosing the superior technology in the previous period nor the current value q_{t+1} for the aggregate shock. However, the agent knows all possible values for $x_t(\mathbf{q}^t)$ and that q_{t+1} is either h or l. Therefore:

$$\Pr(\zeta, s \mid \theta) = \sum_{\mathbf{q}^{t+1} \in Q^{t+1}} \Pr\left(\mathbf{q}^{t+1}\right) \Pr\left[\zeta, s \mid x_t(\mathbf{q}^t), q_{t+1}, \theta\right]$$

Conditioning on specific values for $x_t(\mathbf{q}^t)$, q_{t+1} and θ, the signal s and the sample ζ are independent. Consequently, $\Pr[\zeta, s \mid x_t(\mathbf{q}^t), q_{t+1}, \theta] = \Pr[\zeta \mid x_t(\mathbf{q}^t), \theta] \Pr(s \mid q_{t+1}, \theta)$. Moreover, since the q's are i.i.d.

[10] Recall I assumed without loss of generality that $\Pr(\theta = A) = \Pr(\theta = B)$.

over time, equation (2) can be rewritten in the following way, which separates the two sources of information:

$$\frac{\Pr\left(A \mid \zeta, s\right)}{\Pr\left(B \mid \zeta, s\right)} = \psi(s) \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr\left(\mathbf{q}^t\right) \Pr\left[\zeta \mid x_t(\mathbf{q}^t)\right]}{\sum_{\mathbf{q}^t \in Q^t} \Pr\left(\mathbf{q}^t\right) \Pr\left[\zeta \mid 1 - x_t(\mathbf{q}^t)\right]} \tag{3}$$

Agents choose technology a whenever $\Pr\left(A \mid \zeta, s\right) > \Pr\left(B \mid \zeta, s\right)$. Consequently, I define the following function:

DEFINITION. DECISION FUNCTION:

$$D_{t+1}\left(\zeta, s\right) = \begin{cases} 1 & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} > 1 \\ \sigma(\zeta, s) & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} = 1 \\ 0 & \text{if } \frac{\Pr(A|\zeta,s)}{\Pr(B|\zeta,s)} < 1 \end{cases}$$

This function delivers a value of 1 if technology a is strictly preferred and a value of 0 if technology b is strictly preferred. If the agent is indifferent, he randomizes: technology a is chosen with probability $\sigma(\zeta, s)$. I assume that the tie-breaking rule is symmetric in the following sense: $\sigma(N-\zeta, \beta) = 1-\sigma(\zeta, \alpha)$.[11] For some examples I present later, the randomization probabilities are relevant. In those cases, I specify some randomization rules.

2.6 The Evolution of the System

The model I present is symmetric with respect to the identity of the superior technology. From now on, for expositional clarity, let a be the superior technology. In other words, without loss of generality, assume that $\theta = A$. In that case, in period $t+1$, a fraction $\Pr\left[\zeta \mid x_t(\mathbf{q}^t)\right]$ observes sample ζ. Of those, a fraction q_{t+1} receives signal α and a fraction $(1 - q_{t+1})$ receives signal β. Consequently the fraction of agents choosing the superior technology in $t+1$ is given by:

$$x_{t+1}\left(\mathbf{q}^{t+1}\right) = \sum_{\zeta=0}^{N} \Pr\left[\zeta \mid x_t(\mathbf{q}^t)\right] \Big[q_{t+1} D_{t+1}\left(\zeta, \alpha\right) + \left(1 - q_{t+1}\right) D_{t+1}\left(\zeta, \beta\right)\Big] \tag{4}$$

3. Solution

3.1 Introduction

If $N > 2$, the optimal decision rule of rational agents becomes hard to derive. For this reason, I only provide analytical solutions for the model when $N = 1$ and $N = 2$. In the former case, I show that there

[11] This assumption leads to a tie-breaking rule consistent with the rest of the decision function, as shown in Appendix A.2.

is no social learning. Agents do not benefit from observing one agent from the previous period. As a result, herds do not arise.

In the case where $N = 2$, agents' decisions can be characterized in a simple way. Since $N = 2$ and $s \in \{\alpha, \beta\}$, there are only six possible pairs (ζ, s) an agent can receive. To characterize the decision function, I show first that a sample $\zeta = 1$ is uninformative. Consequently, if the agent observes $\zeta = 1$, the decision is based solely on the signal. Next, I consider the case in which the agent observes two individuals choosing technology a and a signal $s = \beta$; that is, the signal and the sample provide contradictory information. I show that in that case the sample is more informative than the signal; hence the agent chooses technology a. It follows that if $\zeta = 2$ but $s = \alpha$, the agent also chooses a. Finally, since the model is symmetric, a sample of two individuals choosing technology b leads the agent to choose b. The previous analysis allows me to characterize the evolution of the system in a tractable way. This characterization illustrates the dynamics that lead to herds. I conclude this section by presenting necessary and sufficient conditions for herds to occur if $N = 2$.

Finally, I study cases in which N is greater than 2. I show that the decision function may change over time. Moreover, I show that the decision function is not monotonic in ζ for some values of l, h and p. As a result, it is not possible to provide a generic analytical solution for the evolution of the system (4). Nevertheless, I provide examples that show that herds can occur for any odd $N \geq 3$.

3.2 Properties of the Decision Rule for any Sample Size

Before studying agents' behavior for specific sample sizes, I derive some properties of the decision rule that hold for any sample size. These properties simplify the study of agents' decisions. First, I show in Lemma 1 that the symmetry of the model leads to symmetry in the agents' decisions.

LEMMA 1. *If agents behave optimally, the decision function must satisfy:*

(a) $D_{t+1}(N - \zeta, \beta) = 1 - D_{t+1}(\zeta, \alpha)$

(b) If N is even, $D_{t+1}\left(\frac{N}{2}, \alpha\right) = 1$ *and* $D_{t+1}\left(\frac{N}{2}, \beta\right) = 0$

Moreover,

(c) $E\left[x_{t+1}(\mathbf{q}^{t+1})\right] \geq E\left[x_t(\mathbf{q}^t)\right]$

Proof: See Appendix A.2.

Since both technologies are equally attractive ex-ante, the fraction of agents choosing the right technology given $\mathbf{q}^t$ does not depend on the identity of the superior technology. As a result, a sample with the same number of agents choosing a and b is uninformative.

Next, I characterize agents' decisions in period $t = 1$. In that period individuals only observe a signal, since there are no agents from previous periods. Since the signal is informative ($E[q] > \frac{1}{2}$), agents follow it. The results from this fact are summarized in the following proposition:

PROPOSITION 2. *If agents choose optimally,*

(a) The decision function at time $t = 1$ is given by:

$$D(s) = \begin{cases} 1 & \textit{if } s = \alpha \\ 0 & \textit{if } s = \beta \end{cases}$$

(b) $E\left[x_1(\mathbf{q}^1)\right] = E[q]$ and
$E\left[x_t(\mathbf{q}^t)\right] \geq E[q] \ \forall t$

Proof. (a) is immediate since the signal is the only source of information. Regarding (b), note that $E\left[x_1(\mathbf{q}^1)\right] = px_1(h) + (1-p)x_1(l) = ph + (1-p)l = E[q]$. Then, by Lemma 1(c), $E\left[x_t(\mathbf{q}^t)\right]$ is (weakly) increasing over time. Consequently, $E\left[x_t(\mathbf{q}^t)\right] \geq E[q]$ also holds for all $t > 1$. ■

3.3 The Case $N = 1$

If each individual observes the behavior of only one agent from the previous period, there are only four possible combinations of samples and signals. In Proposition 3, I describe agents' optimal behavior in this setting. First, I show that agents that receive a signal that confirms the observed sample follow that observation. Then, I show that if the signal contradicts the sample, agents are indifferent between technologies. As a result, there is no social learning and the system does not converge.

PROPOSITION 3. *For $t > 1$, if agents behave optimally,*

(a) The decision function is given by:

$$D(\zeta, s) = \begin{cases} 1 & \textit{if } (\zeta, s) = (1, \alpha) \\ \sigma(1,\beta) & \textit{if } (\zeta, s) = (1, \beta) \\ 1 - \sigma(1,\beta) & \textit{if } (\zeta, s) = (0, \alpha) \\ 0 & \textit{if } (\zeta, s) = (0, \beta) \end{cases}$$

(b) The evolution of the system is described by:

$$x_{t+1}\left(\mathbf{q}^{t+1}\right) = \sigma(1,\beta)x_t(\mathbf{q}^t) + \left[1 - \sigma(1,\beta)\right] q_{t+1}$$

(c) For any choice of $\sigma(1,\beta)$, there is no social learning on average:

$$E[x_t(\mathbf{q}^t)] = E[q] \ \forall t$$

(d) Herds do not occur.

Proof: See Appendix A.3.

In this case, the informational content of the sample is always exactly as strong as the informational content of the signal. In the next section, I show that if $N = 2$ some samples are more powerful than the signal. This leads to social learning.

3.4 The Case $N = 2$

In this case, agents observe a sample of size 2 and a signal $s \in \{\alpha, \beta\}$. This setting results in six possible combinations of sample and signal: $(0, \alpha)$, $(0, \beta)$, $(1, \alpha)$, $(1, \beta)$, $(2, \alpha)$ and $(2, \beta)$. In what follows, I explain how to determine the value of the decision function for each possible combination.

To begin with, note that by Lemma 1(a), $D(2, \alpha) = 1 - D(0, \beta)$, $D(2, \beta) = 1 - D(0, \alpha)$ and $D(1, \alpha) = 1 - D(1, \beta)$. Consequently, it suffices to find values of the decision function for only three of all six possible combinations.

Next, by Lemma 1(b), if $\zeta = 1$, the sample is uninformative. After observing a mixed sample (one that has the same number of people choosing a and b) agents base their decision solely on the signal. As a result, only full samples (those in which all agents are choosing the same product) remain to be analyzed. In particular, it is of interest to analyze the case in which the sample and the signal provide contradictory information. If an agent observed two people choosing technology a and also received a signal β, how would that agent react?

The following Lemma provides a simple sufficient condition which guarantees that after observing $\zeta = 0$ or $\zeta = 2$, the sample prevails over the signal in the determination of the agents' decisions.

LEMMA 4. *If agents behave optimally,*

$$E\left[x_t(\mathbf{q}^t)\right] \geq E\left[q\right] \quad \textit{implies} \quad D\left(2, \beta\right) = 1$$

Proof: See Appendix A.4.

In Proposition 2, I show that $E\left[x_t(\mathbf{q}^t)\right] \geq E[q]$ $\forall t$. Consequently, the conclusion of Lemma 4 holds for all $t \geq 2$. Theorem 5 completes the characterization of the decision function.

THEOREM 5. *For $t > 1$, if agents behave optimally,*

(a) The decision function is given by:

$$D(\zeta, s) = \begin{cases} 1 & \textit{if } (\zeta, s) \in \{(2, \alpha), (2, \beta), (1, \alpha)\} \\ 0 & \textit{if } (\zeta, s) \in \{(1, \beta), (0, \alpha), (0, \beta)\} \end{cases}$$

(b) *The evolution of the system is described by:*

$$x_{t+1}\left(\mathbf{q}^{t+1}\right) = 2q_{t+1}x_t(\mathbf{q}^t) + (1-2q_{t+1})\left[x_t(\mathbf{q}^t)\right]^2 \tag{5}$$

(c) *Let* $l < \frac{1}{2}$. *If* $x_t(\mathbf{q}^t) \notin \{0,1\}$, *then:*

(a) *If* $q_t = h$, *then* $x_{t+1}(\mathbf{q}^{t+1}) > x_t(\mathbf{q}^t)$.

(b) *If* $q_t = l$, *then* $x_{t+1}(\mathbf{q}^{t+1}) < x_t(\mathbf{q}^t)$.

That is, positive aggregate shocks increase the fraction of agents choosing the superior technology while negative aggregate shocks have the opposite effect.

Proof: See Appendix A.5.

The previous Theorem provides a closed form solution for the evolution of the system. Next, I show that herds occur with positive probability. To do this, I utilize the following Lemma, from Ellison and Fudenberg [1995]:

LEMMA 6. (ELLISON AND FUDENBERG [1995]). *Let* x_t *be a Markov process on* $(0,1)$ *with:*

$$x_{t+1} = \begin{cases} H_1(x_t) & \text{with probability } p \\ H_2(x_t) & \text{with probability } 1-p \end{cases}$$

Suppose that $H_i(x_t) = \gamma_i x_t + o(x_t)$, *with* $\gamma_2 < 1 < \gamma_1$.

(a) *If* $E[\log(\gamma_i)] = p\log(\gamma_1) + (1-p)\log(\gamma_2) > 0$ *then* x_t *cannot converge to* 0 *with positive probability.*

(b) *If* $E[\log(\gamma_i)] = p\log(\gamma_1) + (1-p)\log(\gamma_2) < 0$ *then* $\Pr\{x_t \to 0 \mid x_0 \le \delta\} \ge \epsilon$ *for some strictly positive* δ *and* ϵ.

To see why Lemma 6 holds, consider the simpler case of a linear Markov process $\tilde{x}_{t+1} = \gamma_t \tilde{x}_t$ (where $\gamma_t = \gamma_1$ with probability p and $\gamma_t = \gamma_2$ with probability $1-p$). To analyze the long run behavior of $\tilde{x}_t$, define the associated log-process $\log(\tilde{x}_{t+1}) = \log(\tilde{x}_t) + \log(\gamma_t)$ so that $\log(\tilde{x}_t) = \log(\tilde{x}_0) + \sum_{\tau=0}^{t}\log(\gamma_\tau)$. By the strong law of large numbers, $\log(\tilde{x}_t) \to -\infty$ with probability 1 if $E[\log(\gamma_t)] = p\log(\gamma_1) + (1-p)\log(\gamma_2) < 0$, which implies that $\tilde{x}_t \to 0$ with probability 1. Lemma 6 considers non-linear processes x_t that are approximately linear when $x \approx 0$, showing that they still converge to 0 with positive probability if x_0 is sufficiently close to 0 and $E[\log(\gamma_t)] < 0$.

To see the relationship between the behavior of the system in this paper and Lemma 6, I rewrite (5) in the following way:

$$x_{t+1} = \begin{cases} H_1(x_t) = 2hx_t + (1-2h)\,x_t^2 & \text{with probability } p \\ H_2(x_t) = 2lx_t + (1-2l)\,x_t^2 & \text{with probability } 1-p \end{cases}$$

so that $\gamma_2 = 2l$ and $\gamma_1 = 2h$. Note that $(1-2q)\, x_t^2 = o(x_t)$ for small enough x_t. Consequently, I can state my herding condition:

THEOREM 7.

(a) *If $l \geq \frac{1}{2}$, then herds never occur* ($\Pr\{x_t \to 0\} = 0$)*.*

(b) *If $p\log(h) + (1-p)\log(l) > \log\left(\frac{1}{2}\right)$, then herds never occur* ($\Pr\{x_t \to 0\} = 0$)*.*

(c) *If $p\log(h) + (1-p)\log(l) < \log\left(\frac{1}{2}\right)$, then herds occur with positive probability* ($\Pr\{x_t \to 0\} > 0$)*.*

(d) *For all values of $E[q] < 1$, there exist parameters p, h, l such that herds occur.*

Proof. First, if $l \geq \frac{1}{2}$, it is easy to show that $\Pr\{x_t \to 1\} = 1$. Then, x_t does not converge to 0.

Regarding (b), by Lemma 6, $\Pr\{x_t \to 0\} = 0$.

Next, regarding (c), again by Lemma 6, $\Pr\{x_t \to 0 \mid x_0 \leq \delta\} \geq \epsilon$ for some positive ϵ and δ. Note that x_t can get to any $x \in (0,1)$ in finite time with positive probability from any $x_1 \neq \{0,1\}$. Then, no matter the starting point, $\Pr\{x_t \to 0\} > 0$.

Finally regarding (d), fix some value $E[q] < 1$ and let $h = 1$ so that p and l are related by: $p = \frac{E[q]-l}{1-l}$. Herds occur with positive probability if and only if $(E[q]-l)\log(2) + (1-E[q])\log(2l) < 0$. As $l \to 0$, the first term converges to $E[q]\log(2)$ whereas the second converges to $-\infty$. Consequently, there always exist p, h, l such that herds occur. ■

To understand Theorem 7, suppose that almost everyone in the population chooses technology b (i.e., $x \approx 0$), so that $\zeta = 0$ is the most likely sample. Agents who observe it choose b, so they move with the herd. Next, note that the likelihood of $\zeta = 2$ relative to $\zeta = 1$ approaches 0 as x gets close to 0. Thus, I can disregard agents who observe $\zeta = 2$ and focus only on those who see $\zeta = 1$. Agents get $\zeta = 1$ with a probability of approximately $2x_t$, and in this event choose a only after observing signal α. As a result, $x_{t+1}(\mathbf{q}^{t+1}) \approx 2q_{t+1}x_t(\mathbf{q}^t)$. As stated in the discussion of Lemma 6, l and h affect the behavior of x_t through their logs. When $q_t = h$, x_t can at most double (if h is close to 1). But when $q_t = l$, a low l reduces x_t by a larger proportion (for example, if $l = 0.1$, x_t is divided by approximately 5 when $q_t = l$). In other words, a high h has a weaker effect than a low l.

An alternative way to see this herding condition is to write $\log\left[x_{t+1}(\mathbf{q}^{t+1})\right] \approx \log(2q_{t+1}) + \log\left[x_t(\mathbf{q}^t)\right]$. The long run value of $\log(x_t)$ depends on $E[\log(2q)]$. A low enough value for l makes $E[\log(2q)] < 0$; in this case $\log(x_t) \to -\infty$ with positive probability, which implies $x_t \to 0$. Thus, regardless of how strong the information is on average, herds can still occur with positive probability.

For the case when $p = \frac{1}{2}$, herds occur with positive probability if and only if $l \times h < \frac{1}{4}$. The dark shaded area in Figure 3 shows the combinations of l and h such that herds occur.

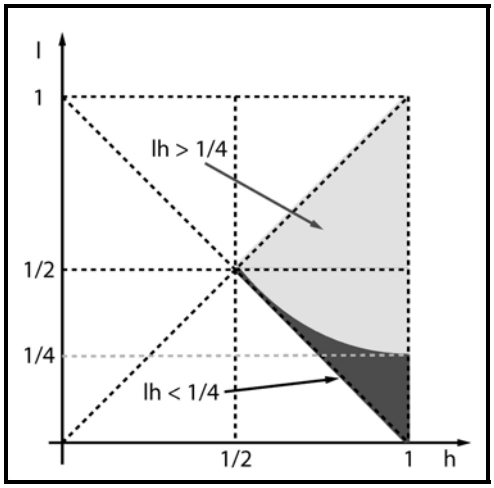

Figure 3: When $p = \frac{1}{2}$, herds occur in the dark shaded area

3.5 The Case $N \geq 3$

As I showed in the previous section, if the sample is of size 2 the optimal decision rule does not change over time. However, if each agent observes more than 2 individuals, the optimal decision rule may change over time; furthermore, it may exhibit counterintuitive traits. I do not provide a general characterization of agents' optimal behavior if $N \geq 3$. In what follows, I present two examples to illustrate the difficulties of obtaining a general characterization of the decision rule and then describe a symmetric environment in which herds occur for sample sizes greater than 2.

EXAMPLE 1. *A sample with more people choosing a given technology can make the agent less inclined to choose that same technology ($N = 3, l = 0.1, h = 0.95$ and $p = \frac{1}{2}$).*

Agents follow their signal in the first period. In period $t = 2$, the optimal decision rule for Example 1 is given by:

$$D_2(\zeta, s) = \begin{cases} 1 & \text{if } \zeta = 3 \\ 0 & \text{if } \zeta = 2 \\ 1 & \text{if } \zeta = 1 \\ 0 & \text{if } \zeta = 0 \end{cases}$$

In period $t = 2$, observing a sample with only *one* person choosing a leads the agent to choose a, whereas observing *two* people choosing a leads the agent to choose b.[12]

The reason for this counterintuitive behavior can be explained as follows. When $N = 2$, $\Pr(\zeta = 2 \mid x)$ is strictly increasing in x. Now, when $N \geq 3$, $\Pr(\zeta = 2 \mid x)$ is no longer monotonic in x. Indeed, for $N = 3$, $\Pr(\zeta = 2 \mid x) = 0$ for $x = 0$; then the probability increases until it attains a maximum at $x = \frac{2}{3}$

[12]In this example, the lack of monotonicity occurs in period $t = 2$. In the periods that follow, the decision function varies.

and then decreases until $\Pr(\zeta = 2 \mid x) = 0$ for $x = 1$. In this way, when $N = 3$, a higher fraction in the population choosing a may actually make observing $\zeta = 2$ less likely. As a result, a sample with a majority of agents choosing a may indicate that a state where a higher fraction of the population chooses a is less likely than another one where a lower fraction chooses a.

On average there is a higher fraction choosing a when A is the true state of the world than there is when B is the true state of the world. This may cause the agent to choose b after observing $\zeta = 2$. This is the case in Example 1. When a is the superior technology, the fraction of the population choosing a is either 0.95 or 0.10, with equal probability. Symmetrically, if b is the superior technology, the fraction of agents choosing a is either 0.05 or 0.90. The agent observes $\zeta = 2$ and infers the fraction of people choosing a. The likelihood of being in a state in which x is 0.05 or 0.10 is practically negligible. A state with $x = 0.90$ is almost twice as likely as one with $x = 0.95$, which makes state of the world B significantly more likely than A. This leads agents to choose b after observing $\zeta = 2$ in the second period of Example 1. Finally, by Lemma 1(b), the agent chooses a after observing $\zeta = 1$.

EXAMPLE 2. *The consequences of a given sample and signal pair can alternate over time* ($N = 3, l = 0.4, h = 0.8$ *and* $p = \frac{1}{2}$).

For $t \geq 2$ (ζ, s)	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$	$t = 7$
$(3, s)$	1	1	1	1	1	1
$(2, \alpha)$	1	1	1	1	1	1
$(2, \beta)$	**0**	**1**	**0**	**1**	**0**	**1**
$(1, \alpha)$	1	0	1	0	1	0
$(1, \beta)$	0	0	0	0	0	0
$(0, s)$	0	0	0	0	0	0

Table 4: Decision Function for Example 2.

Table 4 presents the optimal decision rule for Example 2.[13] I numerically calculate the optimal decision rule for the first 7 periods.[14] In this case, the decision rule changes over time. A sample $\zeta = 2$ is more informative than the signal for periods $3, 5$ and 7. For periods $2, 4$ and 6; the opposite holds. As time goes by, the unconditional expectation $E[x_t(\mathbf{q}^t)]$ increases. However, this increase does not necessarily make the agent more willing to follow a sample $\zeta = 2$. The lack of monotonicity of $\Pr(\zeta = 2 \mid x)$ in x is once again the reason the decision rule fluctuates.

Examples 1 and 2 show that the optimal decision rule may change over time. In spite of that, Propo-

[13] Period $t = 1$ is not presented in this table. As discussed on Proposition 3, agents do not observe any sample in the first period and follow the signal.

[14] In each period, there are 2^t possible values for $x_t(\mathbf{q}^t)$. As a result, numerically calculating the optimal decision rule is computationally demanding, even for low values of t. In numerical calculations, I obtained at most the 20 first periods. I do not know if the alternation presented in Table 4 continues forever.

sition 8 shows that the optimal decision rule can be characterized for some parameter values.

PROPOSITION 8. *Let $N \geq 3$ be odd. If $1 - l > h > 1 - \frac{1}{N}$, p is sufficiently large, l is sufficiently small and agents behave optimally,*

(a) In period $t = 1$, agents follow the signal. Then, for all subsequent periods, agents disregard the signal and follow the most observed choice in their sample:

$$D(\zeta, s) = \begin{cases} 1 & \text{if } \zeta \geq \frac{N+1}{2} \\ 0 & \text{if } \zeta \leq \frac{N-1}{2} \end{cases}$$

(b) The evolution of the system is described by:

$$x_{t+1}(\mathbf{q}^{t+1}) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(\mathbf{q}^t)\right) \quad \forall t > 1$$

(c) Herds occur when $q_1 = l$, and so have probability $1 - p$.

Proof: See Appendix A.6.

Proposition 8 shows that when $N \geq 3$ is odd, there are values of p, l and h such that agents find it optimal to disregard their signals after period $t = 1$. In the first period, the level of aggregate uncertainty determines the fraction choosing the superior technology. Starting in period $t = 2$, agents follow the most observed choice in their sample, so in each period there are only two possible values for x_t: $x_t(h)$ and $x_t(l)$, determined by the level of aggregate uncertainty in the first period. If $q_1 = h$, more than half of the agents in the population choose the superior technology in the first period. In subsequent periods, agents follow the most observed choice in their sample, so the fraction choosing the superior technology increases over time and converges to 1. On the other side, if $q_1 = l$, the fraction choosing the inferior technology increases and converges to 1. Since more than half of the first-period agents choose the inferior technology and subsequent agents follow the most observed choice in their sample, the effect of a bad shock in the first period "snowballs", creating a herd. Herds occur with probability $1 - p$, which is the probability that $q_1 = l$.

Agents disregard their private signal when the informational content of the sample outweighs that of the signal. Under the assumptions of Proposition 8, it is easy to show that this is the case when agents observe a full sample ($\zeta = N$). Now, when a is chosen by most but not all in the sample ($\frac{N+1}{2} \leq \zeta \leq N - 1$), the analysis becomes harder due to the lack of monotonicity of $\Pr(\zeta \mid x)$ mentioned before. The information given by the sample is captured by the likelihood ratio,

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} = \frac{p \Pr[\zeta \mid x_t(h)] + (1-p)\Pr[\zeta \mid x_t(l)]}{p \Pr[\zeta \mid 1 - x_t(h)] + (1-p)\Pr[\zeta \mid 1 - x_t(l)]} \tag{6}$$

Since $N \geq 3$, $\Pr(\zeta \mid x)$ is not monotonic in x for $\frac{N+1}{2} \leq \zeta \leq N-1$. Still, in the environment of Proposition 8, the information contained in any sample in which a is observed more often than b outweighs the information contained in the signal $s = \beta$. To see why this is true, ignore the terms $(1-p)\Pr[\zeta \mid x_t(l)]$ and $p \Pr[\zeta \mid 1 - x_t(h)]$ in (6), since they are small relative to the others. Regarding the remaining terms, note that if $\frac{N+1}{2} \leq \zeta \leq N-1$, $\Pr(\zeta \mid x)$ is strictly decreasing in x when $x > 1 - \frac{1}{N}$. The assumption $1 - l > h > 1 - \frac{1}{N}$ guarantees that $1 - x_t(l) > x_t(h) > 1 - \frac{1}{N}$, which implies that $\Pr[\zeta \mid 1 - x_t(l)] < \Pr[\zeta \mid x_t(h)]$. This observation allows me to obtain a lower bound for the likelihood ratio. In Appendix A.6, I show that if $p \approx 1$ and $l \approx 0$, this bound implies that the information provided in the sample outweighs the information provided by the signal $s = \beta$, and therefore that agents choose technology a when it is the most observed choice, regardless of the signal. By symmetry, agents choose b after observing a sample where b is the most observed choice. In this way, I conclude that the signal plays no role starting in period $t = 2$.

4. Conclusion

I have presented a model of word-of-mouth learning under aggregate uncertainty. In this model, rational agents choose between two competing technologies. Agents learn from past agents' behavior and from a signal whose quality depends on random aggregate shocks.

I derive the agents' optimal decision rule for the case in which each individual observes one agent from the previous period. I show there is no social learning in that case. Next, I derive the agents' optimal decision rule if two agents are observed. In that case, I show that agents base their decision in part on the behavior of past individuals. As a result, bad choices can be perpetuated. I find necessary and sufficient conditions for agents to herd on the inferior technology with positive probability, and I show that herds can occur even with a high average quality of the signal.

Finally, I present examples that show that the decision function for rational agents may change over time if the sample size exceeds two. Consequently, I do not fully characterize the decision function in that case. However, I do present parameter values under which herds occur for any odd sample size greater than two.

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A. Proofs

A.1 Normalization of Payoffs

Under the payoffs presented in Table 1, an individual chooses technology a as long as:

$$\begin{aligned}
E\left[u(a) \mid \zeta, s\right] &> E\left[u(b) \mid \zeta, s\right] \\
\Pr(A \mid \zeta, s)u(a, A) + \Pr(B \mid \zeta, s)u(a, B) &> \Pr(A \mid \zeta, s)u(b, A) + \Pr(B \mid \zeta, s)u(b, B) \\
\Pr(A \mid \zeta, s)\left[u(a, A) - u(b, A)\right] &> \Pr(B \mid \zeta, s)\left[u(b, B) - u(a, B)\right] \\
\Pr(\zeta, s \mid A)\Pr(\theta = A)\left[u(a, A) - u(b, A)\right] &> \Pr(\zeta, s \mid B)\Pr(\theta = B)\left[u(b, B) - u(a, B)\right] \\
\frac{\Pr(\zeta, s \mid A)}{\Pr(\zeta, s \mid B)} &> \frac{\Pr(\theta = B)\left[u(b, B) - u(a, B)\right]}{\Pr(\theta = A)\left[u(a, A) - u(b, A)\right]}
\end{aligned}$$

Under symmetry assumption (1), agents choose technology a as long as:

$$\frac{\Pr(\zeta, s \mid A)}{\Pr(\zeta, s \mid B)} > 1$$

Consequently, the behavior of agents is identical for any payoffs and probabilities such that equation (1) holds. In particular, for the payoffs presented in Table 2, equation (1) holds.

A.2 Lemma 1. Proof

Proof of Lemma 1.

(a) Note that $\psi(\alpha) = \frac{E[q]}{1-E[q]} = \left(\frac{1-E[q]}{E[q]}\right)^{-1} = \frac{1}{\psi(\beta)}$ and that $\Pr\left(\zeta \mid x\right) = \Pr\left(N - \zeta \mid 1 - x\right)$. Consequently:

$$\begin{aligned}
\frac{\Pr\left(A \mid \zeta, \alpha\right)}{\Pr\left(B \mid \zeta, \alpha\right)} &= \psi\left(\alpha\right)\frac{\sum_{\mathbf{q}^t \in Q^t}\Pr(\mathbf{q}^t)\Pr\left[\zeta \mid x_t(\mathbf{q}^t)\right]}{\sum_{\mathbf{q}^t \in Q^t}\Pr(\mathbf{q}^t)\Pr\left[\zeta \mid 1 - x_t(\mathbf{q}^t)\right]} \\
&= \frac{1}{\psi\left(\beta\right)\dfrac{\sum_{\mathbf{q}^t \in Q^t}\Pr(\mathbf{q}^t)\Pr\left[N - \zeta \mid x_t(\mathbf{q}^t)\right]}{\sum_{\mathbf{q}^t \in Q^t}\Pr(\mathbf{q}^t)\Pr\left[N - \zeta \mid 1 - x_t(\mathbf{q}^t)\right]}} \\
&= \left(\frac{\Pr\left(A \mid N - \zeta, \beta\right)}{\Pr\left(B \mid N - \zeta, \beta\right)}\right)^{-1}
\end{aligned}$$

Now, if $\frac{\Pr(A|\zeta,\alpha)}{\Pr(B|\zeta,\alpha)} > 1$, then $D_{t+1}\left(\zeta, \alpha\right) = 1$ and $\frac{\Pr(A|N-\zeta,\beta)}{\Pr(B|N-\zeta,\beta)} < 1$. Thus, $D_{t+1}\left(N - \zeta, \beta\right) = 0$. In the same way, if $\frac{\Pr(A|\zeta,\alpha)}{\Pr(B|\zeta,\alpha)} < 1$, then $D_{t+1}\left(\zeta, \alpha\right) = 0$ and $\frac{\Pr(A|N-\zeta,\beta)}{\Pr(B|N-\zeta,\beta)} > 1$. Thus, $D_{t+1}\left(N - \zeta, \beta\right) = 1$. Regarding the indifference case, I assumed: $\sigma(N - \zeta, \beta) = 1 - \sigma(\zeta, \alpha)$.

As a result:

$$D_{t+1}\left(N - \zeta, \beta\right) = 1 - D_{t+1}\left(\zeta, \alpha\right)$$

(b) Since N is even, $\frac{N}{2} = N - \frac{N}{2}$. Consequently, $\Pr\left[\frac{N}{2} \mid x_t(\mathbf{q}^t)\right] = \Pr\left[N - \frac{N}{2} \mid x_t(\mathbf{q}^t)\right]$. Equation (3) then implies that the sample provides no information,

$$\frac{\Pr\left(A \mid \frac{N}{2}, s\right)}{\Pr\left(B \mid \frac{N}{2}, s\right)} = \psi(s)$$

In this way, the agent bases his decision solely on the signal:

$$D_{t+1}\left(\tfrac{N}{2},\alpha\right)=1 \text{ and } D_{t+1}\left(\tfrac{N}{2},\beta\right)=0$$

(c) Finally, define the average utility in period t by:

$$\overline{U}_t\left(X_t\right)=\frac{1}{2^t}\sum_{\mathbf{q}^t\in Q^t}x_t(\mathbf{q}^t)$$

Next, by Lemma 1 from Banerjee and Fudenberg [2004]:

$$\begin{aligned}\overline{U}_{t+1}\left(X_{t+1}\right)&\geq\overline{U}_t\left(X_t\right)\\ \frac{1}{2^{t+1}}\sum_{\mathbf{q}^{t+1}\in Q^{t+1}}x_{t+1}(\mathbf{q}^{t+1})&\geq\frac{1}{2^t}\sum_{\mathbf{q}^t\in Q^t}x_t(\mathbf{q}^t)\\ E\left[x_{t+1}(\mathbf{q}^{t+1})\right]&\geq E\left[x_t(\mathbf{q}^t)\right]\quad\blacksquare\end{aligned}$$

A.3 Proposition 3. Proof

Proof of Proposition 3. First, I prove this auxiliary Proposition.

PROPOSITION 9. *If $N=1$ and $E[x_t(\mathbf{q}^t)]=E[q]$, then:*

(a) The decision function is as stated in Proposition 3(a).

(b) The evolution of the system is given by:

$$x_{t+1}\left(\mathbf{q}^{t+1}\right)=\sigma(1,\beta)x_t(\mathbf{q}^t)+\left[1-\sigma(1,\beta)\right]q_{t+1}$$

(c) $E[x_{t+1}(\mathbf{q}^{t+1})]=E[q]$

Proof.

(a) If the agent observes $\zeta=1$, the likelihood ratio is given by:

$$\frac{\Pr\left(A\mid 1,s\right)}{\Pr\left(B\mid 1,s\right)}=\psi(s)\frac{\sum_{\mathbf{q}^t\in Q^t}\Pr\left(\mathbf{q}^t\right)x_t(\mathbf{q}^t)}{\sum_{\mathbf{q}^t\in Q^t}\Pr\left(\mathbf{q}^t\right)\left[1-x_t(\mathbf{q}^t)\right]}=\psi(s)\frac{E[q]}{1-E[q]}$$

Consequently,

$$\frac{\Pr\left(A\mid 1,\alpha\right)}{\Pr\left(B\mid 1,\alpha\right)}>1\quad\text{and}\quad\frac{\Pr\left(A\mid 1,\beta\right)}{\Pr\left(B\mid 1,\beta\right)}=1$$

Then, the decision function is given by: $D(1,\alpha)=1, D(1,\beta)=\sigma(1,\beta)$ and by Lemma 1(a), $D(0,\alpha)=1-\sigma(1,\beta)$ and $D(0,\beta)=0$.

(b) It is straightforward from (a) to show:

$$\begin{aligned}x_{t+1}(\mathbf{q}^{t+1})&=\Pr\left[1\mid x_t(\mathbf{q}^t)\right]q_{t+1}D(1,\alpha)+\Pr\left[1\mid x_t(\mathbf{q}^t)\right](1-q_{t+1})D(1,\beta)\\&\quad+\Pr\left[0\mid x_t(\mathbf{q}^t)\right]q_{t+1}D(0,\alpha)+\Pr\left[0\mid x_t(\mathbf{q}^t)\right](1-q_{t+1})D(0,\beta)\\&=x_t(\mathbf{q}^t)q_{t+1}+x_t(\mathbf{q}^t)(1-q_{t+1})\sigma(1,\beta)+\left[1-x_t(\mathbf{q}^t)\right]q_{t+1}\left[1-\sigma(1,\beta)\right]\end{aligned}$$

$$= x_t(\mathbf{q}^t)q_{t+1} + \left[1 - x_t(\mathbf{q}^t)\right] q_{t+1} + \Big[x_t(\mathbf{q}^t)(1 - q_{t+1}) - \left[1 - x_t(\mathbf{q}^t)\right] q_{t+1}\Big]\sigma(1, \beta)$$
$$= q_{t+1} + \left[x_t(\mathbf{q}^t) - q_{t+1}\right] \sigma(1, \beta)$$
$$= \sigma(1, \beta)x_t(\mathbf{q}^t) + [1 - \sigma(1, \beta)] q_{t+1} \tag{7}$$

(c) Finally, the expected value of (7) is given by,

$$\begin{aligned} E[x_{t+1}(\mathbf{q}^{t+1})] &= E\left[\sigma(1, \beta)x_t(\mathbf{q}^t) + [1 - \sigma(1, \beta)] q_{t+1}\right] \\ &= \sigma(1, \beta)E\left[x_t(\mathbf{q}^t)\right] + [1 - \sigma(1, \beta)] E[q_{t+1}] \\ &= \sigma(1, \beta)E[q] + [1 - \sigma(1, \beta)] E[q] \\ &= E[q] \ \blacksquare \end{aligned}$$

Given Proposition 9, I prove Proposition 3 by induction. As shown in Proposition 2, agents follow the signal in the first period and $E\left[x_t(\mathbf{q}^1)\right] = E[q]$. Then, by Proposition 9, all results from Proposition 3 hold for period $t = 2$. Since $E\left[x_t(\mathbf{q}^2)\right] = E[q]$ also, then all results hold for period $t = 3$. By induction, the results are true for any period t.

Finally, regarding (d), note that $l \le x_t(\mathbf{q}^t) \le h$ for all possible values of $\sigma(1, \beta)$. Consequently (7) can never reach 0 or 1. Therefore, herds do not occur. ■

A.4 Lemma 4. Proof

Proof of Lemma 4. First, I show that:

$$\frac{E[x_t(\mathbf{q}^t)^2]}{E\Big[[1 - x_t(\mathbf{q}^t)]^2\Big]} \ge \frac{E[x_t(\mathbf{q}^t)]}{E\left[1 - x_t(\mathbf{q}^t)\right]} \quad \text{if } E[x_t(\mathbf{q}^t)] > \frac{1}{2} \tag{8}$$

Proof.

$$\begin{aligned} E[x_t(\mathbf{q}^t)^2]E\left[1 - x_t(\mathbf{q}^t)\right] &\ge E[x_t(\mathbf{q}^t)]\left[1 + E[x_t(\mathbf{q}^t)^2] - 2E[x_t(\mathbf{q}^t)]\right] \\ E[x_t(\mathbf{q}^t)^2] - 2E[x_t(\mathbf{q}^t)]E[x_t(\mathbf{q}^t)^2] &\ge E[x_t(\mathbf{q}^t)] - 2E[x_t(\mathbf{q}^t)]E[x_t(\mathbf{q}^t)] \\ E[x_t(\mathbf{q}^t)^2]\left(1 - 2E[x_t(\mathbf{q}^t)]\right) &\ge E[x_t(\mathbf{q}^t)]\left(1 - 2E[x_t(\mathbf{q}^t)]\right) \\ \left(E[x_t(\mathbf{q}^t)^2] - E[x_t(\mathbf{q}^t)]\right)\left(1 - 2E[x_t(\mathbf{q}^t)]\right) &\ge 0 \end{aligned}$$

which is true since $E[x_t(\mathbf{q}^t)] > \frac{1}{2}$ and $x_t(\mathbf{q}^t) - x_t(\mathbf{q}^t)^2 \ge 0$. ■

Next, note that:

$$\frac{E[x_t(\mathbf{q}^t)]}{E\left[1 - x_t(\mathbf{q}^t)\right]} \ge \frac{E[q]}{1 - E[q]} \quad \text{if } E[x_t(\mathbf{q}^t)] \ge E[q] \tag{9}$$

Given (8) and (9),

$$\frac{E[x_t(\mathbf{q}^t)^2]}{E\Big[[1 - x_t(\mathbf{q}^t)]^2\Big]} \ge \frac{E[q]}{1 - E[q]} \quad \text{if } E[x_t(\mathbf{q}^t)] \ge E[q] > \frac{1}{2}$$

In fact, the previous statement holds with equality only if both $E[x_t(\mathbf{q}^t)] = E[q]$ and $x_t(\mathbf{q}^t) \in \{0, 1\}$

for all $\mathbf{q}^t \in Q^t$. In the first period, $x_1(\mathbf{q}^1) \notin \{0,1\}$. Starting in the second period[15], $E[x_t(\mathbf{q}^t)] > E[q]$. Consequently, I can assume $\frac{E[x_t(\mathbf{q}^t)^2]}{E\left[[1-x_t(\mathbf{q}^t)]^2\right]} > \frac{E[q]}{1-E[q]}$. As a result,

$$\frac{\Pr(A \mid 2,\beta)}{\Pr(B \mid 2,\beta)} = \frac{1-E[q]}{E[q]} \frac{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t)\left(x_t(\mathbf{q}^t)\right)^2}{\sum_{\mathbf{q}^t \in Q^t} \Pr(\mathbf{q}^t)\left(1-x_t(\mathbf{q}^t)\right)^2} = \frac{1-E[q]}{E[q]} \frac{E\left[x_t(\mathbf{q}^t)^2\right]}{E\left[[1-x_t(\mathbf{q}^t)]^2\right]} > 1 \ \blacksquare$$

A.5 Theorem 5. Proof

Proof of Theorem 5.

(a) By Lemma 1(b), $D_{t+1}(1,\alpha) = 1$ and $D_{t+1}(1,\beta) = 0$. By Lemma 4, $D_{t+1}(2,\beta) = 1$. Since $E[q] > \frac{1}{2}$, $\frac{\Pr(A|2,\alpha)}{\Pr(B|2,\alpha)} > \frac{\Pr(A|2,\beta)}{\Pr(B|2,\beta)} > 1$. Consequently, $D_{t+1}(2,\alpha) = 1$ and so by Lemma 1(a), $D_{t+1}(0,\beta) = 0$ and $D_{t+1}(0,\alpha) = 0$.

(b) Given this decision rule, I can write:

$$\begin{aligned} x_{t+1}\left(\mathbf{q}^{t+1}\right) &= \Pr\left[2 \mid x_t(\mathbf{q}^t)\right] + q_{t+1}\Pr\left[1 \mid x_t(\mathbf{q}^t)\right] \\ &= \left[x_t(\mathbf{q}^t)\right]^2 + 2q_{t+1}\left[x_t(\mathbf{q}^t) - \left[x_t(\mathbf{q}^t)\right]^2\right] \\ &= (1-2q_{t+1})\left[x_t(\mathbf{q}^t)\right]^2 + 2q_{t+1}x_t(\mathbf{q}^t) \end{aligned}$$

(c) Finally, given (*b*),

$$\begin{aligned} x_{t+1}\left(\mathbf{q}^{t+1}\right) - x_t(\mathbf{q}^t) &= (1-2q_{t+1})\left[x_t(\mathbf{q}^t)\right]^2 + 2q_{t+1}x_t(\mathbf{q}^t) - x_t(\mathbf{q}^t) \\ &= (1-2q_{t+1})\left[\left[x_t(\mathbf{q}^t)\right]^2 - x_t(\mathbf{q}^t)\right] \end{aligned}$$

Note $x_t(\mathbf{q}^t) - \left[x_t(\mathbf{q}^t)\right]^2 > 0 \ \forall x_t(\mathbf{q}^t) \notin \{0,1\}$. Then, if $q_{t+1} = h > \frac{1}{2}$, then $x_{t+1}\left(\mathbf{q}^{t+1}\right) - x_t(\mathbf{q}^t) > 0$. If $q_{t+1} = l < \frac{1}{2}$, the opposite holds. $\blacksquare$

A.6 Example of Herds for all odd $N \geq 3$

Proposition 10 presents two properties used later in Proposition 11.

PROPOSITION 10.

(a) Let $N \geq 3$ be odd, $1 - \frac{1}{N} < x_t(h) < 1$ and $\zeta \geq \frac{N+1}{2}$. Then:

$$\left(\frac{1-x_t(h)}{x_t(h)}\right)^{2\zeta - N} \leq \frac{1-x_t(h)}{x_t(h)}$$

(b) Let $1 - \frac{1}{N} < x_t(h) < 1 - x_t(l)$ and $\frac{N+1}{2} \leq \zeta \leq N-1$. Then:

$$(1-x_t(l))^{\zeta}\left(x_t(l)\right)^{N-\zeta} < \left(x_t(h)\right)^{\zeta}\left(1-x_t(h)\right)^{N-\zeta}$$

Proof.

[15] Since $x_1(\mathbf{q}^1) \notin \{0,1\}$, $D_2(2,\beta) = 1$. Consequently, $x_2(\mathbf{q}^2)$ is as given by Theorem 5(b). This implies $E[x_2(\mathbf{q}^2)] > E[q]$ and so by Lemma 1(c), $E[x_t(\mathbf{q}^t)] > E[q]$ for all $t \geq 2$.

(a) First, note that since $N \geq 3$, then $\frac{1}{N} \leq \frac{1}{3}$ and so $x_t(h) > 1 - \frac{1}{N} \geq 1 - \frac{1}{3} = \frac{2}{3}$. This implies that $x_t(h) > \frac{1}{2}$ and so $\frac{1-x_t(h)}{x_t(h)} < 1$.

For all $\zeta \geq \frac{N+1}{2}$, the exponent $2\zeta - N \geq 1$. Consequently, $\left(\frac{1-x_t(h)}{x_t(h)}\right)^{2\zeta - N} \leq \frac{1-x_t(h)}{x_t(h)}$.

(b) Define a function $f(y) = (y)^{\zeta}(1-y)^{N-\zeta}$. Note that $f(y)$ is continuous and differentiable. In fact,

$$\frac{\partial f}{\partial y} = \left[\frac{\zeta}{y} - \frac{N-\zeta}{1-y}\right]\left[(y)^{\zeta}(1-y)^{N-\zeta}\right]$$

I will show that $\frac{\partial f}{\partial y} < 0$ for all $y \in [x_t(h), 1 - x_t(l)]$. Consequently, $f(x_t(h)) > f(1 - x_t(l))$ which is what I wanted to show. Note that $\frac{\partial f}{\partial y} < 0$ if and only if $\frac{\zeta}{y} - \frac{N-\zeta}{1-y} < 0$ that is if and only if $y > \frac{\zeta}{N}$. Then, it suffices to note that:

$$\frac{\zeta}{N} \leq 1 - \frac{1}{N} < x_t(h) < 1 - x_t(l) \ \blacksquare$$

In the example I present here, $x_t(\mathbf{q}^t)$ takes only two values in each period. As I show later, these values are determined by the level of aggregate uncertainty in the first period. Consequently, I denote those two values $x_t(h)$ and $x_t(l)$, depending on whether $q_1 = h$ or $q_1 = l$. The auxiliary Proposition 11 presents two useful conditions for determining agents' behavior in a setting where $x_t(\mathbf{q}^t)$ takes only two values.

PROPOSITION 11. *Assume*

(a) $N \geq 3$ *is odd.*

(b) $1 - \frac{1}{N} < h < 1 - l$

(c) $x_t(\mathbf{q}^t)$ *takes only two values in each period:*

$$x_t(\mathbf{q}^t) = \begin{cases} x_t(h) & \textit{if } q_1 = h \\ x_t(l) & \textit{if } q_1 = l \end{cases}$$

with

(a) $x_t(h) < 1 - x_t(l)$

(b) $x_t(h) \geq h$

(c) $x_t(l) \leq l$

Then:

(a) If $\frac{N+1}{2} \leq \zeta \leq N-1$*:*

$$\frac{\Pr(A \mid \zeta, s)}{\Pr(B \mid \zeta, s)} > \frac{p}{p\frac{1-h}{h} + (1-p)} \frac{p(1-h) + (1-p)(1-l)}{ph + (1-p)l} \tag{10}$$

(b) If $\zeta = N$*:*

$$\frac{\Pr(A \mid \zeta, s)}{\Pr(B \mid \zeta, s)} > \frac{ph^N}{p(1-h)^N + (1-p)} \frac{p(1-h) + (1-p)(1-l)}{ph + (1-p)l} \tag{11}$$

Proof.

(a) If $\frac{N+1}{2} \leq \zeta \leq N-1$, the informational content of the sample is given by:

$$\begin{aligned}
\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} &= \frac{p\,[x_t(h)]^{\zeta}\,[1-x_t(h)]^{(N-\zeta)} + (1-p)\,[x_t(l)]^{\zeta}\,[1-x_t(l)]^{(N-\zeta)}}{p\,[1-x_t(h)]^{\zeta}\,[x_t(h)]^{(N-\zeta)} + (1-p)\,[1-x_t(l)]^{\zeta}\,[x_t(l)]^{(N-\zeta)}} \\
&> \frac{p\,[x_t(h)]^{\zeta}\,[1-x_t(h)]^{(N-\zeta)}}{p\,[1-x_t(h)]^{\zeta}\,[x_t(h)]^{(N-\zeta)} + (1-p)\,[1-x_t(l)]^{\zeta}\,[x_t(l)]^{(N-\zeta)}} \\
&= \frac{p}{p\left[\frac{1-x_t(h)}{x_t(h)}\right]^{2\zeta-N} + (1-p)\left[\frac{1-x_t(l)}{x_t(h)}\right]^{\zeta}\left[\frac{x_t(l)}{1-x_t(h)}\right]^{(N-\zeta)}} \\
&> \frac{p}{p\frac{1-x_t(h)}{x_t(h)} + (1-p)}
\end{aligned}$$

where the last step holds by Proposition 10(a) and Proposition 10(b). Then, since $x_t(h) \geq h$:

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} > \frac{p}{p\frac{1-h}{h} + (1-p)}$$

Finally, since $\psi(\alpha) > \psi(\beta) = \frac{p(1-h)+(1-p)(1-l)}{ph+(1-p)l}$, equation (10) holds.

(b) If $\zeta = N$, the informational content of the sample is also bounded:

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} = \frac{p\,[x_t(h)]^N + (1-p)\,[x_t(l)]^N}{p\,[1-x_t(h)]^N + (1-p)\,[1-x_t(l)]^N} > \frac{p\,[x_t(h)]^N}{p\,[1-x_t(h)]^N + (1-p)} \geq \frac{p\,(h)^N}{p\,(1-h)^N + (1-p)}$$

The inequality is strict since $x_t(l) > 0$ and $[1-x_t(l)]^N < 1$. Again, since $\psi(\alpha) > \psi(\beta) = \frac{p(1-h)+(1-p)(1-l)}{ph+(1-p)l}$, equation (11) holds. ■

PROPOSITION 12. *Assume that*

(a) $N \geq 3$ *is odd.*

(b) $1 - \frac{1}{N} < h < 1 - l$

(c) $x_t(\mathbf{q}^t)$ *is given by:*

$$x_t(\mathbf{q}^t) = \begin{cases} x_t(h) & \textit{if } q_1 = h \\ x_t(l) & \textit{if } q_1 = l \end{cases}$$

with

(a) $x_t(h) < 1 - x_t(l)$

(b) $x_t(h) \geq h$

(c) $x_t(l) \leq l$

(d) The decision function is given by:

$$D_{t+1}(\zeta, s) = \begin{cases} 1 & \textit{if } \zeta \geq \frac{N+1}{2} \\ 0 & \textit{if } \zeta \leq \frac{N-1}{2} \end{cases}$$

Then:

(a) The evolution of the system is given by:

$$x_{t+1}(\mathbf{q}^{t+1}) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(\mathbf{q}^t)\right)$$

(b) In this way, there are only two possible values for x_{t+1}:

$$x_{t+1}(\mathbf{q}^{t+1}) = \begin{cases} x_{t+1}(h) & \text{if } q_1 = h \\ x_{t+1}(l) & \text{if } q_1 = l \end{cases}$$

with

(a) $x_{t+1}(h) < 1 - x_{t+1}(l)$

(b) $x_{t+1}(h) > x_t(h) \geq h$

(c) $x_{t+1}(l) < x_t(l) \leq l$

Proof.

(a) Immediate, given the decision function.

(b) Since the decision does not depend on the signal, the aggregate level of uncertainty in period $t+1$ does not have any impact on the evolution of the system in that period. As a result, there are only two possible values for x_t:

$$x_{t+1}(h) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(h)\right)$$
$$x_{t+1}(l) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(l)\right)$$

(a) $1 - x_{t+1}(l) = 1 - \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(l)\right) = \Pr\left(\zeta \leq \frac{N-1}{2} \mid x_t(l)\right) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid 1 - x_t(l)\right)$. Since $\Pr\left(\zeta \geq \frac{N+1}{2} \mid x\right)$ is increasing in x, then:

$$1 - x_{t+1}(l) = \Pr\left(\zeta \geq \frac{N+1}{2} \mid 1 - x_t(l)\right) > \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(h)\right) = x_{t+1}(h)$$

(b) Define a random variable $Y \sim B(N, x_t(h))$ so that $E[Y] = N x_t(h)$. I want to show that $x_{t+1}(h) > x_t(h)$.[16]

$$\begin{aligned} x_{t+1}(h) &= \Pr\left(\zeta \geq \frac{N+1}{2} \mid x_t(h)\right) = \sum_{\zeta=\frac{N+1}{2}}^{N} \binom{N}{\zeta} x_t(h)^{\zeta} \left[1 - x_t(h)\right]^{N-\zeta} \\ &= \sum_{\zeta=\frac{N+1}{2}}^{N} \binom{N}{\zeta} x_t(h)^{\zeta} \left[1 - x_t(h)\right]^{N-\zeta} \left(\frac{N-\zeta}{N} + \frac{\zeta}{N}\right) \\ &= \frac{1}{N}\left[\sum_{\zeta=\frac{N+1}{2}}^{N} \binom{N}{\zeta} x_t(h)^{\zeta} \left[1 - x_t(h)\right]^{N-\zeta} (N-\zeta) + \sum_{\zeta=\frac{N+1}{2}}^{N} \binom{N}{\zeta} x_t(h)^{\zeta} \left[1 - x_t(h)\right]^{N-\zeta} \zeta\right] \end{aligned}$$

[16] The inequality in the fifth line is a result of $\left(\frac{x_t(h)}{1-x_t(h)}\right)^{N-2\zeta} > 1$ for $\zeta \leq \frac{N-1}{2}$ and $x_t(h) > \frac{1}{2}$.

$$= \frac{1}{N}\left[\sum_{\zeta=0}^{\frac{N-1}{2}}\binom{N}{N-\zeta}x_t(h)^{N-\zeta}\left[1-x_t(h)\right]^{\zeta}\zeta+\sum_{\zeta=\frac{N+1}{2}}^{N}\binom{N}{\zeta}x_t(h)^{\zeta}\left[1-x_t(h)\right]^{N-\zeta}\zeta\right]$$

$$> \frac{1}{N}\left[\sum_{\zeta=0}^{\frac{N-1}{2}}\binom{N}{N-\zeta}x_t(h)^{\zeta}\left[1-x_t(h)\right]^{N-\zeta}\zeta+\sum_{\zeta=\frac{N+1}{2}}^{N}\binom{N}{\zeta}x_t(h)^{\zeta}\left[1-x_t(h)\right]^{N-\zeta}\zeta\right]$$

Consequently,

$$x_{t+1}(h) > \frac{1}{N}\sum_{\zeta=0}^{N}\binom{N}{\zeta}x_t(h)^{\zeta}\left[1-x_t(h)\right]^{N-\zeta}\zeta = \frac{1}{N}E[Y] = \frac{1}{N}Nx_t(h) = x_t(h)$$

By the same argument,

$$x_{t+1}(l) < x_t(l) \leq l \;\blacksquare$$

Given Proposition 12, I prove Proposition 8.

Proof. I prove (a) and (b) by induction. If the assumptions of Proposition 12 hold in period t, then (a) and (b) are true in that period and the same assumptions hold in period $t+1$. As a result, I only need to show that the assumptions of Proposition 12 hold in period $t = 1$.

In period $t = 1$, agents follow the signal they receive. As a result:

$$x_1 = \begin{cases} h & \text{with probability } \; p \\ l & \text{with probability } \; 1-p \end{cases}$$

I need to show that there exist parameters p, h and l such that the information in the sample always outweighs the information provided by a signal $s = \beta$, and therefore agents choose technology a when it is the most observed choice, regardless of the signal. Given equations (10) and (11), it suffices to show that:

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} > \frac{ph^N}{p(1-h)^N+(1-p)}\frac{p(1-h)+(1-p)(1-l)}{ph+(1-p)l} \geq 1$$

$$\frac{\Pr(A \mid \zeta)}{\Pr(B \mid \zeta)} > \frac{p}{p\frac{1-h}{h}+(1-p)}\frac{p(1-h)+(1-p)(1-l)}{ph+(1-p)l} \geq 1$$

Note that both conditions are continuous in p and l. If $p = 1$, the first condition holds with strict inequality, so that if $p = \tilde{p} \approx 1$ the first condition still holds. Moreover, note that lowering l provides even more slack in the first condition. Next, replace p by $\tilde{p}$ in the the second condition. If $l = 0$, the second condition holds with strict inequality, so if $l = \tilde{l} \approx 0$ the second condition — and also the first — still hold.

Finally, regarding (c), $x_{t+1}(h)$ is a continuous function of $x_t(h)$. Moreover, $x_{t+1}(h) > x_t(h)$ for all $x_t(h) \in \left(\frac{1}{2}, 1\right)$ and $x_{t+1}(h) = x_t(h)$ for $x_t(h) \in \{\frac{1}{2}, 1\}$. Consequently, $\lim_{t\to\infty} x_t(h) = 1$. Note that $q_1 = h$ with probability p. Consequently, the fraction of agents choosing the superior technology converges to 1 with probability p. As a result, since $x_t(l) = 1 - x_t(h)$;

$$\Pr\left\{\lim_{t\to\infty} x_t(\mathbf{q}^t) = 0\right\} = 1-p \;\blacksquare$$