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Abstract

We introduce and axiomatize dynamic variational preferences, the dynamic version of the variational preferences we axiomatized in [21], which generalize the multiple priors preferences of Gilboa and Schmeidler [9], and include the Multiplier Preferences inspired by robust control and first used in macroeconomics by Hansen and Sargent (see [11]), as well as the classic Mean Variance Preferences of Markovitz and Tobin. We provide a condition that makes dynamic variational preferences time consistent, and their representation recursive. This gives them the analytical tractability needed in macroeconomic and financial applications. A corollary of our results is that Multiplier Preferences are time consistent, but Mean Variance Preferences are not.

JEL classification: C61; D81

Keywords: Ambiguity Aversion; Model Uncertainty; Recursive Utility; Robust Control; Time Consistency

1 Introduction

In the Multiple Priors (MP) model agents rank acts h using the criterion

$$V(h) \equiv \inf_{p \in C} \mathrm{E}^p [u(h)], \tag{1}$$

where C is a closed and convex subset of the set Δ of all probabilities on states. This model has been axiomatized by Gilboa and Schmeidler [9] with the goal of modeling ambiguity averse agents, who exhibit the Ellsberg-type behavior first observed in the seminal paper of Ellsberg [5].

The nonsingleton nature of C reflects the limited information available to agents, which may not be enough to quantify their beliefs with a single probability, and is instead compatible with a nonsingleton set C of probabilities.

On the other hand, the cautious attitude featured by MP agents can also be viewed as the result of the effect that an adversarial influence, which we may call "Nature," has on the realizations of the state. Under this view, Nature chooses a probability p over states with the objective of minimizing agents' utility, conditional on their choice of an act and under the constraint that the probability p has to be chosen in a fixed set C. This interpretation of the MP model provides an intuitive notion of ambiguity aversion, which can be regarded as the agents' diffidence for any lack of precise definition of the uncertainty involved in a choice, something that provides room for the malevolent influence of Nature.[1]

In a recent paper, [21], we extended the MP representation by generalizing Nature's constraint. Specifically, in our extension the constraint on Nature is given by a cost $c(p)$ associated with the choice of probability, and agents rank acts according to the criterion:

$$V(h) \equiv \inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + c(p)\right), \tag{2}$$

where c is a closed and convex function on Δ. Preferences represented by (2) are called *variational preferences* (VP), and the function c is their *ambiguity index*. In [21] we axiomatize the representation (2) and we discuss in detail its ambiguity interpretation.

The VP representation generalizes the MP representation, which is the special case where there is an infinite cost for choosing outside the set C, with the cost being constant (and hence, without loss of generality, zero) inside that set. In other words, the cost for Nature in the MP model is given by the indicator function $\delta_C : \Delta \to [0, \infty]$ of C, defined as

$$\delta_C(p) \equiv \begin{cases} 0 & \text{if } p \in C, \\ \infty & \text{if } p \notin C, \end{cases} \tag{3}$$

and it is immediate to see that

$$\inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + \delta_C(p)\right) = \inf_{p \in C} \mathrm{E}^p [u(h)].$$

The notion of ambiguity aversion has found an important application in the last years in the literature, pioneered by Hansen and Sargent (see, e.g., [11] and [12] for details and references), that applies the idea of robust control to agents' choices in macroeconomic models. While the initial definition of robust control was different from that of ambiguity aversion, the intuition is

[1] As Hart, Modica, and Schmeidler [13, p. 352] write "In Gilboa and Schmeidler [9] it is shown that preferences ... are represented by functionals of the form $f \mapsto \min_{q \in Q} \sum_s u(f(s)) q(s)$, for some closed convex set $Q \subset \Delta(S)$. So the ambiguity averse decision maker behaves 'as if' there were an opponent who could partially influence occurence of states to his disadvantage (i.e., think of the opponent as choosing $q \in Q$)." This informed opponent interpretation has found support in some recent experimental findings in the psychological and neuroscience literatures (see [18], [19], [17], and [27]).

closely related: an agent prefers a robust control if he is not confident that his (probabilistic) model of the uncertainty is correct, and so he wants to avoid the possibility that a small error in the formulation of the stochastic environment produces a large loss. Ambiguity aversion comes up because the agents' information is too limited to be represented by a single probabilistic model.

In the *multiplier preferences* model, the most important choice model used in this macroeconomic literature (see [11]), the constraint on Nature is represented by a cost c based on a reference probability $q \in \Delta$: Nature can deviate away from q, but the larger the deviation, the higher the cost. In particular, this cost is assumed to be proportional to the relative entropy $R(p\|q)$ between the chosen probability p and the reference probability q; that is,

$$c(p) \equiv \theta R(p\|q),$$

where $\theta > 0$. Multiplier preferences are, therefore, the special case of variational preferences given by

$$V(h) \equiv \inf_{p \in \Delta} \left(\mathrm{E}^p [u(h)] + \theta R(p\|q) \right),$$

and their analytical tractability is important in deriving optimal policies.

Even though the motivation behind multiplier preferences is similar to that used for MP preferences, formally multiplier preferences are not MP preferences. In fact, in [21] we show that they are an example of divergence preferences, a special class of variational preferences featuring tractable cost functions, but which are not MP preferences. Variational preferences are, therefore, the generalization needed in order to encompass both MP and multiplier preferences, as discussed at length in [21].

In view of applications, however, the static analysis of [21] is insufficient and a dynamic extension is required. This is the purpose of the present paper, in which we introduce and axiomatize dynamic variational preferences.

The first observation to make is that, while in a static environment acts are functions from states to consequences, in a dynamic environment they are functions from times and states to consequences. We impose on acts the usual measurability conditions ensuring that agents' choices are consistent with the information they have. As a result, agents' evaluations are conditional to time and state, and they are modelled by a family of (conditional) preferences $\succsim_{t,\omega}$ indexed by time and state pairs (t, ω). In the main results of the paper, Proposition 1 and Theorem 1, we provide necessary and sufficient conditions guaranteeing that agents' preferences at time t are represented by the preference functional $V_t(h) : \Omega \to \mathbb{R}$ given by

$$V_t(h) \equiv \inf_{p \in \Delta} \left(\mathrm{E}^p \left[\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \middle| \mathcal{G}_t \right] + c_t(p \,|\mathcal{G}_t) \right), \tag{4}$$

and we show what restrictions on c_t guarantee time consistency.[2] Under time consistency the representation (4) becomes recursive, and so it has the analytical tractability required in applications.

Besides tractability, time consistency has also an intuitive appeal. In fact, suppose that two acts are the same in every contingency up to the present period, and the first is preferred to the second according to the conditional preference in the next period in every state. Then time consistency requires that the first act should be preferred to the second in the present period. Equivalently, think of a plan as a sequence of conditional choices, so that the choice of a plan in the current period includes a plan of choices in all future periods, conditional on all future contingencies. Then, an agent is time consistent if he never formulates a plan of future choices that he wants to revise later in some event that is conceivable today.

[2] Here β is a discount factor and $\mathcal{G}_t$ represents the information available to the agents at time t.

1.1 The No-Gain Condition and Bayesian Updating

Our present work extends to the VP setting the recent dynamic version of the MP model provided by Epstein and Schneider [6]. These authors give a condition, called rectangularity, that guarantees time consistency of MP preferences. Since rectangularity is a restriction on the sets of probabilities from which Nature can select at every time and state, it is natural that our corresponding condition is formulated as a restriction on cost functions.

Specifically, our condition is given by (11) of Theorem 1. To facilitate the exposition, we present it in a simplified form, dropping the time index (the reader may think of this as the condition for the two-period version of the model). The agent has a partition $\mathcal{G}$ over the set of possible states (see the picture at p. 7). Nature has a cost c_Ω in the first period, so that $c_\Omega(q)$ is Nature's cost of choosing the probability q over the states. To each event G in this partition a new, second period, cost c_G is associated. The announced condition requires that:

$$c_\Omega(q) = \inf_{\{p:p(G)=q(G)\ \forall G\in\mathcal{G}\}} [\beta \sum_{G\in\mathcal{G}} q(G)c_G(q_G) + c_\Omega(p)], \tag{5}$$

where $q(G) \equiv \sum_{\omega\in G} q(\omega)$, β is the discount factor, and

$$q_G(\omega) \equiv \begin{cases} q(\omega)/q(G) & \text{if } \omega \in G, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The condition has a simple interpretation. The choice of probability by Nature over two periods can be thought of as consisting of two steps. The first period choice is a choice of probability over the events that realize in the first period. The second period choice is a choice of probability over states in every event, conditional on that event.

Nature can make this choice in a time consistent way: choose q in the first period, pay the appropriate cost $c_\Omega(q)$, wait for the realization of the second period event G, do nothing, pay nothing, and get the probability q_G on the states in the event G. The total cost of this is the term in the l.h.s. of (5).

Alternatively, Nature can achieve the same result in a time inconsistent way, with total cost given by the r.h.s. of (5). Nature can choose today a probability p that induces the same probability over events in the second period as q does. This constraint is described by the condition $p(G) = q(G)$ for every event G. Nature pays for its choice p the appropriate cost, which is the term $c_\Omega(p)$ in the r.h.s. of (5). After the realization of the event G, the probability over states in that event would be p_G. Nature can now change the conditional probability to q_G, and again pay the appropriate cost, represented by the term $c_G(q_G)$ in the r.h.s. of (5). Overall, in this second more indirect way, Nature achieves the same result as in the first choice: a probability $q(G)$ of every event G in the first period, and a conditional probability q_G if G obtains.

Condition (5) requires that this second, time inconsistent and convoluted, choice is not less costly for Nature. A simple way of stating our main result is therefore the following: *A decision maker is dynamically consistent if and only if (he thinks that) Nature is dynamically consistent.*[3]

In view of all this, we call (5), and more generally (11) of Theorem 1, a "no-gain condition." We will formally prove that the no-gain condition generalizes rectangularity, and it coincides with it when cost functions are indicators δ_C.

[3]This dynamically consistent behavior of Nature reminds of the Principle of Least Action, a fundamental idea in theoretical physics, which for example lies at the heart of both classical and quantum mechanics. In its meta-theoretic form, this principle says that Nature is thrifty in all its actions, and so it acts in the simplest possible way. The dynamic consistency of Nature can be viewed as a form of this important meta-theoretic principle because (5) describes the simplest possible way for Nature to end up with a probability $q(G)$ of every event G in the first period, and a conditional probability q_G if G obtains.

Equation (5) provides a link between cost functions in different periods. One important aspect of this link is that in the second period the probability over states conditional on the event G is the conditional probability q_G as defined by (6), namely according to Bayes' Rule. This link extends to variational preferences the connection between time consistency and Bayes' Rule.

As well known, Subjective Expected Utility preferences are time consistent if and only if their subjective beliefs are updated according to Bayes' Rule. This result is generalized in [6] to MP preferences by showing that they are time consistent if and only if their sets of subjective beliefs are rectangular and updating is done belief by belief (prior by prior in the terminology of the MP model) according to Bayes' Rule. Our Theorem 1 further generalizes all these results by showing that variational preferences are time consistent if and only if their cost functions satisfy the no-gain condition and updating is done according to Bayes' Rule.

Moreover, the recursive structure of the no-gain condition makes it possible to construct by backward induction cost functions that satisfy it. This is shown by Theorem 2, which thus provides a way to construct via (4) examples of variational preferences that are time consistent.

Some papers have recently studied related issues, in particular dynamic aspects of the MP model. We already mentioned Epstein and Schneider [6], which is in turn closely related to Wang [31]. Some aspects of their work have been extended by Ghirardato, Maccheroni, and Marinacci [8] and Hayashi [15]. More recently, Hanany and Klibanoff [10] proposed a dynamic version of the MP model that is dynamically consistent but does not satisfy Consequentialism, while Siniscalchi [28] focused on dynamic MP models that relax Dynamic Consistency. Finally, Ozdenoren and Peck [25] have studied some dynamic games against Nature that lead to ambiguity averse behavior, thus providing a game-theoretic underpinning of the game against Nature interpretation of ambiguity we discussed above and in [21].

The paper is organized as follows. Section 2 introduces the setup and notation, Section 3 presents the axioms needed for our derivation, whereas Section 4 contains the main results of the paper. Section 5 illustrates the main results with two important classes of variational preferences, the multiple priors preferences of Gilboa and Schmeidler [9] and the multiplier preferences of Hansen and Sargent [11]. Finally, Section 6 illustrates the analytical tractability of dynamic variational preferences by showing their convenient differential properties. All proofs are collected in the Appendix.

2 Setup

2.1 Information

Time is discrete and varies over $\mathcal{T} \equiv \{0, 1, ..., T\}$. In our results we model information as an event tree $\{\mathcal{G}_t\}_{t\in\mathcal{T}}$, given and fixed throughout, which is defined on a finite space Ω. The elements of this tree are partitions $\mathcal{G}_t$ of Ω consisting of non-empty sets, with $\mathcal{G}_0 \equiv \{\Omega\}$, $\mathcal{G}_{t+1}$ finer than $\mathcal{G}_t$ for all $t < T$, and $\mathcal{G}_T \equiv \{\{\omega\} : \omega \in \Omega\}$; in particular, $G_t(\omega)$ is the element of $\mathcal{G}_t$ that contains ω. For non-triviality, we assume $T, |\Omega| \geq 2$.

The main interpretation we have in mind for this standard modelling of information is as follows. Given an underlying (and possibly unverifiable) state space S, endowed with a σ-algebra Σ, observations are generated by a sequence of random variables $\{Z_t\}_{t>0}$ taking values on finite observation spaces Ω_t. Each random variable $Z_t : S \to \Omega_t$ is Σ-measurable and for convenience we assume that they are surjective, so that all elements of Ω_t can be viewed as observations generated by Z_t.

The sample space $\prod_{t=1}^{T} \Omega_t$ is denoted by Ω, and its points $\omega = (\omega_1, ..., \omega_T)$ are the possible observation paths generated by the sequence $\{Z_t\}$. Given $t \in \mathcal{T}$, denote by $\{\omega_1, ..., \omega_t\}$ the cylinder

$$\{\omega_1\} \times \cdots \times \{\omega_t\} \times \Omega_{t+1} \times \cdots \times \Omega_T.$$

The event tree $\{\mathcal{G}_t\}$ records the building up of observations and it is given by $\mathcal{G}_0 \equiv \{\Omega\}$,

$$\mathcal{G}_t \equiv \{\{\omega_1, ..., \omega_t\} : \omega_\tau \in \Omega_\tau \text{ for each } \tau = 1, ..., t\},$$

and $\mathcal{G}_T \equiv \{\{\omega\} : \omega \in \Omega\}$. In other words, the atoms of the partition $\mathcal{G}_t$ are the observation paths up to time t and they can be viewed as the nodes of the event tree $\{\mathcal{G}_t\}$.

Denote by $\Delta(\Omega)$ the set of all probability distributions $p : 2^\Omega \to [0,1]$. The elements of $\Delta(\Omega)$ represent the agent's subjective beliefs over the observation paths. Their conditional distributions

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) \equiv \frac{p(\omega_1, ..., \omega_T)}{p(\omega_1, ..., \omega_t)}$$

are called *predictive distributions* and they represent the agent's (subjective) probability that $(\omega_{t+1}, ..., \omega_T)$ will be observed after having observed $(\omega_1, ..., \omega_t)$.[4] Using the standard notation for conditional probabilities, the predictive distributions are given by the collection $\{p(\cdot \mid \mathcal{G}_t)\}_{t>0}$.

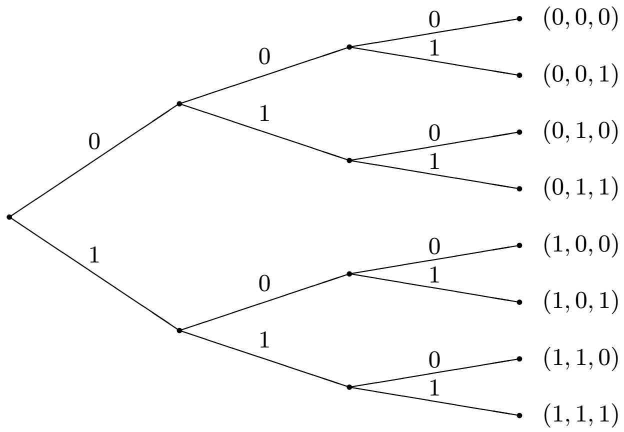
Observe that in the literature on MP preferences, the probabilities $p : 2^\Omega \to [0,1]$ are often called priors and the conditional probabilities $p(\cdot \mid \mathcal{G}_t)$ are called the Bayesian updates of the priors. This terminology is, however, a bit confusing as in Statistics priors are often probabilities on parameters (and posteriors are their Bayesian updates given observations). Here no parametric representation is assumed for the probabilities $p : 2^\Omega \to [0,1]$, and so we prefer not to use the term prior for them.

We now illustrate these notions with few examples.

Example 1 Suppose that observations are given by heads and tails from a given coin. We can set $\Omega_t = \{0,1\}$ for each $t = 1, ..., T$, so that $\Omega = \{0,1\}^T$ is the sample space. A possible $p \in \Delta(\Omega)$ is the one that assigns equal probability to all observation paths ω; that is, $p(\omega) \equiv 2^{-T}$ for each $\omega \in \Omega$. In this case, $p(\omega_1, ..., \omega_t) = 2^{-t}$ and

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \frac{p(\omega_1, ..., \omega_T)}{p(\omega_1, ..., \omega_t)} = 2^{t-T}.$$

For example, if $T = 3$, we have $\Omega = \{0,1\}^3$ and Ω consists of 2^3 states. This case can be illustrated with a simple binomial tree



and the above probability p is such that $p(\omega) = 1/8$ for all $\omega \in \Omega$, while its predictive distributions are:

$$p(\omega_3 \mid \omega_1, \omega_2) = 1/2 \quad \text{and} \quad p(\omega_2, \omega_3 \mid \omega_1) = 1/4.$$

▲

[4] We write $p(\omega_1, ..., \omega_t)$ and $(\omega_{t+1}, ..., \omega_T)$ in place of $p(\{\omega_1, ..., \omega_t\})$ and $\Omega_1 \times \cdots \times \Omega_t \times \{\omega_{t+1}\} \times \cdots \times \{\omega_T\}$.

In the next examples we assume that $\Omega_t = \mathcal{Z}$ for all t, so that $\Omega = \mathcal{Z}^T$. For instance, in the previous example we had $\mathcal{Z} = \{0, 1\}$.

Example 2 Consider a $p \in \Delta(\Omega)$ that makes the sequence $\{Z_t\}$ i.i.d., with common marginal distribution $\pi : 2^{\mathcal{Z}} \to [0, 1]$. In this case, p is a product probability on 2^{Ω} uniquely determined by π as follows:

$$p(\omega) \equiv \prod_{i=1}^{T} \pi(\omega_i) \quad \forall \omega \in \Omega.$$

The predictive distributions are given by:

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \prod_{i=t+1}^{T} \pi(\omega_i),$$

that is, $p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = p(\omega_{t+1}, ..., \omega_T)$. Hence, information is irrelevant for prediction. ▲

Example 3 Consider a $p \in \Delta(\Omega)$ that makes the sequence $\{Z_t\}$ exchangeable, i.e.,

$$p(\omega_1, ..., \omega_T) = p(\omega_{i_1}, ..., \omega_{i_T}) \tag{7}$$

for all permutations $i_1, ..., i_T$. For simplicity, suppose $\mathcal{Z} = \{0, 1\}$ and set

$$v_l^t \equiv \binom{t}{l} p\left(\sum_{i=1}^{t} \omega_i = l\right),$$

where $p\left(\sum_{i=1}^{t} \omega_i = l\right)$ is the probability of having l successes among $t \in \mathcal{T}$ trials. Some algebra shows that here the predictive distributions are given by

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \frac{v_{l+k}^T / \binom{T}{l+k}}{v_l^t / \binom{t}{l}},$$

where $l = \sum_{i=1}^{t} \omega_i$ and $k = \sum_{i=t+1}^{T} \omega_i$. Because of exchangeability, only the quantities l and k matter for the predictive distributions. Here information, as recorded by l and k, is relevant for prediction. ▲

Example 4 Finally, suppose that $p \in \Delta(\Omega)$ makes the sequence $\{Z_t\}$ a homogeneous Markov chain with transition function $\pi : \Omega_{t-1} \times 2^{\mathcal{Z}} \to [0, 1]$ for $t \geq 2$, where $\pi(\omega_{t-1}, \cdot) : 2^{\mathcal{Z}} \to [0, 1]$ is a probability measure on $\mathcal{Z}$ for each $\omega_{t-1} \in \Omega_{t-1}$. Given an initial probability distribution π^0 on $2^{\mathcal{Z}}$, p is uniquely determined by π as follows:

$$p(\omega) \equiv \pi^0(\omega_1) \prod_{i=2}^{T} \pi(\omega_{i-1}, \omega_i) \quad \forall \omega \in \Omega.$$

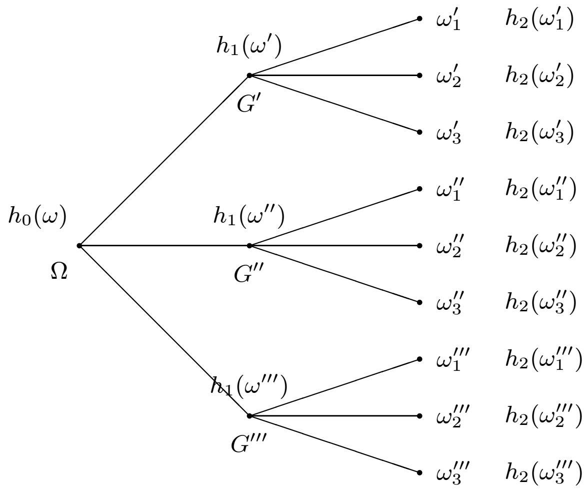
so that,

$$p(\omega_{t+1}, ..., \omega_T \mid \omega_1, ..., \omega_t) = \prod_{i=t+1}^{T} \pi(\omega_{i-1}, \omega_i). \tag{8}$$

Also in this Markov example information matters for prediction. In particular, (8) shows that here the relevant information is given by ω_t. ▲

2.2 Consumption Streams

The acts among which agents choose are here given by consumption processes. Formally, *acts* are X-valued adapted processes of the form $h = (h_0, h_1, ..., h_T)$, where each $h_t : \Omega \to X$ is $\mathcal{G}_t$-measurable and takes values on a convex consumption set X (e.g., $X = \mathbb{R}$ or $\Delta(\mathbb{R})$).



Denote by $\mathcal{H}$ the set of all acts; we equivalently write $h_t(\omega)$ or $h(t, \omega)$ to denote consumption at time t if ω obtains (and sometimes $h(t, G)$ to denote consumption at time t if $G \in \mathcal{G}_t$ occurs). Notice that in our finite setting acts can be regarded as functions defined on $\bigcup_{t \in \mathcal{T}} \mathcal{G}_t$, that is, on the set of all nodes.

We can identify $\mathcal{H}$ with the set of all maps $h : \Omega \to X^{\mathcal{T}}$ such that $h_\tau(\omega) = h_\tau(\omega')$ if $G_\tau(\omega) = G_\tau(\omega')$; in this perspective $h(\omega)$ is the element $(h_0(\omega), h_1(\omega), ..., h_T(\omega)) \in X^{\mathcal{T}}$ for any given ω. For all $\alpha \in [0, 1]$, and all $h, h' \in \mathcal{H}$ we set

$$(\alpha h + (1-\alpha) h')(t, \omega) \equiv \alpha h(t, \omega) + (1-\alpha) h'(t, \omega) \quad \forall (t, \omega) \in \mathcal{T} \times \Omega.$$

If the values of an act $y \in \mathcal{H}$ depend only on time but not on state, that is, for every fixed t

$$y(t, \omega) = y(t, \omega') = y_t \quad \forall \omega, \omega' \in \Omega,$$

with a little abuse of notation we write $y = (y_0, y_1, ..., y_T) \in X^{\mathcal{T}}$. Moreover, if $y_0 = ... = y_T = x$, the act is called constant and, with another little abuse of notation, we denote it by x.

Example 5 Suppose as in Example 1 that $\Omega = \{0, 1\}^T$. A consumption process $h = (h_0, h_1, ..., h_T)$ is such that:

$$\begin{aligned}
h_0(\omega) &= h_0(\omega'), \quad \forall \omega, \omega' \in \Omega, \\
h_1(\omega) &= h_1(\omega'), \quad \forall \omega, \omega' \in \Omega \text{ with } \omega_1 = \omega'_1, \\
&\cdots \\
h_t(\omega) &= h_t(\omega'), \quad \forall \omega, \omega' \in \Omega \text{ with } (\omega_1, ..., \omega_t) = (\omega'_1, ..., \omega'_t), \\
&\cdots
\end{aligned}$$

In other words, h_0 is a constant, h_1 only depends on the first observation, and h_t only depends on the first t observations. ▲

2.3 Notation

We close by introducing some notation, which is usually a bit heavy in dynamic settings. If $p \in \Delta(\Omega)$, we denote by $p_{|\mathcal{G}_t}$ its restriction to the algebra $\mathcal{A}(\mathcal{G}_t)$ generated by $\mathcal{G}_t$, and by $p(\cdot\,|\mathcal{G}_t)$ the conditional probability given $\mathcal{G}_t$.[5] As we already observed, the conditional probabilities $p(\cdot\,|\mathcal{G}_t)$ are called predictive distributions.

For all $t \in \mathcal{T}$, $\Delta(\Omega, \mathcal{G}_t)$ denotes the set of all probabilities on $\mathcal{A}(\mathcal{G}_t)$, hence $\Delta(\Omega, \mathcal{G}_t) = \left\{p_{|\mathcal{G}_t} : p \in \Delta(\Omega)\right\}$. In particular, $\Delta(\Omega, \mathcal{G}_T) = \Delta(\Omega)$.

For each $E \in \mathcal{A}(\mathcal{G}_t)$, we set

$$\Delta(E, \mathcal{G}_t) \equiv \{p \in \Delta(\Omega, \mathcal{G}_t) \mid p(E) = 1\}$$
$$\Delta^{++}(E, \mathcal{G}_t) \equiv \left\{ p \in \Delta(\Omega, \mathcal{G}_t) \;\middle|\; \begin{array}{ll} p(G) > 0 & \forall G \in \mathcal{G}_t : G \subseteq E \\ p(G) = 0 & \forall G \in \mathcal{G}_t : G \nsubseteq E \end{array} \right\}.$$

Denoting by $\operatorname{supp} p$ the support $\{\omega \in \Omega : p(\omega) > 0\}$ of $p \in \Delta(\Omega)$, for each subset E of Ω we have:

$$\Delta(E) = \{p \in \Delta(\Omega) : \operatorname{supp} p \subseteq E\} \quad \text{and} \quad \Delta^{++}(E) = \{p \in \Delta(\Omega) : \operatorname{supp} p = E\}.$$

In particular, $\Delta(G_t(\omega))$ is the set of all predictive distributions that can be obtained by conditioning on $G_t(\omega)$ from probabilities $p \in \Delta(\Omega)$ such that $p(G_t(\omega)) > 0$, while $\Delta^{++}(G_t(\omega))$ is the subset of $\Delta(G_t(\omega))$ derived under the further condition that $p \in \Delta(\Omega)$ be such that $p(\omega') > 0$ for all $\omega' \in G_t(\omega)$.

Similarly, for each $E \in \mathcal{A}(\mathcal{G}_t)$ we have

$$\Delta(E, \mathcal{G}_t) = \left\{p_{|\mathcal{G}_t} : p \in \Delta(E)\right\} \quad \text{and} \quad \Delta^{++}(E, \mathcal{G}_t) = \left\{p_{|\mathcal{G}_t} : p \in \Delta^{++}(E)\right\}.$$

If the vector space $\mathbb{M}(\Omega, \mathcal{G}_t)$ of all measures on $\mathcal{A}(\mathcal{G}_t)$ is endowed with the product topology, then $\Delta^{++}(E, \mathcal{G}_t)$ is the relative interior of the convex set $\Delta(E, \mathcal{G}_t)$ (see Rockafellar [26], to which we refer for the Convex Analysis terminology and notation).

3 Axioms

Let the binary relations $\succsim_{t,\omega}$ on $\mathcal{H}$ represent the agent's preferences at any time-state node. Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

Axiom 1 (Conditional preference—CP) *For each $(t, \omega) \in \mathcal{T} \times \Omega$:*

(i) $\succsim_{t,\omega}$ coincides with $\succsim_{t,\omega'}$ if $G_t(\omega) = G_t(\omega')$.

(ii) If $h(\tau, \omega') = h'(\tau, \omega')$ for all $\tau \geq t$ and $\omega' \in G_t(\omega)$, then $h \sim_{t,\omega} h'$.

(i) says that preferences orderings are "adapted" and allows to write $\succsim_{t,G}$ if $G \in \mathcal{G}_t$. (ii) states that at time t in event G only "continuation acts" matter for choice.

Axiom 2 (Variational preferences—VP) *For each $(t, \omega) \in \mathcal{T} \times \Omega$:*

(i) $\succsim_{t,\omega}$ is complete and transitive.

(ii) For all $h, h' \in \mathcal{H}$ and $y, y' \in X^{\mathcal{T}}$, and for all $\alpha \in (0, 1)$, if $\alpha h + (1-\alpha)y \succsim_{t,\omega} \alpha h' + (1-\alpha)y$ then $\alpha h + (1-\alpha)y' \succsim_{t,\omega} \alpha h' + (1-\alpha)y'$.

(iii) For all $h, h', h'' \in \mathcal{H}$, the sets $\{\alpha \in [0, 1] : \alpha h + (1-\alpha)h' \succsim_{t,\omega} h''\}$ and $\{\alpha \in [0, 1] : h'' \succsim_{t,\omega} \alpha h + (1-\alpha)h'\}$ are closed.

[5] Notice that for all $\omega \in \Omega$ with $p(G_t(\omega)) \neq 0$, $p(\cdot\,|\mathcal{G}_t)(\omega) = p_{G_t(\omega)}$, as defined by (6).

(iv) *For all* $h, h' \in \mathcal{H}$, *if* $(h_0(\omega'), h_1(\omega'), ..., h_T(\omega')) \succsim_{t,\omega} (h'_0(\omega'), h'_1(\omega'), ..., h'_T(\omega'))$ *for all* $\omega' \in \Omega$, *then* $h \succsim_{t,\omega} h'$.

(v) *For all* $h, h' \in \mathcal{H}$, *if* $h \sim_{t,\omega} h'$, *then* $\alpha h + (1-\alpha) h' \succsim_{t,\omega} h$ *for all* $\alpha \in (0,1)$.

(vi) *There exist* $x \succ_{t,\omega} x'$ *in* X *such that for all* $\alpha \in (0,1)$ *there is* $x'' \in X$ *satisfying either* $x' \succ_{t,\omega} \alpha x'' + (1-\alpha) x$ *or* $\alpha x'' + (1-\alpha) x' \succ_{t,\omega} x$.

The requirement here is that at every time-state node the agent has (unbounded) variational preferences; see Maccheroni, Marinacci, and Rustichini [21] for a discussion of (i)-(vi).

Axiom 3 (Risk preference—RP) *For any* $y \in X^{\mathcal{T}}$ *and all* $x, x', x'', x''' \in X$, *if*

$$\left(y_{-\{\tau,\tau+1\}}, x, x'\right) \succsim_{t,\omega} \left(y_{-\{\tau,\tau+1\}}, x'', x'''\right)$$

holds for some $(t,\omega) \in \mathcal{T} \times \Omega$ *and some* $\tau \geq t$, *then it holds for all* $(t,\omega) \in \mathcal{T} \times \Omega$ *and all* $\tau \geq t$.[6]

This is a standard stationarity axiom.

Axiom 4 (Dynamic consistency—DC) *For each* $(t,\omega) \in \mathcal{T} \times \Omega$ *with* $t < T$, *and all* $h, h' \in \mathcal{H}$, *if* $h_\tau = h'_\tau$ *for all* $\tau \leq t$ *and* $h \succsim_{t+1,\omega'} h'$ *for all* $\omega' \in \Omega$, *then* $h \succsim_{t,\omega} h'$.

As Epstein and Schneider [6, p. 6] observe "According to the hypothesis, h and h' are identical for times up to t, while h is ranked (weakly) better in every state at $t+1$. 'Therefore', it should be ranked better also at (t,ω). A stronger and more customary version of the axiom would require the same conclusion given the weaker hypothesis that

$$h_t(\omega) = h'_t(\omega) \text{ and } h \succsim_{t+1,\omega'} h' \text{ for all } \omega' \in G_t(\omega).$$

In fact, given CP, the two versions are equivalent." Again, we refer to [6] for a discussion of dynamic consistency, which might be sometimes controversial in the presence of ambiguity.[7]

A state $\omega'' \in \Omega$ is $\succsim_{t,\omega}$-null if

$$h(\tau', \omega') = h'(\tau', \omega') \text{ for all } \tau' \in \mathcal{T} \text{ and all } \omega' \neq \omega'' \text{ implies } h \sim_{t,\omega} h'.$$

Axiom 5 (Full support—FS) *No state in* Ω *is* $\succsim_{0,\Omega}$*-null.*

4 The Representation

We first extend to the current dynamic setting the notion of ambiguity index c we used in the static setting of [21]. A *dynamic ambiguity index* is a family $\{c_t\}_{t\in\mathcal{T}}$ of functions $c_t : \Omega \times \Delta(\Omega) \to [0,\infty]$ such that for all $t \in \mathcal{T}$:

(i) $c_t(\cdot, p) : \Omega \to [0,\infty]$ is $\mathcal{G}_t$-measurable for all $p \in \Delta(\Omega)$,[8]

(ii) $c_t(\omega, \cdot) : \Delta(\Omega) \to [0,\infty]$ is grounded,[9] closed and convex, with $\operatorname{dom} c_t(\omega,\cdot) \subseteq \Delta(G_t(\omega))$ and $\operatorname{dom} c_t(\omega,\cdot) \cap \Delta^{++}(G_t(\omega)) \neq \varnothing$, for all $\omega \in \Omega$.

[6] Notation: $\left(y_{-\{\tau,\tau+1\}}, x, x'\right) \equiv (y_0, ..., y_{\tau-1}, x, x', y_{\tau+2}, ..., y_T)$ if $\tau < T$ and $(y_0, ..., y_{T-1}, x)$ otherwise.

[7] Inspection of our proofs shows that the weaker version of DC in which $\succsim$ is replaced by $\sim$ is enough to obtain the results of the following section.

[8] Equivalently, $c_t(\omega,\cdot) = c_t(\omega',\cdot)$ for all $\omega, \omega' \in \Omega$ such that $G_t(\omega) = G_t(\omega')$.

[9] That is, $\min_{p\in\Delta(\Omega)} c_t(\omega, p) = 0$.

Observe that the effective domains of the $c_t(\omega, \cdot)$ consist of predictive distributions, that is, of the conditional probabilities on the nodes $G_t(\omega)$. In the terminology more used in the MP model, we would call them the Bayesian updates of the original priors $p \in \Delta(\Omega)$.

In our first result we characterize a dynamic version of variational preferences that do not necessarily satisfy dynamic consistency. Notice that in (9) we consider $\Delta^{++}(\Omega)$ in order to have well defined conditional probabilities $p_{G_t(\omega)}$.

Proposition 1 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, RP, and for each* $(t,\omega) \in \mathcal{T} \times \Omega$ *no state in* $G_t(\omega)$ *is* $\succsim_{t,\omega}$*-null.*

(b) There exist a scalar $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$ *such that, for each* $(t,\omega) \in \mathcal{T} \times \Omega$*,* $\succsim_{t,\omega}$ *is represented by*

$$V_t(\omega, h) \equiv \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}. \tag{9}$$

Moreover, $(\beta', u', \{c'_t\})$ *represents* $\{\succsim_{t,\omega}\}$ *in the sense of (9) if and only if* $\beta' = \beta$*,* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*, and* $\{c'_t\} = \{ac_t\}$*.*

As a result, for all $t \in \mathcal{T}$ and all $h \in \mathcal{H}$, the preference functional $V_t(\cdot, h)$ is a $\mathcal{G}_t$-measurable random variable

$$V_t(h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\mathrm{E}^p \left(\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \,|\mathcal{G}_t \right) + c_t(p\,|\mathcal{G}_t) \right).$$

We call *dynamic variational preferences* the (families of) preferences satisfying CP, VP, RP, and such that no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null. It is natural to wonder what restriction on the dynamic ambiguity index would characterize the dynamic variational preferences that satisfy dynamic consistency. This condition, which we have called the "no-gain condition" in the Introduction, is given in the next theorem, which is the main result of the paper.

Theorem 1 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, RP, FS, and DC.*

(b) There exist a scalar $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$ *such that, for each* $(t,\omega) \in \mathcal{T} \times \Omega$*,* $\succsim_{t,\omega}$ *is represented by*

$$V_t(\omega, h) \equiv \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right) \right) \quad \forall h \in \mathcal{H}, \tag{10}$$

and

$$c_t(\omega, q) = \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G) > 0}} q(G) c_{t+1}(G, q_G) + \min_{\left\{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}\right\}} c_t(\omega, p), \tag{11}$$

for all $q \in \Delta(G_t(\omega))$ *and all* $t < T$*.*

Moreover, $(\beta', u', \{c'_t\})$ *represents* $\{\succsim_{t,\omega}\}$ *in the sense of (10) if and only if* $\beta' = \beta$*,* $u' = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$*, and* $\{c'_t\} = \{ac_t\}$*.*

Therefore, dynamic variational preferences satisfy dynamic consistency if and only if their dynamic ambiguity index has the recursive structure (11), that is, if and only if the no-gain condition is satisfied and updating is done according to Bayes' Rule.

In turn, (11) delivers the recursive representation

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(h)\, dr + \gamma_t(\omega, r) \right) \tag{12}$$

of the agent's preference functional V_t, where

$$\gamma_t(\omega, r) = \min_{\left\{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r\right\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}), \tag{13}$$

(see Lemma 6 in the Appendix).

In view of all this, we call *recursive variational preferences* the dynamic variational preferences satisfying dynamic consistency, and we call *recursive ambiguity indexes* their dynamic ambiguity indexes, that is, the dynamic indexes satisfying the no-gain condition (11) for some $\beta > 0$.

Recall from the Introduction that the recursive formula (11) has a transparent interpretation under the game against Nature interpretation of our setting, in which $\{c_t\}$ is a dynamic cost for Nature. In fact, (11) suggests that the cost for Nature of choosing q at time t in state ω can be decomposed as the sum of: the discounted expected cost of choosing q's conditionals at time $t+1$,[10] plus the cost $\gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right)$ of inducing $q_{|\mathcal{G}_{t+1}}$ as one-period-ahead marginal. By (11) and (12), both Nature's costs and agent's preferences are recursive.

As (12) shows, in our recursive representation the evolution of ambiguity aversion is determined by how the functions $\gamma_t(\omega, \cdot)$ depend on t and ω. This will emerge clearly in the next Subsection. Here we observe that in applications some special specification of such dependence can be useful. For example, in the standard setup $\Omega = \mathcal{Z}^T$ discussed in Section 2.1 we can assume a Markovian structure, where $\gamma_t(\omega)$ depends on ω only through the last observation, or an independent structure, where $\gamma_t(\omega)$ does not depend on t and ω (see Example 6 below).

Behaviorally, these dependence structures can be characterized by suitable stationarity requirements on the preferences $\succsim_{t,\omega}$.

Finally, after the completion of an earlier version of this paper, we learned of independent work by Detlefsen and Scandolo [1], who arrive at a condition related to (11) in studying conditions for the time consistency of risk measures.

4.1 Going Backward

A main advantage of the recursive structure of the no-gain condition (11) is that it permits the construction by backward induction of recursive ambiguity indexes, and so of recursive variational preferences via (12) and (13).

The next result provides the key ingredient for the desired backward induction construction

Proposition 2 *Let $\{c_t\}$ be a dynamic ambiguity index. For all $t < T$ and $\omega \in \Omega$, set*[11]

$$\gamma_t(\omega, r) \equiv \min_{\left\{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r\right\}} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}).$$

The family $\{\gamma_t\}_{t<T}$ of functions $\gamma_t : \Omega \times \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ is such that for all $t < T$:

[10] In fact, $\sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G) c_{t+1}(G, q_G) = \int c_{t+1}(q\,|\mathcal{G}_{t+1})\, dq$.

[11] Here we adopt the convention that the minimum over the empty set is ∞.

(i) $\gamma_t(\cdot, r) : \Omega \to [0, \infty]$ *is* $\mathcal{G}_t$*-measurable for all* $r \in \Delta(\Omega, \mathcal{G}_{t+1})$*.*

(ii) $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ *is grounded, closed and convex, with* $\operatorname{dom} \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$ *and* $\operatorname{dom} \gamma_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1}) \neq \varnothing$*, for all* $\omega \in \Omega$*.*

The index $\gamma_t(\omega, r)$ can be interpreted as the cost for Nature of inducing r as one-period-ahead marginal, as suggested by (12) and (13). Since the properties of $\gamma_t(\omega, \cdot)$ on $\Delta(\Omega, \mathcal{G}_{t+1})$ are analogous to those of a static (or dynamic) ambiguity index on the set of the agent's subjective beliefs, we call *one-period-ahead ambiguity index* a family $\{\gamma_t\}_{t<T}$ of functions that satisfies conditions (i) and (ii) of Proposition 2.

Next we characterize recursive ambiguity indexes by means of one-period-ahead ones, thus giving the desired backward induction construction of recursive ambiguity indexes. Here δ_C is the indicator function defined in (3) and, given $\omega \in \Omega$, d_ω is the Dirac probability assigning mass 1 to ω.

Theorem 2 *Let* $\{c_t\}_{t\in\mathcal{T}}$ *be a family of functions from* $\Omega \times \Delta(\Omega)$ *to* $[0, \infty]$*. The following statements are equivalent:*

(a) $\{c_t\}$ *is a recursive ambiguity index.*

(b) There exist $\beta > 0$ *and a one-period-ahead ambiguity index* $\{\gamma_t\}$ *such that, for all* $\omega \in \Omega$*,*

$$c_T(\omega, \cdot) = \delta_{\{d_\omega\}}, \text{ and for all } t < T$$

$$c_t(\omega, q) = \begin{cases} \beta \sum_{\substack{G\in\mathcal{G}_{t+1} \\ q(G)>0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) & \forall q \in \Delta(G_t(\omega)) \\ \infty & \forall q \in \Delta(\Omega) \setminus \Delta(G_t(\omega)). \end{cases}$$

In this case, $\{\gamma_t\}$ *is unique and satisfies (13).*

The important implication is (b) ⇒ (a), which allows to construct any recursive ambiguity index by backward induction: it suffices to specify at any non-terminal node $G = G_t(\omega)$ a grounded, closed and convex function γ_G on the set of all probabilities on the branches springing from G.

This decomposition of cost functions in one-period-ahead components is a key feature of our derivation. The next example illustrates this feature by showing what happens in a binomial tree if we take at each non-terminal node the relative Gini concentration index $\chi^2(p\|q)$, defined in (19), as one-period-ahead ambiguity index.

Example 6 Consider Example 1 with $T = 2$, that is, $\Omega = \{0, 1\}^2$. We have:

$$\mathcal{G}_1 = \{\{0\}, \{1\}\} \quad \text{and} \quad \mathcal{G}_2 = \{\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}\},$$

where $\{0\} = \{(0, 0), (0, 1)\}$ and $\{1\} = \{(1, 0), (1, 1)\}$. Hence,

$$\begin{aligned} \Delta(\Omega, \mathcal{G}_1) &= \{(r, 1 - r) : r \in [0, 1]\}, \\ \Delta(\{0\}, \mathcal{G}_2) = \Delta(\{0\}) &= \{(r, 1 - r) : r \in [0, 1]\}, \\ \Delta(\{1\}, \mathcal{G}_2) = \Delta(\{1\}) &= \{(r, 1 - r) : r \in [0, 1]\}, \end{aligned}$$

and $\Delta(\Omega, \mathcal{G}_2) = \Delta(\Omega)$. Let $q \in \Delta(\Omega)$ be the uniform distribution with $q(\omega) = 1/4$ for all $\omega \in \Omega$, and set $\varphi(\pi) \equiv 2\pi^2 + 2(1 - \pi)^2 - 1$ for each $\pi \in [0, 1]$. Define

$$\gamma_0(\Omega, p) \equiv \chi^2\left(p\|q_{|\mathcal{G}_1}\right) = \varphi(p(0)) \quad \forall p \in \Delta(\Omega, \mathcal{G}_1),$$

$$\gamma_1(\{0\}, p) \equiv \chi^2\left(p\|q_{\{0\}}\right) = \begin{cases} \varphi(p(0, 0)) & \text{if } p \in \Delta(\{0\}), \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\gamma_1(\{1\},p) \equiv \chi^2(p\|q_{\{1\}}) = \begin{cases} \varphi(p(1,0)) & \text{if } p \in \Delta(\{1\}), \\ \infty & \text{otherwise,} \end{cases}$$

By Theorem 2, using these one-period-ahead ambiguity indexes we can construct a recursive dynamic index, given by:

$$c_1(\{0\},p) = \gamma_1(\{0\},p),$$
$$c_1(\{1\},p) = \gamma_1(\{1\},p),$$

and,

$$\begin{aligned} c_0(\Omega,p) &= \beta\left[p(\{0\})c_1(\{0\},p_{\{0\}}) + p(\{1\})c_1(\{1\},p_{\{1\}})\right] + \gamma_0(\Omega, p_{|\{\{0\},\{1\}\}}) \\ &= \beta\left[(p_{00}+p_{01})\varphi\left(\frac{p_{00}}{p_{00}+p_{01}}\right) + (p_{10}+p_{11})\varphi\left(\frac{p_{10}}{p_{10}+p_{11}}\right)\right] + \varphi(p_{00}+p_{01}), \end{aligned}$$

where we set $p_{ij} = p(i,j)$ for $i,j \in \{0,1\}$ and we adopt the convention $0\varphi(0/0) = 0$. ▲

5 Special Cases

5.1 Multiple Prior Preferences

We now show that Epstein and Schneider [6]'s characterization of dynamic MP preferences is a special case of ours, modulo some minor differences (they do not assume unboundedness and assume a slightly stronger version of dynamic consistency).

MP preferences are the special class of variational preferences satisfying the certainty independence condition of Gilboa and Schmeidler [9]. In the present dynamic setting, this amounts to consider:

MP(ii) For all $h, h' \in \mathcal{H}$, $y \in X^{\mathcal{T}}$, and $\alpha \in (0,1)$, $h \succsim_{t,\omega} h'$ if and only if $\alpha h + (1-\alpha)y \succsim_{t,\omega} \alpha h' + (1-\alpha)y$,

which is a stronger version of VP(ii) (in [21] we discuss the different behavioral implications of these two axioms).

Under the stronger MP(ii), the ambiguity index $c_t(\omega)$ becomes an indicator function, and the no-gain condition (11) coincides with rectangularity, which is the condition that [6] used to characterize recursive MP preferences.

Corollary 1 *Let $\{\succsim_{t,\omega}\}$ be a family of dynamic variational preferences. The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ satisfy MP(ii).

(b) For every t and ω, there exists a closed and convex subset $C_t(\omega)$ of $\Delta(\Omega)$ such that $c_t(\omega) = \delta_{C_t(\omega)}$.

In this case, condition (11) is equivalent to

$$C_t(\omega) = \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) : p^G \in C_{t+1}(G) \;\; \forall G \in \mathcal{G}_{t+1} \text{ and } r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}, \quad (14)$$

for all $\omega \in \Omega$ and $t < T$, where $C_{t+1}(G) = C_{t+1}(\omega')$ for all $\omega' \in G$, and $C_t(\omega)_{|\mathcal{G}_{t+1}}$ is the set of restrictions to the algebra generated by $\mathcal{G}_{t+1}$ of the probabilities in $C_t(\omega)$.

5.2 Multiplier Preferences

Given $p, q \in \Delta(\Omega)$, the *relative entropy* (or *Kullback-Leibler distance*) of p w.r.t. q is

$$R(p\|q) \equiv \begin{cases} \sum_{\omega\in\Omega} p(\omega)\log\dfrac{p(\omega)}{q(\omega)} & \text{if } p \ll q, \\ \infty & \text{otherwise,} \end{cases}$$

with the convention $0\ln(0/a) = 0$ for all $a \geq 0$. Analogously, if $p, q \in \Delta(\Omega, \mathcal{G})$, where $\mathcal{G}$ is a partition of Ω, the *relative entropy* of p w.r.t. q *on* $\mathcal{G}$ is

$$R_{\mathcal{G}}(p\|q) \equiv \begin{cases} \sum_{G\in\mathcal{G}} p(G)\log\dfrac{p(G)}{q(G)} & \text{if } p \ll q, \\ \infty & \text{otherwise,} \end{cases}$$

again with the convention $0\ln(0/a) = 0$ for all $a \geq 0$.

Given a reference probabilistic model $q \in \Delta^{++}(\Omega)$, we call *dynamic multiplier preferences* the family of preferences on $\mathcal{H}$ represented for every t and ω by

$$V_t(\omega, h) \equiv \inf_{p\in\Delta^{++}(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp_{G_t(\omega)} + \theta\beta^{-t}R\left(p_{G_t(\omega)}\|q_{G_t(\omega)}\right)\right) \quad \forall h\in\mathcal{H}. \tag{15}$$

The name is inspired by the robust control approach of Hansen and Sargent [11].[12] They interpret θ as a coefficient of uncertainty aversion, an interpretation we formalize and discuss in [21]. Observe that, by a classical variational formula (see [4, p. 34]), we can equivalently write (15) as:

$$V_t(\omega, h) = -\theta\beta^{-t}\log\left(\int\exp\left(-\sum_{\tau\geq t}\frac{\beta^\tau}{\theta}u(h_\tau)\right)dq_{G_t(\omega)}\right), \tag{16}$$

a very convenient expression in calculations.

Next we show that dynamic multiplier preferences are recursive variational preferences and their (recursive) ambiguity index is

$$c_t(\omega, p) = \theta\beta^{-t}R\left(p_{G_t(\omega)}\|q_{G_t(\omega)}\right) \tag{17}$$

for all $t \in \mathcal{T}$, $\omega \in \Omega$, and $p \in \Delta(\Omega)$.

Theorem 3 *For all $q \in \Delta^{++}(\Omega)$, $\beta > 0$, unbounded affine $u : X \to \mathbb{R}$, and $\theta > 0$, the dynamic multiplier preferences represented by (15) are recursive variational preferences with ambiguity index given by (17). In particular,*

$$V_t(\omega, h) = u(h_t(\omega)) + \min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\beta\int V_{t+1}(h)\,dr + \theta\beta^{-t}R_{\mathcal{G}_{t+1}}\left(r\|\left(q_{G_t(\omega)}\right)_{|\mathcal{G}_{t+1}}\right)\right), \tag{18}$$

for each $h \in \mathcal{H}$, $\omega \in \Omega$, and $t < T$.

The recursive formulation (18) is especially important because it makes it possible to use standard dynamic programming tools in studying optimization problems involving dynamic multiplier preferences. This class of dynamic variational preferences is therefore very tractable, something important for applications.

The recursive nature of multiplier preferences was already observed by Hansen and Sargent (see [11, p. 64]).[13] The contribution of Theorem 3 is to show that this is a very special case of the general recursive representation given in Theorems 1 and 2. As a result, Theorem 3 provides the proper theoretical underpinning for this crucial property of multiplier preferences.

[12] Clearly, the functionals $\beta^t V_t(\omega, h) = \inf_{p\in\Delta^{++}(\Omega)}\left(\int\sum_{\tau\geq t}\beta^\tau u(h_\tau)\,dp_{G_t(\omega)} + \theta R\left(p_{G_t(\omega)}\|q_{G_t(\omega)}\right)\right)$ represent the same preferences.

[13] Skiadas [29] studies the recursive structure of a continuous time version of a robust control preference functional.

5.3 Mean-Variance Preferences

We conclude by observing that Theorem 3 does not hold when we replace the relative entropy with a general convex statistical distance (see [20]). For example, consider the *relative Gini concentration index* (or χ^2*-distance*)

$$\chi^2 (p\|q) \equiv \begin{cases} \sum_{\omega\in\Omega} \dfrac{(p(\omega))^2}{q(\omega)} - 1 & \text{if } p \ll q, \\ \infty & \text{otherwise.} \end{cases} \tag{19}$$

In [21] and [23] we show that $\frac{\theta}{2}\chi^2 (p\|q)$ is the ambiguity index associated with the classic mean-variance preferences. For example, on the domain of monotonicity of such preferences we have:

$$\int f dq - \frac{1}{2\theta}\mathrm{Var}(f) = \min_{p\in\Delta} \left(\int f dp + \frac{\theta}{2}\chi^2 (p\|q) \right),$$

where $q \in \Delta^{++}(\Omega)$ is again a reference probability.

It is easily seen that the dynamic ambiguity index given by

$$c_t(\omega, p) \equiv \frac{\theta}{2}\beta^{-t}\chi^2 \left(p_{G_t(\omega)}\|q_{G_t(\omega)}\right)$$

is not recursive, and so the dynamic variational preferences represented by

$$V_t(\omega, h) \equiv \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + \frac{\theta}{2}\beta^{-t}\chi^2 \left(p_{G_t(\omega)}\|q_{G_t(\omega)}\right) \right)$$

are not dynamically consistent.

It is possible, however, to construct a dynamically consistent version of (monotone) mean-variance preferences along the lines of Example 6 by using the relative Gini concentration index as a one-period-ahead ambiguity index.

6 Differential Properties

Optimization problems are pervasive in economic applications and the differential properties of the involved preference functionals play a key role in their resolution. For this reason we now study the differential properties of our recursive variational preference functionals, and we show that their analytical tractability is adequate for applications.

This extends to the dynamic setting of this paper what we established in [21], where we showed that in the static case variational preference functionals have nice differentiability properties.

In this section we set $X = \Delta(\mathbb{R})$. Let $\mathcal{F}$ be the subset of $\mathcal{H}$ consisting of monetary (i.e., real valued) acts.[14] Throughout this section we consider a recursive variational preference functional $V_t(\omega, \cdot) : \mathcal{H} \to \mathbb{R}$, as given by Theorem 1. We make the standard assumption that the associated utility function $u : X \to \mathbb{R}$ is concave (thus reflecting risk aversion) and strictly increasing on $\mathbb{R}$.

Like Epstein and Wang [7], for $\omega \in \Omega$, $t < T$, and $f \in \mathcal{F}$, we call *one-period-ahead directional derivative* of $V_t(\omega, \cdot)$ at f the functional $V_t'(\omega, f; \cdot) : \mathcal{E}^t \to \mathbb{R}$ defined by

$$V_t'(\omega, f; e) \equiv \lim_{\lambda\downarrow 0} \frac{V_t(\omega, f + \lambda e) - V_t(\omega, f)}{\lambda} \quad \forall e \in \mathcal{E}^t,$$

where $\mathcal{E}^t$ is the subspace of $\mathcal{F}$ consisting of all processes e such that $e_\tau = 0$ if $\tau \neq t, t+1$. These processes represent current and one-period-ahead consumption perturbations.

The functional $V_t(\omega, \cdot)$ is *(one-period-ahead Gateaux) differentiable* at f if $V_t'(\omega, f; \cdot)$ is linear on $\mathcal{E}^t$. In this case, $V_t'(\omega, f; \cdot)$ is the *(Gateaux) differential* of $V_t(\omega, \cdot)$ at f.

[14] Under the usual identification of $z \in \mathbb{R}$ with $d_z \in X$.

Theorem 4 *Let $\omega \in \Omega$ and $t < T$. Then, $V_t(\omega, \cdot)$ is differentiable on $\mathcal{F}$ if and only if u is differentiable on $\mathbb{R}$ and $\gamma_t(\omega, \cdot)$ is essentially strictly convex.*[15] *In particular,*

$$V_t'(\omega, f; e) = u'(f_t(\omega)) e_t(\omega) + \beta \int e_{t+1} u'(f_{t+1}) \, d\rho \quad \forall f \in \mathcal{F}, e \in \mathcal{E}^t, \tag{20}$$

where $\{\rho\} = \arg\min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(f) \, dr + \gamma_t(\omega, r)\right)$.

This result provides a full characterization of differentiability for the recursive variational preference functional $V_t(\omega, \cdot)$, and it provides an explicit formula for evaluating the differential $V_t'(\omega, f; \cdot)$. We proved a static version of this result (as well as of Theorem 5) in [21], and it is important that

Observe that the strict convexity of $\gamma_t(\omega, \cdot)$ holds for all divergence preferences, a large class of variational preferences we introduced in [21] and that includes multiplier preferences. For example, by Theorem 3 (and by some well known properties of the relative entropy, see [4, p. 34]), formula (20) takes the following neat form for dynamic multiplier preferences :

$$V_t'(\omega, f; e) = u'(f_t(\omega)) e_t(\omega) + \beta \frac{\int e_{t+1} u'(f_{t+1}) \exp\left(-\frac{\beta^{t+1}}{\theta} V_{t+1}(f)\right) dq_{G_t(\omega)}}{\int \exp\left(-\frac{\beta^{t+1}}{\theta} V_{t+1}(f)\right) dq_{G_t(\omega)}}$$

for each $f \in \mathcal{F}$ and $e \in \mathcal{E}^t$.

As $V_t(\omega, \cdot)$ is concave, the powerful theory of superdifferentials can be used when $V_t(\omega, \cdot)$ is not differentiable. Besides its intrinsic interest, this is also important conceptually as points of non-differentiability, the so-called "kinks," play an important role in some applications of the multiple priors model and of the closely related Choquet expected utility model (see [3], [7], and [24]).

Denote by $\mathbb{M}(G_t(\omega), \mathcal{G}_{t+1})$ the set of all measures on $\mathcal{A}(\mathcal{G}_{t+1})$ that vanish on each subset of $G_t(\omega)^c$. A *one-period-ahead supergradient* of $V_t(\omega)$ at f is an element (k, m) of $\mathbb{R} \times \mathbb{M}(G_t(\omega), \mathcal{G}_{t+1})$ such that

$$V_t'(\omega, f; e) \leq k e_t(\omega) + \beta \int e_{t+1} dm, \qquad \forall e \in \mathcal{E}^t.$$

The *superdifferential* $\partial V_t(\omega, f)$ of $V_t(\omega, \cdot)$ at f is the set of all one-period-ahead supergradients at f. The superdifferential $\partial V_t(\omega, f)$ is a singleton if and only if $V_t(\omega, \cdot)$ is differentiable at f; in this case, $\partial V_t(\omega, f) = \{V_t'(\omega, f; \cdot)\}$.

The following result is the superdifferential version of Theorem 4.

Theorem 5 *For all $\omega \in \Omega$, $t < T$, and $f \in \mathcal{F}$, $\partial V_t(\omega, f)$ consists of all pairs*

$$(u'(f_t(\omega)), u'(f_{t+1}) \, d\rho) \tag{21}$$

such that $u'(f_t(\omega)) \in \partial u(f_t(\omega))$, $u'(f_{t+1})$ is a $\mathcal{G}_{t+1}$-measurable selection of $\partial u(f_{t+1})$, and $\rho \in \arg\min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(f) \, dr + \gamma_t(\omega, r)\right)$.[16]

Eq. (21) provides an explicit formula for the superdiffential $\partial V_t(\omega, f)$, which is equivalent to (20) when $\partial V_t(\omega, f)$ is a singleton, that is, when $V_t(\omega, \cdot)$ is differentiable at f.

Theorem 5 generalizes Epstein and Wang [7, Lemma 1], and we expect that this result can be used to extend their asset pricing analysis to recursive variational preferences.

Summing up, Theorems 4 and 5 show that dynamic variational preference functionals have nice differentiability properties, something we already established in and this extends

[15] For a formal definition of essential strict convexity see [26, p. 253]. Needless to say, a strictly convex functional is *a fortiori* essentially strictly convex.

[16] Here $\partial u(z)$ is the superdiffential of u at z, while $u'(f_{t+1}) \, d\rho$ is the measure with density $u'(f_{t+1})$ with respect to ρ.

7 Conclusions

Ambiguity adverse behavior is pervasive, and the theory of ambiguity aversion has found applications in macroeconomics, finance, even political analysis.

A widely accepted theory has been so far the theory of multiple priors of [9]. Different approaches, mostly found under the name of robust preferences, have made desirable an extension of this theory to include a larger class of behaviors. The extension, in the static case, has been provided by the theory of variational preferences introduced by [21]. This is, however, a theory of static choice, while most of the applications we have mentioned are in dynamic environments: hence, a further extension to the intertemporal problem is desirable. This paper provides such a theory.

Our main results can be summarized as follows. The first, Proposition 1, characterizes the intertemporal preferences that have a variational representation, the so-called dynamic variational preferences (intuitively, variational decision makers can be viewed as making their choices "as if" they think they are facing a malevolent opponent, which we call Nature).

The second result, Theorem 1, characterizes the dynamic preferences that are time consistent. In particular, a variational decision maker is dynamically consistent if and only if he thinks that Nature is also dynamically consistent.

The third result, Theorem 2, provides a decomposition of the cost function into one step ahead costs, paid by Nature in every period. This decomposition makes it possible to use recursive analysis in studying the dynamic choice problem of a decision maker with variational preferences.

The fourth result, Theorem 3, is an application of Theorem 1 and it shows that the dynamic consistency of the widely used multiplier preferences introduced by Hansen and Sargent is a consequence of our general Theorems 1 and 2.

The final results, Theorem 4 and 5, show that recursive variational preferences have nice differential properties, something crucial for their use in the optimization problems that arise in most economic applications.

We close by observing that, though in the paper we assumed both Ω and T finite, we expect that the extension to the infinite case can be carried out in standard ways. Moreover, even though in our representation theorems we consider standard discounted utility, some results can be extended to include hyperbolic discounting. For example, this can be done by weakening Axiom 3 along the lines of Hayashi [14]. However, the motivation behind hyperbolic discounting is very different from model uncertainty and for this reason here we prefer to use standard discounting in order to better focus the paper.

A Proofs and Related Material

An important tool for the proofs is Convex Analysis, we refer the reader to [26] and [16] for notation, definitions, and results.

Here we just remind that a function $I : C \to (-\infty, \infty]$, defined on a non-empty subset C of $\mathbb{R}^{\Omega}$, is *normalized* if $I(b1_{\Omega}) = b$ for all $b \in \mathbb{R}$ such that $b1_{\Omega} \in C$;[17] it is a *(finite) niveloid* if $I(C) \subseteq \mathbb{R}$ and $I(\psi) - I(\varphi) \leq \sup_{\omega\in\Omega}(\psi(\omega) - \varphi(\omega))$ for all $\psi, \varphi \in C$; it is *grounded* if $\inf_{\psi\in C} I(\psi) = 0$, it is *proper* if it is not identically ∞ and there is an affine function minorizing it. Niveloids are comprehensively studied in Dolecki and Greco [2] and Maccheroni, Marinacci, and Rustichini [22]. When $R \in \{\mathbb{R}, \mathbb{R}^{+}, \mathbb{R}^{++}, \mathbb{R}^{-}, \mathbb{R}^{--}\}$ and $C = R^{\Omega}$, I is a niveloid if and only if I is *monotonic* ($I(\psi) \geq I(\varphi)$ for all $\psi, \varphi \in R^{\Omega}$ such that $\psi \geq \varphi$) and *vertically invariant* ($I(\psi + b) = I(\psi) + b$ for all $\psi \in R^{\Omega}$ and $b \in R$). We will use the following lemmas:

[17] 1_{Ω} is the constant vector $(1, 1, ..., 1)$. Sometimes we will write b instead of $b1_{\Omega}$.

Lemma 1 *Let $J : \mathbb{R}^\Omega \to \mathbb{R}$ be concave, $C \subseteq \mathbb{R}^\Omega$ and $I : C \to [-\infty, \infty]$. The following statements are equivalent:*

(a) $J(\varphi) = \inf_{\psi \in C} (\langle \varphi, \psi \rangle + I(\psi))$ for all $\varphi \in \mathbb{R}^\Omega$;

(b) $I : C \to (-\infty, \infty]$ is proper and $\overline{\text{co}}\, I = -J^*$.[18]

Lemma 2 *Let C be a convex compact subset of $\mathbb{R}^\Omega$, and $I : C \to (-\infty, \infty]$ a proper closed and convex function. Then $\inf_{\psi \in \text{ri}\, C} I(\psi) = \min_{\psi \in C} I(\psi)$ iff $\text{ri}\, C \cap \text{dom}\, I \neq \varnothing$ iff $\inf_{\psi \in \text{ri}\, C} I(\psi) < \infty$.*[19]

Lemma 3 *Let $J : \mathbb{R}^\Omega \to \mathbb{R}$ be a concave normalized niveloid, and $G \subseteq \Omega$. The following statements are equivalent:*

(a) If $\varphi, \psi \in \mathbb{R}^\Omega$ are such that $\varphi_{|G} = \psi_{|G}$, then $J(\varphi) = J(\psi)$;

(b) $J(\varphi + \psi) = J(\varphi) + \psi(G)$ if $\varphi, \psi \in \mathbb{R}^\Omega$ and ψ is constant on G;

(c) $J(\varphi 1_{G^c}) = 0$ for all $\varphi \in \mathbb{R}^\Omega$;[20]

(d) $\text{dom}\, J^* \subseteq \Delta(G)$.

Next lemma is the first important step towards the proof of Proposition 1.

Lemma 4 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ satisfy CP, VP, and RP.

(b) There exists a family $\{c_t(\omega, \cdot) : (t, \omega) \in \mathcal{T} \times \Omega\}$ of grounded, closed and convex functions $c_t(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$, such that $\text{dom}\, c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$ and $c_t(\omega, \cdot) = c_t(\omega', \cdot)$ if $G_t(\omega) = G_t(\omega')$, $\beta > 0$, and an unbounded affine $u : X \to \mathbb{R}$ such that: for every t and ω, $\succsim_{t,\omega}$ is represented by $V_t(\omega, \cdot)$, where

$$V_t(\omega, h) \equiv \min_{p \in \Delta(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right) \quad \forall h \in \mathcal{H}. \tag{22}$$

Moreover, $(\bar{\beta}, \bar{u}, \{\bar{c}_t(\omega, \cdot)\})$ represent $\succsim_{t,\omega}$ in the sense of (22) iff $\bar{\beta} = \beta$, $\bar{u} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$ and $\{\bar{c}_t(\omega, \cdot)\} = \{ac_t(\omega, \cdot)\}$.

Finally, if $|G_t(\omega)| > 1$, the following facts are equivalent:

(i) for all $h \in \mathcal{H}$,

$$V_t(\omega, h) = \inf_{p \in \text{ri}\, \Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right); \tag{23}$$

(ii) no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null;

(iii) $\text{dom}\, c_t(\omega, \cdot) \cap \text{ri}\, \Delta(G_t(\omega)) \neq \varnothing$.

Notice that (22) can be rewritten as

$$V_t(\omega, h) = \min_{p \in \Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_t(\omega, p) \right) \quad \forall h \in \mathcal{H}. \tag{24}$$

Moreover, if $|G_t(\omega)| = 1$, $G_t(\omega)$ is a singleton $\{\omega\}$, $\Delta(G_t(\omega)) = \text{ri}\, \Delta(G_t(\omega))$ and (iii) is automatically satisfied, in this case both (22) and (23) collapse to $V_t(\omega, h) = \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega))$.

For the rest of the paper, we indifferently write $c_t(\omega, \cdot)$, $c_t(\omega)$, or $c_{t,\omega}$, and $V_t(\omega, \cdot)$, $V_t(\omega)$ or $V_{t,\omega}$.

[18] $\overline{\text{co}}\, I$ denotes the closed and convex hull of I, J^* the concave conjugate of J.

[19] $\text{ri}\, C$ denotes the relative interior of C, $\text{dom}\, I$ the effective domain of I.

[20] For every $A \subseteq \Omega$, 1_A is the vector defined by $1_A(\omega) \equiv 1$ if $\omega \in A$, $1_A(\omega) \equiv 0$ if $\omega \notin A$.

Proof. (a) $\Rightarrow$ (b). A variation on the proof of Lemma A.1 of [6] (see also [30]) delivers the first two steps:

Step 1. There exist $\beta > 0$ and an unbounded affine $u : X \to \mathbb{R}$ such that, for all $(t, \omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ on $X^{\mathcal{T}}$ is represented by $U_t(y) \equiv \sum_{\tau \geq t} \beta^{\tau - t} u(y_\tau)$ for all $y \in X^{\mathcal{T}}$.

Step 2. For all $(t, \omega) \in \mathcal{T} \times \Omega$, and all $h \in \mathcal{H}$ there exists $y = y(t, \omega, h) \in X^{\mathcal{T}}$ (indeed constant) such that $y \sim_{t,\omega} h$.

Then it is easy to check that:

Step 3. For all $(t, \omega) \in \mathcal{T} \times \Omega$, the correspondence $V_{t,\omega} : \mathcal{H} \to \mathbb{R}$, defined by $V_{t,\omega}(h) \equiv U_t(y)$ if $h \sim_{t,\omega} y \in X^{\mathcal{T}}$, is a well defined function that represents $\succsim_{t,\omega}$ on $\mathcal{H}$.

Each $h \in \mathcal{H}$ can be regarded as a function $h : \Omega \to X^{\mathcal{T}}$, and $U_t : X^{\mathcal{T}} \to \mathbb{R}$ is an affine function for every $t \in \mathcal{T}$. Set $U_t(h) \equiv U_t \circ h$. By definition, $U_t(h) : \Omega \to \mathbb{R}$ and $U_t(h)(\omega') \equiv U_t(h(\omega')) = \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega'))$ for all $\omega' \in \Omega$. In particular, if $y(\tau, \omega') = y_\tau$ for all $\tau \in \mathcal{T}$ and all $\omega' \in \Omega$, then $U_t(y)(\omega') = \sum_{\tau \geq t} \beta^{\tau - t} u(y_\tau) = U_t(y_0, ..., y_T)$ for all $\omega' \in \Omega$.[21] Moreover, $U_t(\alpha h + (1 - \alpha) h') = \alpha U_t(h) + (1 - \alpha) U_t(h')$ for all $h, h' \in \mathcal{H}$ and $\alpha \in [0, 1]$. Up to a cardinal transformation of u, we can assume $u(X) \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^{++}, \mathbb{R}^-, \mathbb{R}^{--}\}$. For the rest of the proof the case $u(X) = \mathbb{R}^{++}$ is considered (the arguments we use can be easily adapted to the remaining ones).

Step 4. For all $t \in \mathcal{T}$, $\{U_t(h) : h \in \mathcal{H}\} = u(X)^\Omega$.

Proof. The inclusion $\subseteq$ is trivial. If $t < T$ and $\psi \in u(X)^\Omega$, there exists $\varepsilon > 0$ such that $\psi - \varepsilon \in u(X)^\Omega$, choose $x^\varepsilon \in X$ such that $u(x^\varepsilon) = \left(\sum_{T > \tau \geq t} \beta^{\tau - t}\right)^{-1} \varepsilon$. For all $\omega' \in \Omega$, there exists $x^{\psi(\omega')} \in X$ such that $u\left(x^{\psi(\omega')}\right) = \beta^{t - T}(\psi(\omega') - \varepsilon)$. Set

$$h(\tau, \omega') \equiv \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\psi(\omega')} & \text{if } \tau = T. \end{cases}$$

This delivers, $U_t(h(\omega')) = \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega')) = \sum_{T > \tau \geq t} \beta^{\tau - t} u(x^\varepsilon) + \beta^{T - t} u\left(x^{\psi(\omega')}\right) = u(x^\varepsilon)\left(\sum_{T > \tau \geq t} \beta^{\tau - t}\right) + \psi(\omega') - \varepsilon = \psi(\omega')$ for all $\omega' \in \Omega$; as wanted. If $t = T$, set $\varepsilon = 0$ and choose x^ε arbitrarily. $\square$

Step 5. For all $(t, \omega) \in \mathcal{T} \times \Omega$, the correspondence $I_{t,\omega} : u(X)^\Omega \to \mathbb{R}$, defined by $I_{t,\omega}(\psi) \equiv V_{t,\omega}(h)$ if $\psi = U_t(h)$ for some $h \in \mathcal{H}$, is a well defined, monotonic, and normalized function.

Proof. If h and h' in $\mathcal{H}$ are such that $\psi = U_t(h) = U_t(h')$, for all $\omega' \in \Omega$ we have $\sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega')) = \sum_{\tau \geq t} \beta^{\tau - t} u(h'_\tau(\omega'))$ and $h(\omega') \sim_{t,\omega} h'(\omega')$. By VP(iv), $h \sim_{t,\omega} h'$ and $V_{t,\omega}(h) = V_{t,\omega}(h')$. This implies that $I_{t,\omega}$ is a well defined function since for every $\psi \in u(X)^\Omega$ there is $h \in \mathcal{H}$ such that $\psi = U_t(h)$. Monotonicity is proved along the same lines. As to normalization, if $b \in u(X)$, take $x^b \in X$ such that $u\left(x^b\right) = \left(\sum_{\tau \geq t} \beta^{\tau - t}\right)^{-1} b$, and the constant act x^b to obtain $U_t\left(x^b\right)(\omega') = \sum_{\tau \geq t} \beta^{\tau - t} u\left(x^b\right) = b$ for all $\omega' \in \Omega$, then $b 1_\Omega = U_t\left(x^b\right)$ (where x^b is regarded as a constant act) and $I_{t,\omega}(b 1_\Omega) = V_{t,\omega}\left(x^b\right) = U_t\left(x^b, x^b, ..., x^b\right) = b$. $\square$

Step 6. Let $(t, \omega) \in \mathcal{T} \times \Omega$. For every $\psi \in u(X)^\Omega$ and for every $b \in \mathbb{R}$ such that $\psi + b \in u(X)^\Omega$, $I_{t,\omega}(\psi + b) = I_{t,\omega}(\psi) + b$.

Proof. Let $\psi' = U_t(h'), \psi'' = U_t(h'') \in u(X)^\Omega$, $b' = U_t(x'), b'' = U_t(x'') \in u(X)$, VP(ii) guarantees that for all $\alpha \in (0, 1)$, $\alpha h' + (1 - \alpha) x' \sim_{t,\omega} \alpha h'' + (1 - \alpha) x'$ implies $\alpha h' + (1 - \alpha) x'' \sim_{t,\omega} \alpha h'' +$

[21] The identification between acts with consequences depending only on time (and not on state) and elements of $X^{\mathcal{T}}$ corresponds here to the equivalence between constant functions on Ω and real numbers.

$(1-\alpha)x''$, i.e. $V_{t,\omega}(\alpha h' + (1-\alpha)x') = V_{t,\omega}(\alpha h'' + (1-\alpha)x')$ implies $V_{t,\omega}(\alpha h' + (1-\alpha)x'') = V_{t,\omega}(\alpha h'' + (1-\alpha)x'')$, hence $I_{t,\omega}(U_t(\alpha h' + (1-\alpha)x')) = I_{t,\omega}(U_t(\alpha h'' + (1-\alpha)x'))$ implies that $I_{t,\omega}(U_t(\alpha h' + (1-\alpha)x'')) = I_{t,\omega}(U_t(\alpha h'' + (1-\alpha)x''))$, and, finally, $I_{t,\omega}(\alpha\psi' + (1-\alpha)b') = I_{t,\omega}(\alpha\psi'' + (1-\alpha)b')$ implies $I_{t,\omega}(\alpha\psi' + (1-\alpha)b'') = I_{t,\omega}(\alpha\psi'' + (1-\alpha)b'')$. Replacing ψ' with $\psi'/\alpha \in u(X)^\Omega$, ψ'' with $\psi''/\alpha \in u(X)^\Omega$, b' with $b'/(1-\alpha) \in u(X)$, b'' with $b'/(1-\alpha) \in u(X)$, it follows that

$$I_{t,\omega}(\psi' + b') = I_{t,\omega}(\psi'' + b') \text{ implies } I_{t,\omega}(\psi' + b'') = I_{t,\omega}(\psi'' + b'') \tag{25}$$

for all $\psi', \psi'' \in u(X)^\Omega$, $b', b'' \in u(X)$. Let $\psi \in u(X)^\Omega$, then $\min_{\omega'} \psi(\omega') \in u(X)$, but $I_{t,\omega}$ is monotonic and normalized, thus $I_{t,\omega}(\psi) \geq I_{t,\omega}(\min_{\omega'} \psi(\omega')) = \min_{\omega'} \psi(\omega') \in u(X)$, and hence $I_{t,\omega}(\psi) \in u(X)$. Let $b > 0$, there is $\varepsilon > 0$ such that $\psi - \varepsilon \in u(X)^\Omega$ and $I_{t,\omega}(\psi) - \varepsilon \in u(X)$. By normalization and (25) $I_{t,\omega}((\psi - \varepsilon) + \varepsilon) = I_{t,\omega}(\psi) = I_{t,\omega}((I_{t,\omega}(\psi) - \varepsilon) + \varepsilon)$ implies $I_{t,\omega}(\psi + b) = I_{t,\omega}((\psi - \varepsilon) + (\varepsilon + b)) = I_{t,\omega}((I_{t,\omega}(\psi) - \varepsilon) + (\varepsilon + b)) = I_{t,\omega}(\psi) + b$. If $b < 0$, then $I_{t,\omega}(\psi) = I_{t,\omega}((\psi + b) - b) = I_{t,\omega}(\psi + b) - b$, as wanted. □

Moreover, from VP(v), it immediately follows that:

Step 7. Let $(t,\omega) \in \mathcal{T} \times \Omega$. For every $\psi, \psi' \in u(X)^\Omega$ such that $I_{t,\omega}(\psi) = I_{t,\omega}(\psi')$, and every $\alpha \in (0,1)$, $I_{t,\omega}(\alpha\psi + (1-\alpha)\psi') \geq I_{t,\omega}(\psi)$.

Steps 5–7 and the results we prove in [22] imply that: For all $(t,\omega) \in \mathcal{T} \times \Omega$, $I_{t,\omega}$ is a *concave and normalized niveloid* on $u(X)^\Omega$. The restriction of its concave conjugate to $\Delta(\Omega)$, $I^*_{t,\omega}(p) \equiv \inf_{\psi \in u(X)^\Omega} (\langle \psi, p \rangle - I_{t,\omega}(\psi))$ for all $p \in \Delta(\Omega)$, is the unique concave and upper semicontinuous function $I^\#_{t,\omega}$ on $\Delta(\Omega)$ such that $I_{t,\omega}(\psi) = \min_{p \in \Delta(\Omega)} \left(\langle \psi, p \rangle - I^\#_{t,\omega}(p)\right)$ for all $\psi \in u(X)^\Omega$. Moreover, the correspondence $J_{t,\omega} : \mathbb{R}^\Omega \to \mathbb{R}$, defined by $J_{t,\omega}(\varphi) \equiv I_{t,\omega}(\varphi + b) - b$ if $\varphi \in \mathbb{R}^\Omega$ and $b \in \mathbb{R}$ is such that $\varphi + b \in u(X)^\Omega$, is a normalized concave niveloid and its concave conjugate $J^*_{t,\omega}$ coincides with $I^*_{t,\omega}$ on $\Delta(\Omega)$ and takes value $-\infty$ on $\mathbb{R}^\Omega \backslash \Delta(\Omega)$.[22] In particular $J_{t,\omega}(\varphi) = \min_{p \in \Delta(\Omega)} \left(\langle \varphi, p \rangle - I^*_{t,\omega}(p)\right)$ for all $\varphi \in \mathbb{R}^\Omega$. (See [22] for details.) By CP(i), if $G_t(\omega) = G_t(\omega')$, we can choose $I_{t,\omega} = I_{t,\omega'}$ and set $c_{t,\omega}(p) \equiv -I^*_{t,\omega}(p) = -J^*_{t,\omega}(p)$ for all $p \in \Delta(\Omega)$ and all $(t,\omega) \in \mathcal{T} \times \Omega$. Then $c_{t,\omega} : \Delta(\Omega) \to [0,\infty]$ is grounded, closed, and convex for all $(t,\omega) \in \mathcal{T} \times \Omega$; $c_{t,\omega'} = c_{t,\omega}$ if $G_t(\omega) = G_t(\omega')$; for all $(t,\omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by $V_{t,\omega}(h) = I_{t,\omega}(U_t(h)) = J_{t,\omega}(U_t(h)) = \min_{p \in \Delta(\Omega)} \left(\langle U_t(h), p \rangle - I^*_{t,\omega}(p)\right)$, that is

$$V_t(\omega, h) = \min_{p \in \Delta(\Omega)} \left(\sum_{\omega' \in \Omega} p(\omega') \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau(\omega')) + c_t(\omega, p) \right).$$

Step 8. Let $(t,\omega) \in \mathcal{T} \times \Omega$. If $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ and $\varphi^1_{|G_t(\omega)} = \varphi^2_{|G_t(\omega)}$, then $J_{t,\omega}(\varphi^1) = J_{t,\omega}(\varphi^2)$.

Proof. It suffices to show that: if $\psi^1, \psi^2 \in u(X)^\Omega$ and $\psi^1_{|G_t(\omega)} = \psi^2_{|G_t(\omega)}$, then $I_{t,\omega}(\psi^1) = I_{t,\omega}(\psi^2)$. For $i = 1,2$, define

$$h^i(\tau, \omega') \equiv \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T \end{cases}$$

like in Step 4, with the precaution of choosing $x^{\psi^1(\omega')} = x^{\psi^2(\omega')}$ if $\omega' \in G_t(\omega)$ (this is possible since $\psi^1(\omega') = \psi^2(\omega')$). By construction, $h^1(\tau, \omega') = h^2(\tau, \omega')$ for all $\tau \geq t$ and $\omega' \in G_t(\omega)$. CP(ii) implies $h^1 \sim_{t,\omega} h^2$ and $I_{t,\omega}(\psi^1) = I_{t,\omega}(U_t(h^1)) = V_{t,\omega}(h^1) = V_{t,\omega}(h^2) = I_{t,\omega}(\psi^2)$, as wanted. □

By Lemma 3, $\operatorname{dom} c_{t,\omega} = \operatorname{dom} J^*_{t,\omega} \subseteq \Delta(G_t(\omega))$. This concludes the proof of (a) ⇒ (b).

(b) ⇒ (a) and the uniqueness properties of $(\beta, u, \{c_t(\omega, \cdot)\})$ can be easily checked (though the verification is a bit long).

[22] $J_{t,\omega}$ is the unique vertically invariant function that extends $I_{t,\omega}$ to $\mathbb{R}^\Omega$.

Step 9. Let $(t,\omega) \in \mathcal{T} \times \Omega$ be such that $|G_t(\omega)| > 1$. A state $\omega'' \in G_t(\omega)$ is $\succsim_{t,\omega}$-null if and only if $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$.

Proof. We show that if $\omega'' \in G_t(\omega)$ is $\succsim_{t,\omega}$-null, then $J_{t,\omega}(\psi^1) = J_{t,\omega}(\psi^2)$ for every $\psi^1, \psi^2 \in \mathbb{R}^\Omega$ such that $\psi^1_{|G_t(\omega)\setminus\{\omega''\}} = \psi^2_{|G_t(\omega)\setminus\{\omega''\}}$. By Lemma 3, $\operatorname{dom} c_{t,\omega} = \operatorname{dom} J^*_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$. Again, it is sufficient to show it for $\psi^1, \psi^2 \in u(X)^\Omega$. For $i = 1, 2$, define

$$h^i(\tau,\omega') \equiv \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T \end{cases} \quad \text{and } g^i(\tau,\omega') \equiv \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\psi^i(\omega')} & \text{if } \tau = T \text{ and } \omega' \neq \omega'' \\ x^\varepsilon & \text{if } \tau = T \text{ and } \omega' = \omega'' \end{cases}$$

where x^ε and $x^{\psi^i(\omega')}$ are defined like in Step 4, with the precaution of choosing $x^{\psi^1(\omega')} = x^{\psi^2(\omega')}$ if $\omega' \in G_t(\omega) \setminus \{\omega''\}$ (this is possible since $\psi^1(\omega') = \psi^2(\omega')$). Since $g^1(\tau,\omega') = g^2(\tau,\omega')$ for all $\tau \geq t$ and $\omega' \in G_t(\omega)$, CP(ii) implies $g^1 \sim_{t,\omega} g^2$, while $\succsim_{t,\omega}$-nullity of ω'' implies $h^i \sim_{t,\omega} g^i$ for $i = 1, 2$. Therefore, $h^1 \sim_{t,\omega} h^2$, and $I_{t,\omega}(\psi^1) = I_{t,\omega}(U_t(h^1)) = V_{t,\omega}(h^1) = V_{t,\omega}(h^2) = I_{t,\omega}(\psi^2)$. The converse is easily checked. □

Therefore, if a state ω'' in $G_t(\omega)$ is $\succsim_{t,\omega}$-null, then $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$, and $V_t(\omega, h) \neq \infty = \inf_{p \in \operatorname{ri}\Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_{t,\omega}(p) \right)$ for some (indeed all) $h \in \mathcal{H}$. That is (i) ⇒ (ii). Conversely, if $V_t(\omega, h) \neq \inf_{p \in \operatorname{ri}\Delta(G_t(\omega))} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau)\, dp + c_{t,\omega}(p) \right)$ for some $h \in \mathcal{H}$, then, by Lemma 2, $\operatorname{ri}\Delta(G_t(\omega)) \cap \operatorname{dom} c_{t,\omega} = \varnothing$. If, per contra, $\operatorname{dom} c_{t,\omega}$ is not contained in $\Delta(G_t(\omega) \setminus \{\omega''\})$ for some $\omega'' \in G_t(\omega)$, then for all $\omega' \in G_t(\omega)$ there exists $p^{\omega'} \in \operatorname{dom} c_{t,\omega}$ with $\omega' \in \operatorname{supp} p^{\omega'}$, then $|G_t(\omega)|^{-1} \sum_{\omega' \in G_t(\omega)} p^{\omega'} \in \operatorname{ri}\Delta(G_t(\omega)) \cap \operatorname{dom} c_{t,\omega}$, which is absurd. Then $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega) \setminus \{\omega''\})$ for some ω'', which must be $\succsim_{t,\omega}$-null. This is (ii) ⇒ (i). The equivalence between (i) and (iii) descends immediately from Lemma 2. ■

Lemma 5 *If $\{\succsim_{t,\omega}\}$ satisfy CP, FS, and DC, then for each t and ω, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null provided $|G_t(\omega)| > 1$.*

Proof. Assume, *per contra*, that there exist $\omega^\circ \in \Omega$ and $t^\circ \in \mathcal{T}$ such that $|G_{t^\circ}(\omega^\circ)| > 1$ and $G_{t^\circ}(\omega^\circ)$ contains a $\succsim_{t^\circ,\omega^\circ}$-null state. W.l.o.g., ω° is $\succsim_{t^\circ,\omega^\circ}$-null. By FS, $t^\circ > 0$ and

$$h(\tau',\omega') = h'(\tau',\omega') \text{ for all } \tau' \in \mathcal{T} \text{ and all } \omega' \neq \omega^\circ \text{ implies } h \sim_{t^\circ,\omega^\circ} h'. \tag{26}$$

Clearly, $|G_{t^\circ-1}(\omega^\circ)| \geq |G_{t^\circ}(\omega^\circ)| > 1$. Next we show that ω° is $\succsim_{t^\circ-1,\omega^\circ}$-null. In a finite number of steps this leads to an absurd.

Assume that $h(\tau',\omega') = h'(\tau',\omega')$ for all $\tau' \in \mathcal{T}$ and all $\omega' \neq \omega^\circ$. By (26) and CP(i), $h \sim_{t^\circ,\omega} h'$ for all $\omega \in G_{t^\circ}(\omega^\circ)$. Moreover, if $\omega \in G_{t^\circ-1}(\omega^\circ) \setminus G_{t^\circ}(\omega^\circ)$, then $G_{t^\circ}(\omega)$ does not contain ω°, and $h(\tau',\omega') = h'(\tau',\omega')$ for all $\tau' \in \mathcal{T}$ and all $\omega' \in G_{t^\circ}(\omega)$. By CP(ii), $h \sim_{t^\circ,\omega} h'$ for all $\omega \in G_{t^\circ-1}(\omega^\circ) \setminus G_{t^\circ}(\omega^\circ)$. Therefore, $h \sim_{t^\circ,\omega} h'$ for all $\omega \in G_{t^\circ-1}(\omega^\circ)$. Since $|G_{t^\circ-1}(\omega^\circ)| > 1$ and $h_{t^\circ-1}$ is $\mathcal{G}_{t^\circ-1}$ measurable, choose $\omega'' \in G_{t^\circ-1}(\omega^\circ) - \{\omega^\circ\}$ to obtain $h(t^\circ-1,\omega^\circ) = h(t^\circ-1,\omega'') = h'(t^\circ-1,\omega'') = h'(t^\circ-1,\omega^\circ)$ and conclude $h(t^\circ-1,\omega^\circ) = h'(t^\circ-1,\omega^\circ)$ and $h \sim_{t^\circ,\omega} h'$ for all $\omega \in G_{t^\circ-1}(\omega^\circ)$, DC implies that $h \sim_{t^\circ-1,\omega^\circ} h'$. That is ω° is $\succsim_{t^\circ-1,\omega^\circ}$-null. As wanted. ■

A.1 Proof of Proposition 1

Axiom 6 (Strong full support—SFS) *For each $(t,\omega) \in \mathcal{T} \times \Omega$, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null.*

For technical reasons we prove a slightly more general version of Proposition 1.

Proposition 3 *The following statements are equivalent:*

(a) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, RP, and no state in* $G_t(\omega)$ *is* $\succsim_{t,\omega}$*-null if* $G_t(\omega)$ *is not a singleton.*

(b) There exist a scalar $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$*, such that: for each* $(t,\omega) \in \mathcal{T} \times \Omega$*,* $\succsim_{t,\omega}$ *is represented by the functional* $V_t(\omega,\cdot)$ *defined by (23).*

(c) $\{\succsim_{t,\omega}\}$ *satisfy CP, VP, RP, and SFS.*

Moreover, $(\bar{\beta},\bar{u},\{\bar{c}_t\})$ *represent* $\succsim_{t,\omega}$ *in the sense of (b) iff* $\bar{\beta} = \beta$*,* $\bar{u} = au + b$ *for some* $a > 0$ *and* $b \in \mathbb{R}$ *and* $\{\bar{c}_t\} = \{ac_t\}$*.*

Proof. (a) $\Leftrightarrow$ (b) immediately descends from Lemma 4.

(c) $\Rightarrow$ (a) is trivial.

(b) $\Rightarrow$ (c). Since (b) $\Rightarrow$ (a), if $G_t(\omega)$ is not a singleton, no state in $G_t(\omega)$ is $\succsim_{t,\omega}$-null. Let $G_t(\omega)$ be a singleton $\{\omega\}$. Then $\Delta(G_t(\omega)) = \{d_\omega\}$, and $V_t(\omega,h) = \sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau(\omega))$ for all $h \in \mathcal{H}$. Since u is unbounded, there are $x^1, x^2 \in X$ such that $u(x^1) > u(x^2)$. Consider the acts

$$h^i(\tau,\omega') \equiv \begin{cases} x^1 & \text{if } (\tau,\omega') \neq (T,\omega) \\ x^i & \text{if } (\tau,\omega') = (T,\omega) \end{cases}$$

$h^1(\tau,\omega') = h^2(\tau,\omega')$ for all $\tau \in \mathcal{T}$ and all $\omega' \neq \omega$. If ω where $\succsim_{t,\omega}$-null, we would have $h^1 \sim_{t,\omega} h^2$, but $V_t(\omega,h^1) = \sum_{\tau\geq t}\beta^{\tau-t}u(x^1) > \sum_{T>\tau\geq t}\beta^{\tau-t}u(x^1) + \beta^{T-t}u(x^2) = V_t(\omega,h^2)$. Therefore ω is not $\succsim_{t,\omega}$-null. ■

Since for every t and ω, $\operatorname{ri}\Delta(G_t(\omega)) = \{p_{G_t(\omega)} : p \in \operatorname{ri}\Delta(\Omega)\}$, (23) is equivalent to

$$V_t(\omega,h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp_{G_t(\omega)} + c_t(\omega,p_{G_t(\omega)})\right) \quad \forall h \in \mathcal{H}. \tag{27}$$

Consider the $\mathcal{G}_t$ measurable functions $V_t(\cdot,h) : \Omega \to \mathbb{R}$ and $c_t(\cdot,p) : \Omega \to [0,\infty]$, (27) becomes

$$V_t(\cdot,h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)}\left(\mathrm{E}^p\left(\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,|\mathcal{G}_t\right)(\cdot) + c_t(\cdot,p\,|\mathcal{G}_t(\cdot))\right) \quad \forall h \in \mathcal{H}, \tag{28}$$

or

$$V_t(h) = \inf_{p\in \operatorname{ri}\Delta(\Omega)}\left(\mathrm{E}^p\left(\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,|\mathcal{G}_t\right) + c_t(p\,|\mathcal{G}_t)\right) \quad \forall h \in \mathcal{H}. \tag{29}$$

By Lemma 5, if $\{\succsim_{t,\omega}\}$ satisfies CP, VP, RP, DC and FS, then it admits this representation.

A.2 Dynamic Consistency

Lemma 6 *Let* $\{\succsim_{t,\omega}\}$ *be a family of preferences on* $\mathcal{H}$ *for which there exist a scalar* $\beta > 0$*, an unbounded affine function* $u : X \to \mathbb{R}$*, and a dynamic ambiguity index* $\{c_t\}$*, such that: for each* $(t,\omega) \in \mathcal{T} \times \Omega$*,* $\succsim_{t,\omega}$ *is represented by:*

$$V_t(\omega,h) \equiv \min_{p\in\Delta(G_t(\omega))}\left(\int\sum_{\tau\geq t}\beta^{\tau-t}u(h_\tau)\,dp + c_t(\omega,p)\right) \quad \forall h \in \mathcal{H}.$$

The following statements are equivalent:

(a) $\{\succsim_{t,\omega}\}$ *satisfy DC.*

(b) For all $t < T$*,* $\omega \in \Omega$*, and* $q \in \Delta(G_t(\omega))$*,*

$$c_t(\omega,q) = \beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p). \tag{30}$$

(c) For all $t < T$ and $\omega \in \Omega$,

$$c_t(\omega, \cdot) = \overline{\text{co}}\, \varrho_t(\omega, \cdot) \tag{31}$$

where $\varrho_t(\omega, q) \equiv \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} q(G) c_{t+1}(G, q_G) + \inf_{p \in \text{ri}\,\Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p)$, *for all* $q \in \text{ri}\,\Delta(G_t(\omega))$.

(d) $V_t(\omega, h) = u(h_t(\omega)) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int \beta V_{t+1}(h)\, dr + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right)$ *for all* $t < T$, $\omega \in \Omega$, *and* $h \in \mathcal{H}$.

Proof. Like in the proof of Lemma 4, up to a cardinal transformation of u, we can assume $u(X) \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{R}^{++}, \mathbb{R}^-, \mathbb{R}^{--}\}$. For the rest of the proof the case $u(X) = \mathbb{R}^{++}$ is considered (the arguments we use can be easily adapted to the remaining ones). For all $t \in \mathcal{T}$, $\omega \in \Omega$, $\varphi \in \mathbb{R}^\Omega$, $y \in \mathcal{X}^T$ define

$$J_t(\omega, \varphi) \equiv \min_{p \in \Delta(G_t(\omega))} (\langle \varphi, p \rangle + c_t(\omega, p)) = \inf_{p \in \text{ri}\,\Delta(G_t(\omega))} (\langle \varphi, p \rangle + c_t(\omega, p)) \tag{32}$$

and $U_t(y) \equiv V_t(\omega, y) = \sum_{\tau \geq t} \beta^{\tau - t} u(y_\tau)$. Then $J_t : \mathbb{R}^\Omega \to \mathbb{R}^\Omega(\mathcal{G}_t)$, where $\mathbb{R}^\Omega(\mathcal{G}_t)$ the set of all $\mathcal{G}_t$-measurable functions. Notice that: (32) coincides with the property $\text{dom}\, c_t(\omega, \cdot) \cap \text{ri}\,\Delta(G_t(\omega)) \neq \varnothing$ of dynamic ambiguity indexes; $u(X)^\Omega = \{U_t \circ h : h \in \mathcal{H}\}$ (see Lemma 4); $V_t(\omega, h) = J_t(\omega, U_t \circ h)$ for all $(t, \omega, h) \in \mathcal{T} \times \Omega \times \mathcal{H}$ and if $t < T$

$$U_t \circ h = u \circ h_t + \beta (U_{t+1} \circ h) \text{ and } V_t(h) = u \circ h_t + J_t(\beta(U_{t+1} \circ h)).^{23} \tag{33}$$

Step 1. Let $t < T$ and $\omega \in \Omega$, $J_t(\omega, \xi) = \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int \xi dr + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right) = \inf_{r \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})} \left(\int \xi dr + \inf_{p \in \text{ri}\,\Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \right)$ for all $\xi \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$.[24]

Proof. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$, and by $\Delta\mathcal{G}$ the set $\Delta(G_t(\omega), \mathcal{G}_{t+1})$ (brutally: the probabilities on $\mathcal{G}_{t+1}$ with support in $\{G_1, ..., G_g\}$). For all $\xi = \sum_{G \in \mathcal{G}_{t+1}} \xi_G 1_G \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$, $J_t(\omega, \xi) = \min_{p \in \Delta(G_t(\omega))} \left[\sum_{\omega \in \Omega} p(\omega) \xi(\omega) + c_{t,\omega}(p) \right] = \min_{p \in \Delta(G_t(\omega))} \left[\sum_{i=1}^g \xi_{G_i} p(G_i) + c_{t,\omega}(p) \right] = \min_{r \in \Delta\mathcal{G}} \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} \left[\sum_{i=1}^g \xi_{G_i} p(G_i) + c_{t,\omega}(p) \right]$ $= \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^g r(G_i) \xi_{G_i} + \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_{t,\omega}(p) \right)$ the observation that if $r \in \Delta(\Omega, \mathcal{G}_{t+1}) \setminus \Delta\mathcal{G}$ there is $G \in \mathcal{G}_{t+1}$ such that $G \nsubseteq G_t(\omega)$ with $r(G) > 0$ and hence there is no $p \in \Delta(G_t(\omega))$ such that $p_{|\mathcal{G}_{t+1}} = r$ delivers the first equality. The second is proved in the same way. □

Step 2. Let $t < T$ and $\omega \in \Omega$. The function $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$, defined by

$$\gamma_t(\omega, r) \equiv \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p) \quad \forall r \in \Delta(\Omega, \mathcal{G}_{t+1}),^{24}$$

is closed, convex, grounded, and $\text{dom}\, \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$.

Proof. If $r \in \Delta(\Omega, \mathcal{G}_{t+1}) \setminus \Delta(G_t(\omega), \mathcal{G}_{t+1})$, there is $G \in \mathcal{G}_{t+1}$ such that $G \nsubseteq G_t(\omega)$ with $r(G) > 0$, then there is no $p \in \Delta(G_t(\omega))$ such that $p_{|\mathcal{G}_{t+1}} = r$ and $\gamma_t(\omega, r) = \infty$. Therefore $\text{dom}\, \gamma_t(\omega, r) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1}) = \Delta\mathcal{G}$. For $\xi \in \mathbb{R}^\Omega(\mathcal{G}_{t+1})$, by Step 1, $J_t(\omega, \xi) = \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int \xi dr + \gamma_t(\omega, r) \right)$. Hence, $J_t(\omega, 0) = 0$ implies that $\min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \gamma_t(\omega, r) = 0$, and $\gamma_t(\omega, \cdot)$ is grounded. Let $r, s \in \Delta\mathcal{G}$ and $\alpha \in (0, 1)$, then $\gamma_t(\omega, \alpha r + (1 - \alpha) s) = \min_{p \in \Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = \alpha r + (1 - \alpha) s} c_t(\omega, p) \leq \min_{\substack{p, q \in \Delta(G_t(\omega)) \\ p_{|\mathcal{G}_{t+1}} = r, q_{|\mathcal{G}_{t+1}} = s}} c_t(\omega, \alpha p + (1 - \alpha) q) \leq \min_{\substack{p, q \in \Delta(G_t(\omega)) \\ p_{|\mathcal{G}_{t+1}} = r, q_{|\mathcal{G}_{t+1}} = s}} (\alpha c_t(\omega, p) + (1 - \alpha) c_t(\omega, q)) = \alpha \gamma_t(\omega, r) + (1 - \alpha) \gamma_t(\omega, s)$. Therefore $\gamma_t(\omega, \cdot)$ is convex. Let $b \in \mathbb{R}$ and $r_n \in \Delta\mathcal{G}$, be such that $r_n \to r$ and $\gamma_t(\omega, r_n) \leq b$ for all $n \geq 1$. For all n there exists $\bar{p}^n$ such that $\gamma_t(\omega, r_n) =$

[23] In fact, for all $\omega \in \Omega$, $(U_t \circ h)(\omega) = u(h_t(\omega)) + \sum_{\tau \geq t+1} \beta^{\tau - t} u(h_\tau(\omega)) = (u \circ h_t)(\omega) + \beta \sum_{\tau \geq t+1} \beta^{\tau - (t+1)} u(h_\tau(\omega)) = (u \circ h_t)(\omega) + \beta U_{t+1}(h(\omega)) = (u \circ h_t)(\omega) + \beta(U_{t+1} \circ h)(\omega)$ and $V_t(\omega, h) = J_t(\omega, U_t \circ h) = J_{t,\omega}(\beta(U_{t+1} \circ h) + u \circ h_t) = J_{t,\omega}\left((\beta(U_{t+1} \circ h) + u \circ h_t) 1_{G_t(\omega)}\right) = J_{t,\omega}\left((\beta(U_{t+1} \circ h) + u(h_t(\omega))) 1_{G_t(\omega)}\right) = J_{t,\omega}(\beta(U_{t+1} \circ h) + u(h_t(\omega))) = J_{t,\omega}(\beta(U_{t+1} \circ h)) + u(h_t(\omega))$.

[24] With the convention that the minimum over the empty set is ∞.

$\min_{p^n\in\Delta(G_t(\omega)):p^n_{|\mathcal{G}_{t+1}}=r_n} c_t(\omega,p^n) \le c_t(\omega,\bar{p}^n) \le b$ and $\bar{p}^n_{|\mathcal{G}_{t+1}} = r_n$. Take a convergent subsequence $\bar{p}^{n_j} \to \bar{p}$ of $\bar{p}^n$, since $c_t(\omega,\cdot)$ is closed $c_t(\omega,\bar{p}) \le b$, moreover, $\bar{p}(G) = \lim_j \bar{p}^{n_j}(G) = \lim_j r_{n_j}(G) = r(G)$ for all $G \in \mathcal{G}_{t+1}$. In turn this implies $\gamma_t(\omega,r) \le c_t(\omega,\bar{p}) \le b$ and $\gamma_t(\omega,\cdot)$ is closed. □

This implies that:

Step 3. Let $t < T$ and $\omega \in \Omega$. The function $\nu_t(\omega,\cdot) : \Delta(\Omega) \to [0,\infty]$ defined by $\nu_t(\omega,q) \equiv \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right)$ for all $q \in \Delta(\Omega)$ is grounded, closed and convex with $\operatorname{dom}\nu_t(\omega,\cdot) \subseteq \Delta(G_t(\omega))$.

Step 4. Let $t < T$ and $\omega \in \Omega$. The function $\eta_t(\omega,\cdot) : \Delta(G_t(\omega)) \to [0,\infty]$ defined by $\eta_t(\omega,q) \equiv \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G)$ for all $q \in \Delta(G_t(\omega))$, is closed and convex.

Proof. For later use (in the proof of Theorem 2) we just assume that c_{t+1} satisfies (i) and (ii) of the definition of dynamic ambiguity index (not that $\{c_t\}$ is an ambiguity index itself). We show that $\eta_t(\omega,\cdot)$ is the closure of its convex restriction $\kappa_t(\omega,\cdot)$ to $\operatorname{ri}\Delta(G_t(\omega))$. Let $\mathcal{G} \equiv \{G \in \mathcal{G}_{t+1} : G \subseteq G_t(\omega)\}$. For all $q,p \in \operatorname{ri}\Delta(G_t(\omega))$, $\alpha \in (0,1)$, and $G \in \mathcal{G}$, let $\mu_G \equiv \alpha q(G) + (1-\alpha)p(G)$ and $\alpha_G \equiv \alpha q(G)/\mu_G \in (0,1)$. This delivers $(\alpha q + (1-\alpha)p)_G = \alpha_G q_G + (1-\alpha_G)p_G$ and then $\kappa_{t,\omega}(\alpha q + (1-\alpha)p) = \sum_{G\in\mathcal{G}} \mu_G c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right) = \sum_{G\in\mathcal{G}} \mu_G c_{t+1,G}(\alpha_G q_G + (1-\alpha_G)p_G) \le \sum_{G\in\mathcal{G}} \mu_G(\alpha_G c_{t+1,G}(q_G) + (1-\alpha_G)c_{t+1,G}(p_G)) = \sum_{G\in\mathcal{G}} \alpha q(G)c_{t+1,G}(q_G) + (1-\alpha)p(G)c_{t+1,G}(p_G) = \alpha\kappa_{t,\omega}(q) + (1-\alpha)\kappa_{t,\omega}(p)$, hence $\kappa_{t,\omega}$ is convex. For all $G \in \mathcal{G}$, there is $p^G \in \operatorname{ri}\Delta(G) \cap \operatorname{dom} c_{t+1}(G,\cdot)$. Therefore, choosing $\{q(G) : G \in \mathcal{G}\}$ such that $\sum_{G\in\mathcal{G}} q(G) = 1$ and $q(G) > 0$ for all $G \in \mathcal{G}$, the probability $r \equiv \sum_{G\in\mathcal{G}} q(G)p^G \in \operatorname{dom}\kappa_{t,\omega}$ and $\kappa_{t,\omega}$ is proper.[25] Take $p \in \operatorname{ri}(\operatorname{dom}\kappa_{t,\omega})$ and $q \in \Delta(G_t(\omega))$. If $G \in \mathcal{G}_{t+1}$ and $q(G) > 0$ then $G \subseteq G_t(\omega)$. In this case, the function $f(\alpha) \equiv \alpha_G$ has strictly positive first derivative w.r.t. α in $(0,1)$ and $\lim_{\alpha\uparrow 1} f(\alpha) = 1$; since $p \in \operatorname{ri}(\operatorname{dom}\kappa_{t,\omega})$, then $p_G \in \operatorname{dom} c_{t+1}(G,\cdot)$, and [26, Cor. 7.5.1] implies $\lim_{\alpha\uparrow 1} c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right) = \lim_{\alpha\uparrow 1} c_{t+1,G}(\alpha_G q_G + (1-\alpha_G)p_G) = \lim_{\alpha\uparrow 1} c_{t+1,G}(f(\alpha)q_G + (1-f(\alpha))p_G) = c_{t+1,G}(q_G)$. Else if $G \in \mathcal{G}$ and $q(G) = 0$, then $(\alpha q + (1-\alpha)p)_G = p_G$ for all $\alpha \in (0,1)$, and hence $\lim_{\alpha\uparrow 1} c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right) = \lim_{\alpha\uparrow 1} c_{t+1,G}(p_G) = c_{t+1,G}(p_G)$, with $c_{t+1,G}(p_G) < \infty$ since $p_G \in \operatorname{dom} c_{t+1}(G,\cdot)$. Then, by [26, Thm. 7.5], $(\overline{\mathrm{co}}\,\kappa_{t,\omega})(q) = \lim_{\alpha\uparrow 1} \kappa_{t,\omega}((1-\alpha)p + \alpha q) = \lim_{\alpha\uparrow 1} \sum_{G\in\mathcal{G}} (\alpha q + (1-\alpha)p)(G)c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right) = \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1,G}(q_G) = \eta_{t,\omega}(q)$ for all $q \in \Delta(G_t(\omega))$. □

Step 5. Let $t < T$, $\omega \in \Omega$ and DC be satisfied. If $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ and $J_{t+1,\omega'}(\varphi^1) = J_{t+1,\omega'}(\varphi^2)$ for all $\omega' \in G_t(\omega)$, then $J_{t,\omega}(\beta\varphi^1) = J_{t,\omega}(\beta\varphi^2)$.

Proof. First assume $\varphi^1, \varphi^2 \in u(X)^\Omega$. There exists $\varepsilon > 0$ such that $\varphi^i - \varepsilon \in u(X)^\Omega$, choose $x^\varepsilon \in X$ such that $u(x^\varepsilon) = \left(\sum_{T>\tau\ge t}\beta^{\tau-t}\right)^{-1}\beta\varepsilon$. For all $\omega' \in \Omega$, there exists $x^{\varphi^i(\omega')} \in X$ such that $u\left(x^{\varphi^i(\omega')}\right) = \beta^{1-T+t}\left(\varphi^i(\omega') - \varepsilon\right)$. Consider the acts h^1, h^2 defined by

$$h^i(\tau,\omega') \equiv \begin{cases} x^\varepsilon & \text{if } \tau < T \\ x^{\varphi^i(\omega')} & \text{if } \tau = T. \end{cases}$$

It is easy to check that there is $k \in \mathbb{R}$ such that $U_t\left(h^i(\omega')\right) = \beta\varphi^i(\omega')$ and $U_{t+1}\left(h^i(\omega')\right) = \varphi^i(\omega') + k$ for all $\omega' \in \Omega$ and $i = 1,2$. Then $V_{t+1,\omega'}(h^1) = J_{t+1,\omega'}\left(U_{t+1}(h^1)\right) = J_{t+1,\omega'}(\varphi^1 + k) = J_{t+1,\omega'}(\varphi^1) + k$, and $V_{t+1,\omega'}(h^2) = J_{t+1,\omega'}(\varphi^2) + k$ for all $\omega' \in \Omega$. For all $\omega' \in G_t(\omega)$, $J_{t+1,\omega'}(\varphi^1) = J_{t+1,\omega'}(\varphi^2)$, then $h^1 \sim_{t+1,\omega'} h^2$; moreover $h^1_\tau = h^2_\tau$ for all $\tau \le t$, then DC implies $h^1 \sim_{t,\omega} h^2$ and $J_{t,\omega}(\beta\varphi^1) = J_{t,\omega}\left(U_t(h^1)\right) = V_{t,\omega}(h^1) = V_{t,\omega}(h^2) = J_{t,\omega}(\beta\varphi^2)$. As wanted. Finally, if $\varphi^1, \varphi^2 \in \mathbb{R}^\Omega$ are such that $J_{t+1,\omega'}(\varphi^1) = J_{t+1,\omega'}(\varphi^2)$ for all $\omega' \in G_t(\omega)$, there exist $\psi^1, \psi^2 \in u(X)^\Omega$ and $b \in \mathbb{R}$ such that $\varphi^i = \psi^i + b$, then $J_{t+1,\omega'}(\psi^1) = J_{t+1,\omega'}(\psi^2)$ for all $\omega' \in G_t(\omega)$. Therefore, $J_{t,\omega}(\beta\varphi^1) = J_{t,\omega}(\beta\psi^1 + \beta b) = J_{t,\omega}(\beta\psi^1) + \beta b = J_{t,\omega}(\beta\psi^2) + \beta b = J_{t,\omega}(\beta\varphi^2)$. □

[25] Notice that for all $G \in \mathcal{G}$, $r(G) = q(G)$ and $r_G = p^G$.

Step 6. Let $t<T$. If $\{\succsim_{t,\omega}\}$ satisfy DC, then $J_t(\beta J_{t+1}(\varphi))=J_t(\beta\varphi)$ for all $\varphi\in\mathbb{R}^\Omega$.

Proof. Choose $\omega\in\Omega$, and remember that $J_{t+1}(\varphi)=\sum_{G\in\mathcal{G}_{t+1}}J_{t+1}(G,\varphi)1_G\in\mathbb{R}^\Omega(\mathcal{G}_{t+1})$. For all $\omega'\in G_t(\omega)$, $\operatorname{dom}c_{t+1,\omega'}\subseteq\Delta(G_{t+1}(\omega'))$, then $J_{t+1}(\omega',\varphi)=J_{t+1}(\omega',J_{t+1,\omega'}(\varphi)1_\Omega)=J_{t+1}\left(\omega',J_{t+1}(G_{t+1}(\omega'),\varphi)1_{G_{t+1}(\omega')}\right)=J_{t+1}(\omega',J_{t+1}(\varphi))$, then, by Step 5 above, $J_t(\omega,\beta\varphi)=J_t(\omega,\beta J_{t+1}(\varphi))$. The proof is concluded by the observation that this is true for all $\omega\in\Omega$. □

Step 7. $\{\succsim_{t,\omega}\}$ satisfy DC if and only if $J_t(\beta J_{t+1}(\varphi))=J_t(\beta\varphi)$ for all $t<T$ and $\varphi\in\mathbb{R}^\Omega$.

Proof. By the previous step we just have to prove necessity. Let $\omega\in\Omega$ and $t<T$. Assume $\varphi^1,\varphi^2\in\mathbb{R}^\Omega$ are such that $J_{t+1,\omega'}(\varphi^1)\geq J_{t+1,\omega'}(\varphi^2)$ for all $\omega'\in G_t(\omega)$, then $J_{t,\omega}(\beta\varphi^1)=J_{t,\omega}(\beta J_{t+1}(\varphi^1))=J_{t,\omega}\left(\beta J_{t+1}(\varphi^1)1_{G_t(\omega)}\right)\geq J_{t,\omega}\left(\beta J_{t+1}(\varphi^2)1_{G_t(\omega)}\right)=J_{t,\omega}(\beta J_{t+1}(\varphi^2))=J_{t,\omega}(\beta\varphi^2)$, i.e. $J_{t,\omega}(\beta\varphi^1)\geq J_{t,\omega}(\beta\varphi^2)$. Let $h^1,h^2\in\mathcal{H}$ be such that $h^1_\tau=h^2_\tau$ for all $\tau\leq t$ and $h^1\succsim_{t+1,\omega'}h^2$, for all $\omega'\in\Omega$, we want to show that $h^1\succsim_{t,\omega}h^2$. Since $h^1\succsim_{t+1,\omega'}h^2$, for all $\omega'\in\Omega$, then $J_{t+1,\omega'}(U_{t+1}\circ h^1)=V_{t+1,\omega'}(h^1)\geq V_{t+1,\omega'}(h^2)=J_{t+1,\omega'}(U_{t+1}\circ h^2)$ for all $\omega'\in\Omega$, whence (set $\varphi^i=U_{t+1}\circ h^i$, $i=1,2$) $J_{t,\omega}(\beta(U_{t+1}\circ h^1))\geq J_{t,\omega}(\beta(U_{t+1}\circ h^2))$. But $h^1_t=h^2_t$, then, by (33), $V_t(\omega,h^1)=J_{t,\omega}(\beta(U_{t+1}\circ h^1))+u(h^1_t(\omega))\geq J_{t,\omega}(\beta(U_{t+1}\circ h^2))+u(h^2_t(\omega))=V_t(\omega,h^2)$, i.e. $V_t(\omega,h^1)\geq V_t(\omega,h^2)$. □

Step 8. (a) $\Leftrightarrow$ (d).

Proof. By Step 7, $\{\succsim_{t,\omega}\}$ satisfy DC iff
$J_{t,\omega}(\beta J_{t+1}(\varphi))=J_{t,\omega}(\beta\varphi)$ for all $t<T,\varphi\in\mathbb{R}^\Omega,\omega\in\Omega$ iff
$J_{t,\omega}(\beta J_{t+1}(\psi+b))=J_{t,\omega}(\beta(\psi+b))$ for all $t<T$, $\psi\in u(X)^\Omega$, $b\in\mathbb{R}$, $\omega\in\Omega$ iff
$J_{t,\omega}(\beta J_{t+1}(\psi))+\beta b=J_{t,\omega}(\beta\psi)+\beta b$ for all $t<T$, $\psi\in u(X)^\Omega$, $b\in\mathbb{R}$, $\omega\in\Omega$ iff
$J_{t,\omega}(\beta J_{t+1}(\psi))=J_{t,\omega}(\beta\psi)$ for all $t<T$, $\psi\in u(X)^\Omega$, $\omega\in\Omega$ iff
$J_{t,\omega}(\beta J_{t+1}(U_{t+1}\circ h))=J_{t,\omega}(\beta(U_{t+1}\circ h))$ for all $t<T$, $h\in\mathcal{H}$, $\omega\in\Omega$ iff
$J_{t,\omega}(\beta V_{t+1}(h))+u(h_t(\omega))=J_{t,\omega}(\beta(U_{t+1}\circ h))+u(h_t(\omega))$ for all $t<T$, $h\in\mathcal{H}$, $\omega\in\Omega$ iff
$\min_{r\in\Delta(\Omega,\mathcal{G}_{t+1})}\left(\int\beta V_{t+1}(h)\,dr+\min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r}c_t(\omega,p)\right)+u(h_t(\omega))=V_t(\omega,h)$ for all $t<T$, $h\in\mathcal{H}$, $\omega\in\Omega$, by $\mathcal{G}_{t+1}$ measurability of $V_{t+1}(h)$, Step 1, and (33). □

Step 9. For all $t<T$, $\omega\in\Omega$, and $\varphi\in\mathbb{R}^\Omega$

$$J_{t,\omega}(\beta J_{t+1}(\varphi))=\min_{q\in\Delta(G_t(\omega))}\left(\langle\beta\varphi,q\rangle+\beta\sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}}q(G)c_{t+1}(G,q_G)+\min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}}c_t(\omega,p)\right)\tag{34}$$

Proof. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set $\{G \in \mathcal{G}_{t+1} : G \subseteq G_t(\omega)\}$. By Steps 1 and 2,

$$J_{t,\omega}(\beta J_{t+1}(\varphi)) = J_t\left(\omega, \sum_{G \in \mathcal{G}_{t+1}} \beta J_{t+1}(G, \varphi) 1_G\right)$$

$$= \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\sum_{G \in \mathcal{G}_{t+1}} r(G) \beta J_{t+1}(G, \varphi) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} r(G_i) \beta J_{t+1}(G_i, \varphi) + \gamma_t(\omega, r)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} r(G_i)\beta \min_{p^i \in \Delta(G_i)} \left[\sum_{\bar{\omega} \in \Omega} p^i(\bar{\omega})\varphi(\bar{\omega}) + c_{t+1}(G_i, p^i)\right] + \gamma_t(\omega, r)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} \min_{p^i \in \Delta(G_i)} r(G_i)\beta \left[\sum_{\bar{\omega} \in \Omega} p^i(\bar{\omega})\varphi(\bar{\omega}) + c_{t+1}(G_i, p^i)\right] + \gamma_t(\omega, r)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} \min_{p^i \in \Delta(G_i)} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i)p^i(\bar{\omega})\varphi(\bar{\omega}) + \beta r(G_i)c_{t+1}(G_i, p^i)\right] + \gamma_t(\omega, r)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \sum_{i=1}^{g} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i)p^i(\bar{\omega})\varphi(\bar{\omega}) + \beta r(G_i)c_{t+1}(G_i, p^i)\right] + \gamma_t(\omega, r)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{i=1}^{g} \left[\sum_{\bar{\omega} \in \Omega} \beta r(G_i)p^i(\bar{\omega})\varphi(\bar{\omega}) + \beta r(G_i)c_{t+1}(G_i, p^i)\right] + \gamma_t(\omega, r)\right)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{i=1}^{g} \sum_{\bar{\omega} \in \Omega} \beta r(G_i)p^i(\bar{\omega})\varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i)c_{t+1}(G_i, p^i) + \gamma_t(\omega, r)\right)\right)$$

$$= \min_{r \in \Delta\mathcal{G}} \left(\min_{p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{\bar{\omega} \in \Omega} \left(\sum_{i=1}^{g} r(G_i)p^i(\bar{\omega})\right) \beta\varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i)c_{t+1}(G_i, p^i) + \gamma_t(\omega, r)\right)\right)$$

$$= \min_{r \in \Delta\mathcal{G}, p^1 \in \Delta(G_1), ..., p^g \in \Delta(G_g)} \left(\sum_{\bar{\omega} \in \Omega} \left(\sum_{i=1}^{g} r(G_i)p^i(\bar{\omega})\right) \beta\varphi(\bar{\omega}) + \sum_{i=1}^{g} \beta r(G_i)c_{t+1}(G_i, p^i) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = r} c_t(\omega, p)\right)$$

$$= \min_{q \in \Delta(G_t(\omega))} \left(\sum_{\bar{\omega} \in \Omega} q(\bar{\omega})\beta\varphi(\bar{\omega}) + \sum_{\substack{i=1,...,g \\ q(G_i)>0}} \beta q(G_i)c_{t+1}(G_i, q_{G_i}) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p)\right).$$

Last equality holds since $\Delta(G_t(\omega)) = \left\{\sum_{i=1}^{g} r(G_i)p^i : r \in \Delta\mathcal{G},\ p^i \in \Delta(G_i)\ \forall i = 1, ..., g\right\}$, and $q \in \Delta(G_t(\omega))$ can be written as $q = \sum_{i=1,...,g} r(G_i)p^i$ with $r \in \Delta\mathcal{G}$ and $p^i \in \Delta(G_i)$ if and only if $r = q_{|\mathcal{G}_{t+1}}$ and $p^i = q_{G_i}$ for all $i = 1, ..., g$ with $q(G_i) = r(G_i) > 0$ (clearly p^i can be chosen arbitrarily in $\Delta(G_i)$ if $q(G_i) = r(G_i) = 0$). □

Steps 7 and 9 imply that $\{\succsim_{t,\omega}\}$ satisfy DC iff for all $t < T$, $\omega \in \Omega$, and $\varphi \in \mathbb{R}^\Omega$

$$J_{t,\omega}(\beta\varphi) = \min_{q \in \Delta(G_t(\omega))} \left(\langle \beta\varphi, q \rangle + \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G)c_{t+1}(G, q_G) + \min_{p \in \Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_t(\omega, p)\right);$$

Eq. (32) and Lemma 1 guarantee that this is equivalent to $c_t(\omega, \cdot) = \overline{\text{co}}(\eta_t(\omega, \cdot) + \nu_t(\omega, \cdot))$ where $\eta_t(\omega, \cdot)$ and $\nu_t(\omega, \cdot)$ are defined in Steps 4 and 3. These steps also guarantee closure and convexity of $\eta_t(\omega, \cdot)$ and $\nu_t(\omega, \cdot)$. That is (a) ⇔ (b).

(a) ⇔ (c) can be proved in a similar way. ■

Remark 1 *In particular, for a dynamic ambiguity index $\{c_t\}$ conditions (30) and (31) are equivalent.*

A.3 Proof of Theorem 1

(a) $\Rightarrow$ (b) By Proposition 3 and Lemma 5 there exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t,\omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by $V_t(\omega, h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right)\right)$ for all $h \in \mathcal{H}$. Lemma 6 guarantees that (11) holds.

(b) $\Rightarrow$ (a) Assume that there exist a scalar $\beta > 0$, an unbounded affine function $u : X \to \mathbb{R}$, and a dynamic ambiguity index $\{c_t\}$, such that: for each $(t,\omega) \in \mathcal{T} \times \Omega$, $\succsim_{t,\omega}$ is represented by $V_t(\omega, h) = \inf_{p\in\Delta^{++}(\Omega)} \left(\int \sum_{\tau\geq t} \beta^{\tau-t} u(h_\tau)\, dp_{G_t(\omega)} + c_t\left(\omega, p_{G_t(\omega)}\right)\right)$ for all $h \in \mathcal{H}$. By Proposition 1, $\{\succsim_{t,\omega}\}$ satisfy CP, VP, RP, and FS., and so, by (11) and Lemma 6, $\{\succsim_{t,\omega}\}$ satisfy DC.

Uniqueness of the representation follows again from Proposition 1. ■

A.4 Proof of Proposition 2

(i) is trivial. Step 2 of the proof of Lemma 6 shows that $\gamma_t(\omega, \cdot)$ is grounded, closed and convex, with $\operatorname{dom} \gamma_t(\omega, \cdot) \subseteq \Delta(G_t(\omega), \mathcal{G}_{t+1})$, for all $t < T$ and all $\omega \in \Omega$. It only remains to show that and $\operatorname{dom} \gamma_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1}) \neq \varnothing$. Take $p^\circ \in \operatorname{ri} \Delta(G_t(\omega)) \cap \operatorname{dom} c_t(\omega, \cdot)$, then $r^\circ = p^\circ_{|\mathcal{G}_{t+1}} \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})$ and $\gamma_t(\omega, r^\circ) = \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r^\circ} c_t(\omega, p) \leq c_t(\omega, p^\circ) < \infty$. ■

A.5 Proof of Theorem 2

(a) $\Rightarrow$ (b) is an immediate consequence of the definition of recursive ambiguity index and Proposition 2.

(b) $\Rightarrow$ (a) The proof that $\{c_t\}$ is a dynamic ambiguity index is by backward induction. Clearly, c_T satisfies (i) and (ii) of the definition of dynamic ambiguity index. Next we assume that c_{t+1} $(0 \leq t < T)$ satisfies (i) and (ii) of the definition of dynamic ambiguity index, and show that c_t satisfies them.

By assumption, $c_{t+1} : \Omega \times \Delta(\Omega) \to [0, \infty]$ is such that:

(i) $c_{t+1}(\cdot, p) : \Omega \to [0, \infty]$ is measurable w.r.t. $\mathcal{G}_{t+1}$ for all $p \in \Delta(\Omega)$,

(ii) $c_{t+1}(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$ is grounded, closed and convex, with $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_{t+1}(\omega))$ and $\operatorname{dom} c_{t+1}(\omega, \cdot) \cap \Delta^{++}(G_{t+1}(\omega)) \neq \varnothing$, for all $\omega \in \Omega$.

Clearly, for all $\omega \in \Omega$, the function $c_t(\omega, \cdot)$ appearing in (b) is well defined (since c_{t+1} satisfies (i)) and $c_t(\omega, \cdot) = c_t(\omega', \cdot)$ if $G_t(\omega) = G_t(\omega')$.

Step 4 of the proof of Lemma 6 shows that for all $\omega \in \Omega$ the function $\eta_t(\omega, \cdot) : \Delta(G_t(\omega)) \to [0, \infty]$ defined by $\eta_t(\omega, q) \equiv \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G) c_{t+1}(G, q_G)$ for all $q \in \Delta(G_t(\omega))$ is closed and convex. Since $q \mapsto q_{|\mathcal{G}_{t+1}}$ is affine (and continuous) from $\Delta(\Omega)$ to $\Delta(\Omega, \mathcal{G}_{t,+1})$ and $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ is grounded, closed and convex, with effective domain in $\Delta(G_t(\omega), \mathcal{G}_{t+1})$, then $q \mapsto \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right)$ is closed and convex on $\Delta(\Omega)$ and its effective domain is contained in $\Delta(G_t(\omega))$. We conclude that, for all $\omega \in \Omega$, the function $c_t(\omega, \cdot)$ appearing in (b) is closed and convex, from $\Delta(\Omega)$ to $[0, \infty]$, with $\operatorname{dom} c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$.

Next we show that $c_t(\omega, \cdot)$ is grounded. Choose arbitrarily $\omega \in \Omega$, there exists $r \in \Delta(\Omega, \mathcal{G}_{t+1})$ such that $r(G_t(\omega)) = 1$ and $\gamma_t(\omega, r) = 0$; moreover, for all $G \in \mathcal{G}_{t+1}$ there exists $p^G \in \Delta(G)$ such that $c_{t+1}(G, p^G) = 0$, set $q \equiv \sum_{G\in\mathcal{G}_{t+1}} r(G)\, p^G$ to obtain $c_t(\omega, q) = \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G) c_{t+1}(G, q_G) + \gamma_t\left(\omega, q_{|\mathcal{G}_{t+1}}\right) = \sum_{\substack{G\in\mathcal{G}_{t+1}\\ r(G)>0}} r(G) c_{t+1}(G, p^G) + \gamma_t(\omega, r) = 0$.

It remains to show that $\operatorname{ri} \Delta(G_t(\omega)) \cap \operatorname{dom} c_t(\omega, \cdot) \neq \varnothing$ for all $\omega \in \Omega$. Choose arbitrarily $\omega \in \Omega$, there exists $r \in \Delta^{++}(G_t(\omega), \mathcal{G}_{t+1})$ such that $\gamma_t(\omega, r) < \infty$; moreover, for all $G \in \mathcal{G}_{t+1}$ there exists $p^G \in \operatorname{ri} \Delta(G)$ such that $c_{t+1}(G, p^G) < \infty$, set $q \equiv \sum_{G\in\mathcal{G}_{t+1}} r(G)\, p^G$ to obtain $q \in \operatorname{ri} \Delta(G_t(\omega))$

and $c_t(\omega,q) = \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \gamma_t(\omega, q_{|\mathcal{G}_{t+1}}) = \sum_{\substack{G\in\mathcal{G}_{t+1}\\ r(G)>0}} r(G)c_{t+1}(G,p^G) + \gamma_t(\omega,r) < \infty$. This concludes the proof that $\{c_t\}$ is a dynamic ambiguity index.

Moreover, notice that $\min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} \left(\sum_{\substack{G\in\mathcal{G}_{t+1}\\ p(G)>0}} p(G)c_{t+1}(G,p_G)\right) = 0$ for all $\omega \in \Omega$, $t < T$, and $q \in \Delta(G_t(\omega))$ (it is enough to take, for all $G \in \mathcal{G}_{t+1}$, $p^G \in \Delta(G)$ such that $c_{t+1}(G,p^G) = 0$ and set $p \equiv \sum_{G\in\mathcal{G}_{t+1}} q(G)\,p^G$). Therefore,

$$
\begin{aligned}
&\beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)\\
&= \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} \left(\beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ p(G)>0}} p(G)c_{t+1}(G,p_G) + \gamma_t(\omega, p_{|\mathcal{G}_{t+1}})\right)\\
&= \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \gamma_t(\omega, q_{|\mathcal{G}_{t+1}}) + \beta \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} \left(\sum_{\substack{G\in\mathcal{G}_{t+1}\\ p(G)>0}} p(G)c_{t+1}(G,p_G)\right)\\
&= \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)c_{t+1}(G,q_G) + \gamma_t(\omega, q_{|\mathcal{G}_{t+1}}) = c_t(\omega,q) \quad \forall \omega\in\Omega, t<T, q\in\Delta(G_t(\omega)).
\end{aligned}
$$

Hence $\{c_t\}$ satisfies condition (11) and it is a recursive ambiguity index.

Finally, the above equalities deliver $\gamma_t(\omega, q_{|\mathcal{G}_{t+1}}) = \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_t(\omega,p)$ for all $\omega \in \Omega$, $t < T$, and $q \in \Delta(G_t(\omega))$, which implies (13) for $r \in \Delta(G_t(\omega), \mathcal{G}_{t+1})$. If $r \notin \Delta(G_t(\omega), \mathcal{G}_{t+1})$, then $\gamma_t(\omega,r) = \infty = \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} c_t(\omega,p)$ (the first equality descending from the definition of one-period-ahead ambiguity index, the second from the convention we adopted for minima over the empty set). We can conclude that (13) holds for all $r \in \Delta(\Omega, \mathcal{G}_{t+1})$ and that $\{\gamma_t\}$ is unique. ■

A.6 Proof of Corollary 1

It is easy to see that the effect of MP(ii) on the representation provided by Proposition 1 is guaranteeing that, for every $t \in \mathcal{T}$ and $\omega \in \Omega$, $c_t(\omega,p) = \delta_{C_t(\omega)}(p)$, for a closed and convex subset $C_t(\omega) \subseteq \Delta(\Omega)$.

The relation $\operatorname{dom} c_{t,\omega} \subseteq \Delta(G_t(\omega))$ implies $C_t(\omega) \subseteq \Delta(G_t(\omega))$. Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$, and write indifferently C_i or $C_{t+1}(G_i)$. Let $\omega \in \Omega$ and $t < T$. Condition (11) is equivalent to

$$
\begin{aligned}
C_t(\omega) &= \left\{ q\in\Delta(G_t(\omega)) \;\middle|\; \beta \sum_{\substack{G\in\mathcal{G}_{t+1}\\ q(G)>0}} q(G)\delta_{C_{t+1}(G)}(q_G) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} \delta_{C_t(\omega)}(p) = 0 \right\}\\
&= \left\{ q\in\Delta(G_t(\omega)) \;\middle|\; \beta \sum_{\substack{i=1,...,g\\ q(G_i)>0}} q(G_i)\delta_{C_i}(q_{G_i}) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} \delta_{C_t(\omega)}(p) = 0 \right\}\\
&= \left\{ \sum_{i=1,...,g} r(G_i)p^i \;\middle|\; \begin{array}{l} r\in\Delta\mathcal{G},\ p^i\in\Delta(G_i)\ \forall i=1,...,g,\\ \beta \sum_{\substack{i=1,...,g\\ r(G_i)>0}} r(G_i)\delta_{C_i}(p^i) + \min_{p\in\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=r} \delta_{C_t(\omega)}(p) = 0 \end{array} \right\}\\
&= \left\{ \sum_{i=1,...,g} r(G_i)p^i \;\middle|\; \begin{array}{l} r\in\Delta\mathcal{G},\ p^i\in\Delta(G_i)\ \forall i=1,...,g,\\ \delta_{C_i}(p^i) = 0 \text{ for all } i \text{ s.t. } r(G_i)>0,\\ \delta_{C_t(\omega)}(p) = 0 \text{ for some } p\in\Delta(G_t(\omega)) : p_{|\mathcal{G}_{t+1}} = r \end{array} \right\}\\
&= \left\{ \sum_{i=1,...,g} r(G_i)p^i \;\middle|\; \begin{array}{l} r\in\Delta\mathcal{G},\ p^i\in\Delta(G_i)\ \forall i=1,...,g,\\ p^i\in C_i \text{ for all } i=1,...,g,\\ r\in C_t(\omega)_{|\mathcal{G}_{t+1}} \end{array} \right\}
\end{aligned}
$$

$$= \left\{ \sum_{i=1,...,g} r(G_i) p^i \,\middle|\, \begin{array}{l} p^i \in C_{t+1}(G_i) \text{ for all } i = 1, ..., g, \\ r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \end{array} \right\}$$

$$= \left\{ \sum_{G \in \mathcal{G}_{t+1}} p^G r(G) \,\middle|\, p^G \in C_{t+1}(G) \ \forall G \in \mathcal{G}_{t+1} \text{ and } r \in C_t(\omega)_{|\mathcal{G}_{t+1}} \right\}. \quad \blacksquare$$

A.7 Proof of Theorem 3

W.l.o.g., set $\theta = 1$ and denote by q° the reference probability of the statement. The properties of the relative entropy (see, e.g., [20]) guarantee that $\{c_t\}$ (as defined by (17)) is a dynamic ambiguity index. By Theorem 1, we only have to show that $\{c_t\}$ satisfies (11) or the equivalent (31), see Remark 1.

Next we show that, for all $t < T$, $\omega \in \Omega$, and $q \in \operatorname{ri} \Delta(G_t(\omega))$,

$$c_{t,\omega}(q) = \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} q(G) c_{t+1,G}(q_G) + \inf_{p \in \operatorname{ri}\Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p). \tag{35}$$

For all $p \in \operatorname{ri}\Delta(G_t(\omega))$, $c_{t,\omega}(p) = \frac{1}{\beta^t} \sum_{\omega' \in G_t(\omega)} p_{G_t(\omega)}(\omega') \log \frac{p_{G_t(\omega)}(\omega')}{q^\circ_{G_t(\omega)}(\omega')} = \frac{1}{\beta^t} \sum_{\omega' \in G_t(\omega)} p_{\omega'} \log \frac{p_{\omega'}}{q^\circ_{G_t(\omega),\omega'}}$ where $p_{\omega'} \equiv p(\omega')$ and $q^\circ_{G_t(\omega),\omega'} \equiv q^\circ_{G_t(\omega)}(\omega')$. For all $G \in \mathcal{G}_{t+1}$ such that $G \subseteq G_t(\omega)$ and all $p \in \operatorname{ri}\Delta(G)$, $c_{t+1,G}(p) = \frac{1}{\beta^{t+1}} \sum_{\omega' \in G} p_{G,\omega'} \log \frac{p_{G,\omega'}}{q^\circ_{G,\omega'}} = \frac{1}{\beta^{t+1}} \sum_{\omega' \in G} p_{\omega'} \log \frac{p_{\omega'}}{\left(q^\circ_{G_t(\omega)}\right)_{G,\omega'}}$. To simplify the notation, set $S = G_t(\omega)$, $\bar{q} = q^\circ_S$, $\mathcal{G} = \{G \in \mathcal{G}_{t+1} : G \subseteq G_t(\omega)\}$ (notice that $\mathcal{G}$ is a partition of S). Choose arbitrarily $q \in \operatorname{ri}\Delta(G_t(\omega))$, $\beta^{t+1} c_{t+1,G}(q_G) = \sum_{s \in G} \frac{q_s}{q(G)} \log \frac{q_s}{q(G)} \frac{\bar{q}(G)}{\bar{q}_s} = \frac{1}{q(G)} \sum_{s \in G} q_s \log \frac{q_s}{\bar{q}_s} + \frac{1}{q(G)} \sum_{s \in G} q_s \log \frac{\bar{q}(G)}{q(G)} = \frac{1}{q(G)} \sum_{s \in G} q_s \log \frac{q_s}{\bar{q}_s} - \log \frac{q(G)}{\bar{q}(G)}$. Then, for all $q \in \operatorname{ri}\Delta(G_t(\omega))$, $\beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} q(G) c_{t+1,G}(q_G) = \beta \sum_{G \in \mathcal{G}} q(G) \frac{1}{\beta^{t+1}} \left(\frac{1}{q(G)} \sum_{s \in G} q_s \log \frac{q_s}{\bar{q}_s} - \log \frac{q(G)}{\bar{q}(G)} \right) = \frac{1}{\beta^t} \left(\sum_{G \in \mathcal{G}} \sum_{s \in G} q_s \log \frac{q_s}{\bar{q}_s} - \sum_{G \in \mathcal{G}} q(G) \log \frac{q(G)}{\bar{q}(G)} \right) = \frac{1}{\beta^t} \sum_{s \in S} q_s \log \frac{q_s}{\bar{q}_s} - \frac{1}{\beta^t} \sum_{G \in \mathcal{G}} q(G) \log \frac{q(G)}{\bar{q}(G)}$, i.e.

$$\beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ G \subseteq G_t(\omega)}} q(G) c_{t+1,G}(q_G) = c_{t,\omega}(q) - \frac{1}{\beta^t} \sum_{G \in \mathcal{G}} q(G) \log \frac{q(G)}{\bar{q}(G)}. \tag{36}$$

Moreover, for all $q \in \operatorname{ri}\Delta(G_t(\omega))$, $\inf_{p \in \operatorname{ri}\Delta(G_t(\omega)): p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) = \inf_{p \in \operatorname{ri}\Delta(S): p_{|\mathcal{G}} = q_{|\mathcal{G}}} c_{t,\omega}(p)$ is the value of the problem

$$\begin{cases} \inf \dfrac{1}{\beta^t} \sum_{s \in S} p_s \log \dfrac{p_s}{\bar{q}_s} \\ \text{sub} \\ p_s > 0 \quad \forall s \in S \\ \sum_{s \in S} p_s = 1 \\ \sum_{s \in G} p_s = q(G) \quad \forall G \in \mathcal{G}. \end{cases} \tag{37}$$

We solve the easier problem

$$\begin{cases} \inf \sum_{s \in S} p_s \log \dfrac{p_s}{\bar{q}_s} \\ \text{sub} \\ \sum_{s \in G} p_s = q(G) \quad \forall G \in \mathcal{G} \end{cases} \tag{38}$$

and observe that the solution p° is unique, it is a strictly positive vector (this is also required for the existence of $\log \frac{p^\circ_s}{\bar{q}_s}$), $\sum_{s \in S} p^\circ_s = \sum_{G \in \mathcal{G}} \sum_{s \in G} p^\circ_s = \sum_{G \in \mathcal{G}} q(G) = 1$, and obviously the constant β^t has no effect. Thus p° is the solution of problem (37). The Lagrangian of problem (38) is $\mathcal{L}(p, \lambda) = \sum_{s \in S} p_s \log \frac{p_s}{\bar{q}_s} - \sum_{G \in \mathcal{G}} \lambda_G \left(\sum_{s \in G} p_s - q(G) \right)$, denoting by $G(s)$ the element of G containing s, the first order conditions are

$$\begin{cases} \log \dfrac{p_s}{\bar{q}_s} + 1 - \lambda_{G(s)} = 0 \quad \forall s \in S \\ \sum_{s \in G} p_s = q(G) \quad \forall G \in \mathcal{G} \end{cases} \tag{39}$$

simple manipulation yields

$$\begin{cases} p_s = \bar{q}_s \exp\left(\lambda_{G(s)} - 1\right) & \forall s \in S \\ \sum_{s\in G} p_s = q(G) & \forall G \in \mathcal{G} \end{cases} \tag{40}$$

then the observation that $G(s) = G(w)$ for all $s \in G(w)$ implies

$$\begin{cases} p_s = \bar{q}_s \exp\left(\lambda_{G(s)} - 1\right) & \forall s \in S \\ \sum_{s\in G(w)} \bar{q}_s \exp\left(\lambda_{G(w)} - 1\right) = q(G(w)) & \forall w \in S \end{cases} \tag{41}$$

and

$$\begin{cases} \exp\left(\lambda_{G(w)} - 1\right) = \dfrac{q(G(w))}{\bar{q}(G(w))} & \forall w \in S \\ p_s = \bar{q}_s \dfrac{q(G(s))}{\bar{q}(G(s))} & \forall s \in S. \end{cases} \tag{42}$$

The solution is

$$\begin{cases} \lambda_G^\circ = 1 + \log \dfrac{q(G)}{\bar{q}(G)} & \forall G \in \mathcal{G} \\ p_s^\circ = \bar{q}_s \dfrac{q(G(s))}{\bar{q}(G(s))} & \forall s \in S \end{cases} \tag{43}$$

which plugged into the value function $\frac{1}{\beta^t}\sum_{s\in S} p_s \log \frac{p_s}{\bar{q}_s}$ delivers $\inf_{p\in \operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) = \frac{1}{\beta^t}\sum_{s\in S} p_s^\circ \log \frac{p_s^\circ}{\bar{q}_s} = \frac{1}{\beta^t}\sum_{s\in S} \bar{q}_s \frac{q(G(s))}{\bar{q}(G(s))} \log \frac{q(G(s))}{\bar{q}(G(s))} = \frac{1}{\beta^t}\sum_{G\in\mathcal{G}}\sum_{s\in G} \bar{q}_s \frac{q(G)}{\bar{q}(G)} \log \frac{q(G)}{\bar{q}(G)}$, finally

$$\inf_{p\in \operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) = \frac{1}{\beta^t}\sum_{G\in\mathcal{G}} q(G) \log \frac{q(G)}{\bar{q}(G)} \tag{44}$$

which together with Eq. (36) delivers Eq. (35).

Setting, for all $t < T$, $\omega \in \Omega$, and $q \in \Delta(\Omega)$,

$$\varrho_{t,\omega}(q) = \begin{cases} \beta \sum_{\substack{G\in\mathcal{G}_{t+1} \\ G\subseteq G_t(\omega)}} q(G) c_{t+1,G}(q_G) + \inf_{p\in \operatorname{ri}\Delta(G_t(\omega)):p_{|\mathcal{G}_{t+1}}=q_{|\mathcal{G}_{t+1}}} c_{t,\omega}(p) & \text{if } q \in \operatorname{ri}\Delta(G_t(\omega)) \\ \infty \quad \text{otherwise} \end{cases}$$

the function $\varrho_{t,\omega}$ coincides with the closed and convex function $c_{t,\omega}$ on ri $(\Delta(G_t(\omega)))$. Take $q \in \operatorname{ri}\left(\operatorname{dom} \varrho_{t,G_t(\omega)}\right) = \operatorname{ri}\left(\Delta(G_t(\omega))\right)$, by [26, Thm. 7.5], for all $p \in \Delta(G_t(\omega))$,

$$\overline{\operatorname{co}}\, \varrho_{t,\omega}(p) = \lim_{\lambda\uparrow 1} \varrho_{t,\omega}((1-\lambda)q + \lambda p) = \lim_{\lambda\uparrow 1} c_{t,\omega}((1-\lambda)q + \lambda p) = c_{t,\omega}(p). \tag{45}$$

Since Eq. (45) is *a fortiori* true if $p \notin \Delta(G_t(\omega))$, condition (31) holds, as wanted.

To complete the proof we need to prove (18). Let $c_{t,\omega}(p) \equiv \beta^{-t} R\left(p_{G_t(\omega)} \| q^\circ_{G_t(\omega)}\right)$ for all $(t,\omega,p) \in \mathcal{T} \times \Omega \times \Delta(\Omega)$. Fix $\omega \in \Omega$ and $t < T$. Step 4 of the proof of Lemma 6 shows that there is a suitable $p \in \operatorname{ri}(\Delta(G_t(\omega)))$ such that for all $q \in \Delta(G_t(\omega))$

$$\sum_{\substack{G\in\mathcal{G}_{t+1} \\ q(G)>0}} q(G)\, c_{t+1,G}(q_G) = \lim_{\alpha\uparrow 1} \sum_{\substack{G\in\mathcal{G}_{t+1} \\ G\subseteq G_t(\omega)}} (\alpha q + (1-\alpha)p)(G)\, c_{t+1,G}\left((\alpha q + (1-\alpha)p)_G\right).$$

Moreover, by definition of $c_{t,\omega}$, $\infty > c_{t,\omega}(q) = \lim_{\alpha\uparrow 1} c_{t,\omega}(\alpha q + (1-\alpha)p)$. Since $\{c_t\}$ is a recursive ambiguity index we have $\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = c_{t,\omega}(q) - \sum_{\substack{G\in\mathcal{G}_{t+1} \\ q(G)>0}} q(G)\, c_{t+1,G}(q_G)$, since both summands are finite, $\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = \lim_{\alpha\uparrow 1}\left(c_{t,\omega}(q\alpha p) - \sum_{G\in\mathcal{G}} (q\alpha p)(G)\, c_{t+1,G}\left((q\alpha p)_G\right)\right)$ where $q\alpha p = \alpha q + (1-\alpha)p$, but $q\alpha p \in \operatorname{ri}(\Delta(G_t(\omega)))$ and Eq. (36) delivers

$$\begin{aligned} \gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) &= \lim_{\alpha\uparrow 1} \frac{1}{\beta^t} \sum_{\substack{G\in\mathcal{G}_{t+1} \\ G\subseteq G_t(\omega)}} (\alpha q + (1-\alpha)p)(G) \log \frac{(\alpha q + (1-\alpha)p)(G)}{q^\circ_{G_t(\omega)}(G)} \\ &= \frac{1}{\beta^t} \sum_{\substack{G\in\mathcal{G} \\ q(G)>0}} q(G) \log \frac{q(G)}{q^\circ_{G_t(\omega)}(G)} = \frac{1}{\beta^t} \sum_{G\in\mathcal{G}_{t+1}} q(G) \log \frac{q(G)}{q^\circ_{G_t(\omega)}(G)} \end{aligned}$$

for all $q \in \Delta(G_t(\omega))$. By Proposition 2, $\gamma_{t,\omega}\left(q_{|\mathcal{G}_{t+1}}\right) = \infty$ if $q \in \Delta(\Omega) \setminus \Delta(G_t(\omega))$. Therefore, $\gamma_{t,\omega}(r) = \beta^{-t} R_{\mathcal{G}_{t+1}}\left(r \| \left(q^{\circ}_{G_t(\omega)}\right)_{|\mathcal{G}_{t+1}}\right)$ for all $r \in \Delta(\Omega, \mathcal{G}_{t+1})$. By (12), this implies (18). ■

A.8 Proofs of Theorems 4 and 5

We first prove Theorem 5. Choose $\omega \in \Omega$, $t < T$, and $f \in \mathcal{F}$. First observe that if e and $\bar{e}$ belong to $\mathcal{E}^t$, $e(t,\omega) = \bar{e}(t,\omega)$, and $e(t+1,\omega') = \bar{e}(t+1,\omega')$ for all $\omega' \in G_t(\omega)$, then, by CP, $V_t(\omega, f+e) = V_t(\omega, f+\bar{e})$. Therefore, $\partial V_t(\omega, f) = \{(k,m) \in \mathbb{R} \times \mathbb{M}(G_t(\omega), \mathcal{G}_{t+1}) : V_t'(\omega, f; e) \leq k e_t(\omega) + \beta \int e_{t+1} dm$ for all $e \in \mathcal{E}(t,\omega)\}$ where $\mathcal{E}(t,\omega)$ is the set of all $\{\mathcal{G}_t\}$-adapted processes e such that $e(\tau,\omega') = 0$ if $\tau \neq t, t+1$ or $\omega' \notin G_t(\omega)$.

For all $e \in \mathcal{E}(t,\omega)$: If $t = T-1$, then $V_{t+1}(\omega', f+e) = V_T(\omega', f+e) = u(f_T(\omega') + e_T(\omega'))$ for all $\omega' \in \Omega$; set $\varphi(\omega') = 0$ for all $\omega' \in \Omega$, and get $V_{t+1}(\omega', f+e) = u(f_{t+1}(\omega') + e_{t+1}(\omega')) + \varphi(\omega')$. Else $V_{t+1}(\omega', f+e) = u(f_{t+1}(\omega') + e_{t+1}(\omega')) + \min_{p \in \Delta(\Omega, \mathcal{G}_{t+2})}\left(\beta \int V_{t+2}(f+e)\, dp + \gamma_{t+1}(\omega', p)\right) = u(f_{t+1}(\omega') + e_{t+1}(\omega')) + \min_{p \in \Delta(\Omega, \mathcal{G}_{t+2})}\left(\beta \int V_{t+2}(f)\, dp + \gamma_{t+1}(\omega', p)\right)$ for all $\omega' \in \Omega$, where the last equality descends from CP and the fact that $f_\tau + e_\tau = f_\tau$ for all $\tau \geq t+2$. $\mathcal{G}_{t+1}$-measurability of $V_{t+1}(\cdot, f+e)$, f_{t+1}, and e_{t+1} implies $\mathcal{G}_{t+1}$-measurability of the function φ defined by $\varphi(\omega') = \min_{p \in \Delta(\Omega, \mathcal{G}_{t+2})}\left(\beta \int V_{t+2}(f)\, dp + \gamma_{t+1}(\omega', p)\right)$ for all $\omega' \in \Omega$. Also in this case,

$$V_{t+1}(\omega', f+e) = u(f_{t+1}(\omega') + e_{t+1}(\omega')) + \varphi(\omega') \quad \forall \omega' \in \Omega. \tag{46}$$

Denote by $\mathcal{G} = \{G_1, ..., G_g\}$ the set of all elements of $\mathcal{G}_{t+1}$ contained in $G_t(\omega)$, by $\Delta\mathcal{G}$ (resp. $\mathbb{M}(\mathcal{G})$) the set $\Delta(G_t(\omega), \mathcal{G}_{t+1})$ (resp. $\mathbb{M}(G_t(\omega), \mathcal{G}_{t+1})$), by $\vec{f} = (f_0, f_1, ..., f_g)$ the vector $(f_t(\omega), f_{t+1}(G_1), ..., f_{t+1}(G_g))$ for all $f \in \mathcal{F}$, by $\vec{m} = (m_1, ..., m_g)$ the vector $(m(G_1), ..., m(G_g))$ for all $m \in \mathbb{M}(\mathcal{G})$, and by $\vec{\varphi} = (\varphi_1, ..., \varphi_g)$ the vector $(\varphi(G_1), ..., \varphi(G_g))$.

Notice that $e \mapsto \vec{e}$ defines a linear isomorphism between $\mathcal{E}(t,\omega)$ and $\mathbb{R}^{g+1}$, and set for all $\vec{e} = (e_0, ..., e_g) \in \mathbb{R}^{g+1}$

$$\begin{aligned}
F(e_0, ..., e_g) = V_t(\omega, f+e) &= u(f_t(\omega) + e_t(\omega)) + \min_{r \in \Delta\mathcal{G}}\left(\beta \int V_{t+1}(f+e)\, dr + \gamma_t(\omega, r)\right) \\
&= u(f_0 + e_0) + \min_{r \in \Delta\mathcal{G}}\left(\beta \int [u(f_{t+1}(\omega') + e_{t+1}(\omega')) + \varphi(\omega')]\, dr(\omega') + \gamma_t(\omega, r)\right) \\
&= u(f_0 + e_0) + \min_{r \in \Delta\mathcal{G}}\left(\sum_{i=1}^{g} \beta (u(f_i + e_i) + \varphi_i) r_i + \gamma_t(\omega, r)\right).
\end{aligned}$$

Moreover, $(k,m) \in \partial V_t(\omega, f)$ iff $\lim_{\lambda \downarrow 0} \lambda^{-1}[V_t(\omega, f+\lambda e) - V_t(\omega, f)] \leq k e_t(\omega) + \beta \int e_{t+1} dm$ for all $e \in \mathcal{E}(t,\omega)$ iff $\lim_{\lambda \downarrow 0} \lambda^{-1}\left[F(\lambda \vec{e}) - F\left(\vec{0}\right)\right] \leq k e_0 + \sum_{i=1}^{g} \beta m_i e_i$ for all $\vec{e} \in \mathbb{R}^{g+1}$ iff $(k, \beta \vec{m})$ belongs to the superdifferential of Convex Analysis $\partial F\left(\vec{0}\right)$ of F at $\vec{0}$.

For each $j = 0, 1, ..., g$, consider:

- the concave function $\phi_j : \mathbb{R}^{g+1} \to \mathbb{R}$ defined by $\phi_j(\vec{e}) \equiv \beta_j (u(f_j + e_j) + \varphi_j)$ for all $\vec{e} \in \mathbb{R}^{g+1}$, with the convention $\varphi_0 \equiv 0, \beta_0 \equiv 1, \beta_j \equiv \beta$ if $j = 1, ..., g$;
- the row vector A_j corresponding to the projection on the j-th component;
- the function $\beta_j u + \beta_j \varphi_j : \mathbb{R} \to \mathbb{R}$.

Then $\phi_j(\vec{e}) = (\beta_j u + \beta_j \varphi_j) \circ (A_j + f_j)(\vec{e})$ and, by [16, Vol. I Thm. VI.4.2.1],

$$\partial\phi_j(\vec{e}) = A_j^T \partial(\beta_j u + \beta_j \varphi_j)((A_j + f_j)(\vec{e}))$$
$$= A_j^T \beta_j \partial u(f_j + e_j) = \left\{ \left[\begin{array}{c} 0 \\ ... \\ \beta_j u'(f_j + e_j) \\ ... \\ 0 \end{array} \right] \middle| \; u'(f_j + e_j) \in \partial u(f_j + e_j) \right\}.$$

Consider the function $\Phi : \mathbb{R}^{g+1} \to \mathbb{R}$ defined by $\Phi(\vec{v}) \equiv v_0 + \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} v_i r_i + \gamma_t(\omega, r)\right)$. It is easy to check that Φ is concave, monotonic, and

$$\partial\Phi(\vec{v}) = \left\{ \left[\begin{array}{c} 1 \\ \rho_1 \\ ... \\ \rho_g \end{array} \right] \middle| \; \rho \in \arg\min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} v_i r_i + \gamma_t(\omega, r) \right) \right\}.$$

For all $\vec{e} \in \mathbb{R}^{g+1}$, $F(\vec{e}) = \Phi(\phi_0(\vec{e}), \phi_1(\vec{e}), ..., \phi_g(\vec{e}))$ and, setting $\rho_0 = 1$, [16, Vol. I Thm. VI.4.3.1] delivers:

$$\partial F(0) = \left\{ \sum_{i=0}^{g} \rho_i \left[\begin{array}{c} 0 \\ ... \\ \beta_i u'(f_i) \\ ... \\ 0 \end{array} \right] \middle| \; \begin{array}{l} \rho \in \arg\min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} (\beta(u(f_i) + \varphi_i)) r_i + \gamma_t(\omega, r)\right) \\ u'(f_i) \in \partial u(f_i) \quad \forall i = 0, ..., g \end{array} \right\}$$
$$= \left\{ \left[\begin{array}{c} u'(f_0) \\ \beta u'(f_1)\rho_1 \\ ... \\ \beta u'(f_g)\rho_g \end{array} \right] \middle| \; \begin{array}{l} \rho \in \arg\min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} (\beta(u(f_i) + \varphi_i)) r_i + \gamma_t(\omega, r)\right) \\ u'(f_i) \in \partial u(f_i) \quad \forall i = 0, ..., g \end{array} \right\},$$

whence

$$\partial V_t(\omega, f) = \left\{ (u'(f_t(\omega)), m) \;\middle|\; \begin{array}{l} m(G) = u'(f_{t+1}(G))\rho(G) \quad \forall G \in \mathcal{G}_{t+1}, \\ \rho \in \arg\min_{r \in \Delta\mathcal{G}} \left(\beta \int (u(f_{t+1}(\omega')) + \varphi(\omega')) \, dr(\omega') + \gamma_t(\omega, r)\right), \\ u'(f_t(\omega)) \in \partial u(f_t(\omega)), \; u'(f_{t+1}(G)) \in \partial u(f_{t+1}(G)) \quad \forall G \in \mathcal{G}_{t+1} \end{array} \right\},$$

which together with (46) (i.e. $u(f_{t+1}) + \varphi = V_{t+1}(f)$) delivers (21), and concludes the proof of Theorem 5.

The proof of Theorem 4 starts with the observation that, for every $\omega \in \Omega$, $t < T$, and $f \in \mathcal{F}$, $V_t'(\omega, f; \cdot)$ is linear iff $F'\left(\vec{0}; \cdot\right)$ is linear iff $\partial F\left(\vec{0}\right)$ is a singleton iff $\partial V_t(\omega, f)$ is a singleton. If u is differentiable and $\gamma_t(\omega)$ is essentially strictly convex, then [26, Thm. 26.3] guarantees that

$$I(v_1, ..., v_g) \equiv \min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} v_i r_i + \gamma_t(\omega, r) \right) \quad \forall (v_1, ..., v_g) \in \mathbb{R}^g \tag{47}$$

is differentiable, and hence Φ and F are differentiable. Conversely, if u is not differentiable, just take $\vec{f} \in \mathbb{R}^{g+1}$ with f_0 a point of non-differentiability of u to find a nonsingleton $\partial F\left(\vec{0}\right)$. Then differentiability of $V_t(\omega)$ implies differentiability of u. If $\gamma_t(\omega)$ is not essentially strictly convex, then (again by [26, Thm. 26.3]) the functional I defined by (47) is not differentiable, and there exists $\left(v_1^\circ, ..., v_g^\circ\right) \in \mathbb{R}^g$ such that there are two different ρ and $\bar{\rho} \in \partial I\left(v_1^\circ, ..., v_g^\circ\right) = \arg\min_{r \in \Delta\mathcal{G}} \left(\sum_{i=1}^{g} v_i^\circ r_i + \gamma_t(\omega, r)\right)$. Since $u(\mathbb{R})$ is unbounded, there are $f \in \mathcal{F}$ and $b \in \mathbb{R}$ such that

$\beta(u(f_i)+\varphi_i)=v_i^\circ+b$ for $i=1,...,g$. Then $\rho,\bar{\rho}\in\arg\min_{r\in\Delta\mathcal{G}}\left(\sum_{i=1}^g(\beta(u(f_i)+\varphi_i))\,r_i+\gamma_t(\omega,r)\right)$, moreover $u'(z)\neq 0$ for all $z\in\mathbb{R}$ (u is strictly monotonic, concave, and differentiable), and hence

$$\begin{bmatrix} u'(f_0) \\ \beta u'(f_1)\rho_1 \\ \dots \\ \beta u'(f_g)\rho_g \end{bmatrix} \text{ and } \begin{bmatrix} u'(f_0) \\ \beta u'(f_1)\bar{\rho}_1 \\ \dots \\ \beta u'(f_g)\bar{\rho}_g \end{bmatrix}$$

are two distinct elements of $\partial F\left(\vec{0}\right)$, which is absurd. ■

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