# Achieving Efficiency in Repeated Partnerships via Information Design*

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**Abstract**

Two partners contribute to a common project over time. The value of the project is determined by the aggregate effort of the partners and by a common productivity parameter that each partner is privately informed about. At each instant, the two partners observe a noisy public signal of total effort. An equilibrium of this game is Markov if effort choices of agents depend only on the beliefs about the value of the project and on calendar time. I characterize the unique symmetric linear Markov equilibrium as the solution to a nonlinear boundary value problem. The equilibrium features a mutual *encouragement effect*, as agents exaggerate their effort in order to signal their private information, which counteracts free-riding incentives. Indeed, if the project lasts sufficiently long, the diffused information structure approximates the first-best in terms of welfare. If, instead of distributed private information, one agent has all the information about the productivity parameter, the excessive signaling effect is accentuated. As a result, the centralized information structure can yield output levels above the first best.

# 1 Introduction

Many key economic activities—R&D ventures, startups, and contributions to political campaigns to name but a few—require the sustained collaboration of several agents. Of-

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ten, such collaborative projects suffer from the well-known free-riding problem: agents do not internalize the externalities they create on others, leading to efficiency losses. In addition to free riding, however, collaborations also entail asymmetric information among participating agents. For instance, in joint ventures, the parties might have different expectations about market conditions; and in startup teams, founders might have different perceptions about the value of their product. In these settings, agents can learn from each other while interacting. Although the literature has studied the free-riding phenomenon extensively, the effect of asymmetric information and learning on free-riding incentives is still not as well-understood.[1] In particular, it is not known $(i)$ how dynamic effort incentives are impacted by the presence of asymmetric information, and $(ii)$ whether information asymmetries can be used as design tools to mitigate free-riding.

In order to answer these questions, this paper provides a finite-horizon continuous-time collaboration model in which agents have private information about the prospects of their joint project.[2] At every instant agents exert costly effort to contribute to a public good that is shared among agents, and hence, the *free-riding* problem arises. The value of the public good depends on both total effort and on the prospect (the type) of the project. An agent's effort is unobservable to other agents: instead, in every instant, the agents observe a common noisy signal of aggregate contemporaneous effort.

Many economic settings share the features described above. For instance, consider a startup engaged in developing a new product with a pre-specified release date. The founders of the startup have different expectations about the success of the project. Even though they can observe how much time each founder spends in the office, no one is ever sure about the exact working hours of a given founder. Moreover, founders can work at any instant, so time runs continuously. Similar observations can be made for countries working together to reduce carbon emissions, and for communities contributing to a public construction project. In all such situations, a hard-working member of a dynamic partnership may boost the other members' confidence about the viability of their project, while a lazy member might create the opposite effect. Thus, signaling private information about the common state is central in shaping the dynamic incentives of the agents.

Asymmetric information and imperfect monitoring create substantial challenges for

[1] In static settings, Holmström (1982) and Legros and Matthews (1993) show how penalties based on total output can mitigate free-riding, and in a repeated partnership setting Radner, Myerson and Maskin (1986) shows that various equilibrium dynamics can be supported by the threat of future non-cooperation.

[2] A finite horizon captures the idea of deadlines, which fits some of the applications naturally. In section 2.5, I analyze the infinite horizon limit and show that the main findings from the finite horizon carry over.

the analysis. For tractability, we focus on a Linear-Quadratic-Gaussian model and linear Markov equilibria. That is, we assume that: (*i*) information is Gaussian; (*ii*) actions affect the evolution of the state in a linear fashion; and (*iii*) costs are quadratic.[3] In this setup there is a natural way of capturing Markovian behavior: behavior is Markovian if individual effort choices depend only on the agents' beliefs about the prospect of the project and calendar time.[4] Because effort choices are not perfectly observable, the agents' beliefs about the prospects are private information. As agents condition their actions on private beliefs, other agents will have private beliefs about these beliefs, which leads agents to have private beliefs about other agents private beliefs about their private beliefs, and so on, ad infinitum.[5] At first glance, this infinite regress seems to make an agent's inference problem intractable. However, in linear environments there is an elegant way to represent an agent's private belief: it can be written as a weighted average of the initial private information and the public posterior mean about the project's prospect conditional on public information only. This representation allows us to study linear Markov equilibria, i.e., equilibria where strategies are time dependent linear functions of initial private information and the public posterior mean. Moreover, tractability of Linear-Quadratic-Gaussian model allows us to compare different information structures.

We establish the existence and uniqueness of a symmetric linear Markov equilibrium and characterize it as the solution of a nonlinear boundary value problem. Although this dynamical system does not admit a closed form solution, we derive several qualitative properties of equilibrium behavior. In particular, the equilibrium features a mutual *encouragement effect*: agents have an incentive to work harder in order to influence the beliefs of others about their private information, and consequently affect their future effort choices. Sufficiently close to the deadline, the *encouragement effect* vanishes and the equilibrium play resembles that of the static game. If the project is long enough, even though approximate learning happens early in the relationship, the *encouragement effect* vanishes only at "infinity".[6]

To understand the *encouragement effect*, recall the reason for free-riding: individual effort has positive externalities for the others. More specifically, each agent chooses effort by comparing his marginal cost to the marginal benefit that accrues to himself, as

[3]There is a natural state in this problem which would be clear in next sentences.

[4]Note that these strategies are non-stationary, and this is a consequence of the finite horizon assumption. When the horizon is infinite, it is without loss of generality to focus on stationary strategies.

[5]This is known as the "forecasting the forecasts of others" problem as described in Townsend (1983).

[6]Strictly speaking, this phenomenon happens only if the discount rate is low enough.

opposed to all of the agents. Dispersed information about the prospects of the project provides a way to increase the marginal benefit of effort for each agent through the following channel: better outcomes in the current period lead to more optimistic beliefs about the prospects, which in turn lead to higher effort by the other agents in the future. Therefore, by choosing a higher effort level in the current period, an agent not only improves the distribution of output that will accrue to himself in the current period, but also improves the future beliefs, and consequently the future effort choices, of his collaborators.

It is not clear a priori whether the *encouragement effect* increases the agents' welfare. It is true that agents exert more effort compared to the case of complete information; however, there is a possibility that agents exert an inefficiently high level of effort. In addition, agents bear the cost of making effort choices under uncertainty. These trade-offs naturally lead to the following questions: How should one design such information asymmetry? Suppose a social designer knows the true prospects of the project. Should the designer reveal the state to every agent or decentralize the information by endowing each agent with some information and letting them learn from each other? Should there be a leader who knows the true prospects of the project?

In principle, the answer to the first question posed above requires optimizing over the set of all possible information structures, and this is not a simple task. It turns out, however, that from an efficiency standpoint, one simple information structure is approximately optimal. Namely, the decentralized information structure described above is approximately efficient when the time horizon is long enough. More precisely, because learning happens fast, as the horizon grows larger the relevant notion of efficiency is that of ex-post efficiency. And for all periods (after learning has occurred) except for a very small time interval close to the deadline, the outcome obtained under decentralized information is the efficient one of the static complete information case. As the horizon grows large, the efficiency loss from the initial interval of time before learning and from the small interval of time before the deadline can be made arbitrarily small, so the result follows. On the other hand, from an output maximization standpoint, the decentralized information structure might be dominated by the information structure with a leader who knows the true prospects of the project. That is, depending on parameters, the encouragement effect might make the leader work so hard to get the others to work hard as well that the resulting total output is higher than in the decentralized case.

## 1.1 Literature Review

This paper is related to the following four strands of the literature: $(i)$ continuous-time games, $(ii)$ dynamic Bayesian games, $(iii)$ repeated partnerships and contribution games, and $(iv)$ information design in dynamic environments.

In the literature on continuous-time games, multi agent version of Kyle model by Back, Cao and Willard (2000) and Bonatti, Cisternas and Toikka (2017)(henceforth BCT) are the closest to this paper.[7] In particular BCT consider a finite horizon dynamic oligopoly model with incomplete information (private values) and give sufficient conditions for existence of a linear Markov equilibrium. Moreover, BCT identify how signaling and learning incentives affect equilibrium behavior. In contrast, in this paper valuations are interdependent and the stage game is a game of common interest. Moreover, the main goal of this paper is to compare different information structures in the presence of asymmetric information and learning, and this is not the focus of BCT.

The literature on dynamic Bayesian games is well established.[8] These papers typically focus on games where agents communicate their private information via cheap talk messages in the beginning of every period, actions are perfectly observed, valuations are private and players are patient.[9] An important question is whether cooperation can be achieved in such environments or not. This has been addressed in the literature from a "Folk Theorem" perspective, which in particular considers highly history-dependent strategies. In contrast, this paper shows that even if communication channels are absent, actions are imperfectly monitored and valuations are interdependent, approximate efficient outcome can still be achieved in the long run, in *Markovian strategies*.

Admati and Perry (1991) and Marx and Matthews (2000) show that if agents contribute to a public project over time, a Markov-perfect equilibrium outcome is not efficient due to free-riding. Hence, dynamics *per se* do not solve the free-rider problem, at

[7] Sannikov (2007) started the analysis of continuous-time repeated games with Brownian noise. Faingold and Sannikov (2011) studied reputation dynamics with one long-run player interacting with a sequence of short-run players and characterize sequential equilibria as solutions to non-linear ODE. Daley and Green (2012) and Dilmé (2014) studied signaling incentives of a long-run agent with binary types facing a sequence of short-run players. Keeping the long-run and short-run player assumptions Cisternas (2017) established conditions for the validity of the first-order approach in symmetric Gaussian models. Kyle (1985) introduced insider-trading models to study strategic use of information in financial markets. Back (1992) provided a continuous-time version of Kyle (1985) model. Foster and Viswanathan (1996) and Back, Cao and Willard (2000) provided multi-agent versions of the model.

[8] It was initiated by Aumann and Maschler (1995), and includes several important contributions, like Athey and Bagwell (2008), Hörner and Lovo (2009), Escobar and Toikka (2013), and Hörner, Takahashi and Vieille (2015), Pęski and Toikka (2017).

[9] Escobar and Llanes (2015) analyzed the effects of communication on equilibrium behavior and whether cooperation can be achieved without communication.

least when one focuses on Markovian behavior. Yildirim (2006) and Georgiadis (2014) were able to mitigate free-riding in dynamic contribution games by having effort choices become strategic complements over time. This is achieved by assuming that payoffs only occur if contributions reach a given threshold. In this paper, effort choices also become strategic complements over time, but through a different channel: private beliefs. Agents exaggerate their choices in order manipulate each other's beliefs, hence influencing future effort choices. This channel is also present in Cetemen, Hwang and Kaya (2017), who considered an environment in which learning is exogenous and private beliefs matter only after deviations from equilibrium. In contrast, Bonatti and Hörner (2011) considers a dynamic moral hazard model in which agents try to achieve a one-time breakthrough. The arrival rate of the breakthrough depends on how much effort agents exert and the quality of the project. In their model agent's current effort and the others' future efforts are strategic substitutes, because the game ends after the first breakthrough. This fact leads to inefficiencies in the form of procrastination.[10]

Hermalin (1998) and Komai, Stegeman and Hermalin (2007) identify channels by which a leader can improve overall welfare of team if aggregate output of the team depends both on the aggregate effort and on an unobserved state, which is known by the leader.[11] In these papers, the leader is an agent who has superior information about the unobserved state. Both papers consider the case of sequential moves and perfect monitoring: the leader exerts effort first, and then the rest of agents make their choices. As such, the leader's effort provides extra information, which reduces the payoff loss resulting from uncertainty. In the latter paper, by working hard, a leader can encourage other members to work hard. By contrast, in this paper both channels coexist and the result of their interaction is that the introduction of a leader *does not* improve overall welfare, although it might result in higher total output.

There is also a recent literature on how to design an information disclosure policy in dynamic environments. Hörner and Lambert (2015) study the design of optimal rating system (which maximizes the agent's effort) in a career-concerns framework with a stationary Gaussian environment.[12] Pei (2015) also studies the effect of information

[10] Guo and Roesler (2016) and Dong (2016) extended Bonatti and Hörner (2011) by allowing agents to have asymmetric information. See also Campbell, Ederer and Spinnewijn (2014), which analyzes the interaction between free-riding and strategic information revelation.

[11] See also Hermalin (2007), and Komai and Stegeman (2010).

[12] Ekmekci (2011) shows that, in the canonical reputation model, if one allows for rating systems there exist a finite rating system under which a long-run player can approximately guarantee his Stackelberg payoff under all histories.

disclosure by a third party in a career-concerns framework, however, Pei (2015) models uncertainty with Poisson-bandits instead of Brownian motion. Ely (2017) examines information disclosure in a principal-agent setting without transfers.[13] Similar to this paper in the multi-agent settings, Halac, Kartik and Liu (2017) and Bimpikis, Ehsani and Mostagir (2014) ask the question of how to design a contest with the goal of achieving breakthroughs. Similar to above papers this paper uses coarse information policies and does not identify the optimal information structure. Competitive nature of contests and how the uncertainty modeled in these papers makes the results and the analysis different from this paper.

This paper is organized as follows: Section 2 describes the main model and the results. Section 3 describes the possible information structures. Section 4 discusses possible extensions and robustness of the results. All proofs are provided in the Appendix.

## 2 Model

We consider a partnership game played in continuous time over a fixed time horizon $[0, T]$. There are 2 agents, labeled 1 and 2. Each agent has private information about the profitably of the joint project.[14] For each agent $i$, let $\theta_i$ denote his private information, assumed to be normally distributed with mean $\mu_i$ and variance $g_i$. That is, $\theta_i \sim \mathcal{N}(\mu_i, g_i)$. The environment is assumed to be symmetric: $g_1 = g_2$ and $\mu_1 = \mu_2$. The agents' private information is persistent and drawn once at the beginning of the game. The profitability of the project are given by $\Theta := \theta_1 + \theta_2$. At every instant $t$, each agent $i$ chooses an effort level $a_{i,t}$ which is not observable by the other agent. Instead, each agent observes a common public signal $(Y_t)_{t\geq 0}$, whose evolution is given by the following stochastic differential equation

$$dY_t = \left(\sum_{i=1}^{N} a_{i,t}\right) dt + \sigma dZ_t, \quad Y_0 = 0, \tag{1}$$

where $Z$ is a standard Brownian Motion, that is independent of the productivity of the project $(\theta_1, \theta_2)$, and $\sigma > 0$. Agents incur a quadratic cost of effort. Each agent is risk neutral and they share the aggregate output equally. A pure strategy of an agent determines how much effort the agent should exert at every instant.

Specifically, a pure strategy for agent $i$ is a stochastic process, that is measurable

[13] See also Orlov, Skrzypacz and Zryumov (2016).

[14] The extension to more than two agents is straightforward, hence omitted.

with respect to the public $\left((Y_t)_{t\geq 0}\right)$ and the agent's private information (previous effort choices and $\theta_i$). Since our monitoring structure has full support ($\sigma > 0$), agents cannot detect whether the other agent has deviated from a given strategy.[15] Therefore, it is enough to focus on strategies that depend only on the process $(Y_t)_{t\geq 0}$ and on $\theta_i$. This is the analogue of public strategies in repeated games of imperfect monitoring.

More precisely, some technical restrictions are required on the set of available strategies in order for this game to be well defined. Let $\left(\mathcal{F}_t^Y\right)_{t\geq 0}$ denote the public filtration. We require strategies $(a_{i,t})_{t\geq 0}$ to be progressively measurable with respect to the filtration generated by $\left(\left(\mathcal{F}_t^Y\right)_{t\geq 0}, \theta_i\right)$ and square integrable $\left(\mathbb{E}\left[\int_0^t a_{i,s}^2 ds\right] < \infty,\ \forall t \geq 0\right)$, call these strategies admissible. From now on, a strategy will always mean an admissible strategy.[16] Given a strategy profile $(a_1, a_2)$, agent $i$'s payoff is given by

$$\mathbb{E}^{(a_1,a_2)}\left[\int_0^T e^{-rs}\left(\frac{1}{2}\Theta\sum_i a_{i,s} - \frac{1}{2}a_{i,s}^2\right)ds\right] \tag{2}$$

where $r \geq 0$ is the common discount factor. Note that agents only observe payoffs at the deadline $T$, so they do not observe flow payoffs before $T$. However, see Remarks I and II below for alternative equivalent formulations of the problem. Because of the full support assumption, it is without loss of generality to focus on Nash Equilibrium as the solution concept. Indeed, the set of equilibrium outcomes does not change if we impose sequential rationality on off-path behavior.

A strategy $(a_{i,t})_{t\geq 0}$ is *linear* if it has the following representation

$$a_{i,t} = \alpha_t\theta^i + \int_0^t f_s^t dY_s \quad \forall t \in [0, T], \tag{3}$$

where $\alpha : [0, T] \to \mathbb{R},\ f : [0, T]^2 \to \mathbb{R}$ are Borel measurable functions. A strategy profile is *symmetric linear* if the functions $\alpha$ and $f^t$ are the same for each player. The focus on *linear* strategies allows us to have a tractable analysis by simplifying the filtering problem

[15] This follows from Girsanov theorem (*e.g* Karatzas and Shreve (1991) page 191). Agents jointly control the mean of a diffusion process, therefore take any two admissible strategy profiles then the corresponding measure they create over the sample paths of $(Y_t)_{t\geq 0}$ are absolutely continuous respect to each other.

[16] To be precise, the game takes place on a probability space $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$. State space $\Omega = \mathbb{R}^2 \times C[0, T]$, denotes the set of all possible paths that can be generated by $(Y_t)_{t\geq 0}$ and possible realizations of the states $(\theta_i, \theta_j)$. The strategy is a function $a : \mathbb{R}_+ \times C(\mathbb{R}_+) \to \mathbb{R}$, where $C$ denotes the space of continuous functions. Strategy for player $i$ is progressively measurable with respect to $\mathcal{F}_t^i$. $\mathcal{F}_t$ is the canonical $\sigma$-algebra on $C[0, t]$.

of each agent.[17]

**Remark I** An alternative model that leads to the same predictions is as follows. Agents jointly control a state variable $X$ and observe the process $(X_t)_{t\geq 0}$, which evolves according to the following stochastic differential equation

$$dX_t = \left(rX_t + \sum_i a_{i,t}\right) dt + \sigma dZ_t, \quad X_0 = 0.$$

Payoffs are realized at time $T$ and the payoff of each agent is

$$\frac{1}{2}e^{-rT}\mathbb{E}\left[X\Theta\right] - \int_0^t e^{-rt}\frac{1}{2}a_{i,t}^2 dt.$$

$X$ can be interpreted as the capital stock and the $dX_t$ is the flow investment to capital. The dividends depend both on the capital stock and on market conditions ($\Theta$). This model is similar to gradual revision games introduced by Iijima and Kasahara (2015). The difference here is that we allow for asymmetric information among the agents.

**Remark II** Another information structure that gives the same equilibrium dynamics is as follows. Each agent observes a conditionally independent private signal about $\Theta$,

$$s_i = \Theta + \epsilon_i$$

where $\epsilon_i$ is a Normal random variable with mean 0 and variance $g$, and $\left(\epsilon_i, \epsilon_j, \Theta\right)$ are independent.[18] Since agents are risk neutral, the object of interest is the conditional expectation of the profitability of the project, conditional on all the signals of the agents. Denote this value by $\Theta_p$. Because of the normality assumptions, $\Theta_p$ is an affine function of the signals and, by normalizing the signals, one can assume $\Theta_p = s_1 + s_2$, which brings us back to our initial formulation.

Having described the model, we now proceed to analysis. It has four main steps: ($i$) derivation of the agents' beliefs, ($ii$) formulation of the best-response problem as an optimal control problem, ($iii$) Hamilton-Jacobi-Bellman (HJB) equations of agents and derivation the dynamical system governing the equilibrium dynamics, and ($iv$) showing

[17] It will become clear in the next section how linearity helps the filtering problem. Theoretically, if we do not focus on linear strategies, one can use *Kushner-Stratonovich* filter instead of *Kalman* filter. However, by doing so we lose the quadratic structure of pay-offs which is essential for the analysis.

[18] Independence is not crucial – joint normality would do.

the dynamical system described in step (*iii*) has a solution and it is unique.

## 2.1 Beliefs and Equilibrium Notion

Under the assumption that agent $j$ uses linear strategies, agent $i$ can always entangle the effects of its own effort choice and the public component of agent $j's$ effort choice from the public signal. Therefore, agent $i$ observes the following signal about agent $j's$ private information

$$dY_t^i := dY_t - a_{i,t}dt - \int_0^t f_s^t dY_s dt = \alpha_t \theta_j dt + \sigma dZ_t.$$

Then agent $i's$ first order belief about $\theta_j$ becomes a standard linear filtering problem. An application of Kalman Filter gives us agent $i's$ first order belief about $\theta_j$. Denote this posterior mean as $m_t^{i,j} := \mathbb{E}[\theta_t^j | \mathcal{F}_t^i]$ and $\gamma_t := \mathbb{E}[(\theta_t^j - m_t^{i,j})^2 | \mathcal{F}_t^i]$ denote the posterior variance of $m_t^{i,j}$.

**Lemma 1** *Under any linear strategy profile, agent i's first order belief about $\theta_j$ evolves as*

$$\begin{aligned} dm_t^{i,j} &= \frac{\alpha_t \gamma_t}{\sigma^2}\left(dY_t^i - \alpha_t m^{i,j} dt\right) \\ \dot{\gamma}_t &= -\left(\frac{\alpha_t \gamma_t}{\sigma}\right)^2 \end{aligned}$$

*with the boundary conditions $m_0^{i,j} = \mu_0$, $\gamma_0 = g$.*

One important aspect of the posterior variance is that it is a decreasing deterministic function of calendar time and only depends on $\alpha_t$.[19]

It is convenient to introduce a fictitious player (the outside observer) who only observes the process $(Y_t)_{t \geq 0}$ and knows the strategies of the agents. Define $\mu_t$ as the posterior mean and $\left(\gamma_t^o\right)$ as the posterior variance of the outside observer. Formally the public posterior mean is

$$\mu_t := \frac{1}{2}\mathbb{E}\left[\Theta_t | \mathcal{F}_t^Y, (a_{1,t}, a_{2,t})_{t=0}^T\right],$$

and the public variance is

$$\gamma_t^o := \mathbb{E}\left[\left(\Theta - 2\mu_t\right)^2 | \mathcal{F}_t^Y, (a_{1,t}, a_{2,t})_{t=0}^T\right].$$ [20]

[19] This system has a closed form solution, see Appendix A.

[20] Under the proposed strategies the outside observer observes the following equivalent process $d\hat{Y}_t =$

Similar to Lemma 1, public posterior mean and the variance are derived applying Kalman filter on the public signal. The following Lemma shows that under the assumption that agents use symmetric linear strategies, an agent's private belief can be written as a weighted average of the public mean and his initial private information.

**Lemma 2** *Under symmetric linear strategies, for each agent i, private belief has the following decomposition*

$$m_t^{i,j} = z_t \mu_t + (1 - z_t)\,\theta_i, \tag{4}$$

*where* $z_t = 2\frac{\gamma_t}{\gamma_t^o} \in [1,2]$ *is a deterministic non-decreasing function over time.*

One implication of Lemma 2 is that it allows us to circumvent higher order beliefs considerations. Here is some heuristics of this problem. If we were to use strategies of the following form

$$\alpha_t \theta_i + \beta_t m_t^{i,j},$$

rather than the form we assume, the filtering problem of agents would be intractable. As agent $i$ conditions his action on $m_t^{i,j}$, due to imperfect monitoring agent $j$ will have private beliefs about $m_t^{i,j}$, which leads agent $i$ to have private beliefs about agents $j's$ private beliefs about $m_t^{i,j}$, and so on ad infinitum. The filtering problem becomes intractable since each agent has to condition on infinite belief hierarchies.[21]

Lemma 2 is only true on the equilibrium path. In fact, if agent $i$ deviates from the equilibrium strategy, then $\mu_t$ is no longer an unbiased estimate of $\Theta$.[22] After a deviation by agent $i$, the private belief of agent $i$ can still be written as weighted average of initial private information and a counterfactual belief rather than the public belief. This counterfactual belief takes into account that agent $i$ deviated from the prescribed strategy.

**Lemma 3** *Under symmetric linear strategies the private belief, for each agent i, has the following*

$\alpha_t \Theta + \sigma dZ_t$. Since $\int_0^t f_s^t dY_s$ only depends on the public information, the outside observer can always filter it out from the public signal. Observe that $\gamma_t^o$ is the posterior variance of $\Theta$, and not of $\frac{1}{2}\Theta$ – this is purely for notational purposes.

[21]This is known as the "forecasting the forecasts of others" problem as described in Townsend (1983). Foster and Viswanathan (1996) suggested writing strategies as functions of signals rather than the private estimates to circumvent this problem. In section 2.3, We show that on the equilibrium path $\alpha_t \theta_i + \beta_t \mu_t$ has an equivalent representation as $\eta_t \left(\theta_i + m_t^{i,j}\right)$. Observe that in this formulation actions only depend on private estimate of each agent about $\Theta$.

[22]Notice that Lemma 2 is still true for other agent (agent $j$), since given the full support assumption agent $j$ cannot detect deviations.

*decomposition*

$$m_t^{i,j} = z_t \hat{\mu}_t^i + (1 - z_t)\,\theta_i,$$

*where* $z_t = \frac{\gamma_t}{\gamma_t^o}$

$$d\hat{\mu}_t^i = \alpha_t \lambda_t (2 - z_t)(\theta_i - \hat{\mu}_t^i)dt + \lambda_t \sigma dZ_t^i, \quad \hat{\mu}_t^i = \mu_0.$$

*Moreover,*

$$\lambda_t = \frac{\alpha_t \gamma_t}{2\sigma^2} \quad \text{and} \quad dZ_t^i = \frac{dY_t^i - \alpha_t \left( z_t \hat{\mu}_t^i + (1 - z_t)\,\theta_i \right) dt}{\sigma}$$

*where* $Z_t^i$ *is a standard Brownian Motion with respect to agent i's information.*

The belief $\hat{\mu}_t^i$ is agent $i's$ private information, since it conditions on the previous effort choices by agent $i$. Nevertheless, on path we have $\mu_t = \mu_t^i, \quad \forall t \in [0, T]$.

**Markov Strategies** A strategy is Markov if it depends only on the agent's belief about $\Theta$ and calendar time. This is the natural analogue of Markov strategies in games of complete information. Markovian behavior and linearity implies that strategies must take the following form

$$a_{i,t} = \alpha_t^i \theta + \beta_t^i \mu_t.$$ [23]

As a final minor technical restriction, we assume that $\alpha_t$ and $\beta_t$ are continuously differentiable functions.

**Equilibrium Notion**: Given the focus on linear Markov strategies, a strategy can be represented as the pair of functions $(\alpha, \beta)$. A profile $((\alpha^1, \beta^1), (\alpha^2, \beta^2))$ which induces a Nash equilibrium of the game is called as linear Markov equilibrium. [24]

Given the linear quadratic structure of the model focusing on linear strategies is natural, because static best responses are linear. Yet, it is not clear whether a non-linear equilibrium exists or not. Also, it is possible to show that if other agent uses linear strategies, even in off-path agent has a best response in the following class

$$a_{i,t} = \alpha_t^i \theta_i + \beta_t^i \hat{\mu}_t^i.$$

[23]Given our definition of Markov strategy, in this model linear Markov strategies have the following characterization. A symmetric linear strategy is Markov if and only if there exists $(\alpha_t, \beta_t)$ such that $a_{i,t} = \alpha_t \theta + \beta_t \mu_t$. See Appendix A for a proof. This characterization is first derived by Bonatti, Cisternas and Toikka (2017).

[24]This definition does not impose any form of sequential rationality assumption, however by the full support assumption, it does not restrict the set equilibrium outcomes.

## 2.2 Agent $i's$ Optimal Control Problem

This section analyzes agent $i$'s best response problem, assuming agent $j$ uses a linear Markov strategy. Since we are focusing on linear strategies, the best response problem of agent $i$ is a stochastic linear regulator problem.[25] Using the triple $(\theta_i, \mu_t, \hat{\mu}_t)$ as a state, agent $i'$s problem is to:

$$\sup_{a_{i,t}\in L^2[0,T]} \mathbb{E}[\int_0^T \left(\frac{1}{2}e^{-rt}\left(\theta_i + m_t^{i,j}\right)\left(a_{i,t} + \alpha_t m_t^{i,j} + \beta_t \mu_t\right) - \frac{1}{2}\left(a_{i,t}\right)^2\right) dt$$

subject to

$$\begin{aligned} d\mu_t &:= \lambda_t\left(\alpha_t\left(m_t^{i,j} - \mu_t\right) + a_t - \mu_t(\alpha_t + \beta_t)\right) dt + \lambda_t \sigma dZ_t^i \quad \mu_0 = \mu, \\ d\hat{\mu}_t^i &:= \lambda_t\left(\alpha_t\left(\theta - \hat{\mu}_t^i\right) + \alpha_t(m_t^{i,j} - \hat{\mu}_t^i)\right) + \lambda_t \sigma dZ_t^i \quad \hat{\mu}_0^i = \mu, \\ m_t^{i,j} &= z_t \hat{\mu}_t^i + (1 - z_t)\theta_i, \end{aligned}$$

where $\lambda_t := \frac{\alpha_t \gamma_t}{2\sigma^2}$ measures how fast the public belief changes.[26] From the perspective of agent $i$, $\mu_t$ has a drift, since agent $i$ observe the true realization of the $\theta_i$ and has better information $m_t^{i,j}$ about the $\theta_j$ compared to the outside observer. From the perspective of the outside observer, $\mu_t$ is a martingale. By deviating from the equilibrium action, agent $i$ can change the drift of $\mu_t$ linearly. In contrast, $\hat{\mu}_t^i$ is independent of agent $i's$ actions and only depends on the agent $j's$ actions, therefore it is exogenous for agent $i$. Using

[25] See Chapter 3 of Bensoussan (2004) or Chapter 6.3 of Yong and Zhou (1999) .

[26] The evaluation of $d\hat{\mu}_t^i$ is given in Lemma 3. For the evolution of $d\mu_t$ recall that (assuming linear strategies)

$$\begin{aligned} d\mu_t &= \lambda_t[dY_t - 2\left(\alpha_t \mu_t + \int_0^t f_s^t dY_s\right) dt] \\ &= \lambda_t[dY_t^i + \left(a_t - 2\alpha_t \mu_t - \beta \mu_t\right) dt] \\ &= \lambda_t \sigma dZ_t^i + \lambda_t\left[\alpha_t\left(m_t^{i,j} - \mu_t\right) + a_t - \mu_t(\alpha_t + \beta_t)\right] dt. \end{aligned}$$

standard methods we can write an agent's HJB equation as follows:

$$\begin{aligned} rV\left(\theta,\mu,\hat{\mu},t\right) &= \sup_{a\in R}\frac{1}{2}\left(\left(2-z_t\right)\theta+z_t\hat{\mu}_t\right)\left(a+\alpha_t\left(z_t\hat{\mu}_t+(1-z_t)\theta\right)+\mu_t\beta_t\right)-\frac{1}{2}a^2 &(5)\\ &+ V_\mu d(\mu)+V_{\hat{\mu}}d(\hat{\mu})+\frac{\left(\lambda_t\sigma_t\right)^2}{2}\left(V_{\mu\mu}+V_{\hat{\mu}\hat{\mu}}+2V_{\hat{\mu}\mu}\right)+V_t &(6)\\ d(\mu) &:= \lambda_t\left(\alpha_t\left(m_t^{i,j}-\mu_t\right)+a_t-\mu_t(\alpha_t+\beta_t)\right) &(7)\\ d(\hat{\mu}) &:= \lambda_t\left(\alpha_t\left(\theta-\hat{\mu}_t\right)+\alpha_t(m^{i,j}-\hat{\mu}_t)\right), &(8)\end{aligned}$$

where $d(\mu)$ denotes the drift of $d\mu$ and $d\left(\hat{\mu}\right)$ denotes the drift of $d\hat{\mu}$. Observe that the maximization problem above is strictly concave in $a$ because costs are convex and gains are linear in $a$. This implies that there exists a unique maximizer $a^*$. It can be found using the first-order condition,

$$a^*=\frac{1}{2}\left(\theta\left(2-z_t\right)+z_t\hat{\mu}_t\right)+\lambda_t V_\mu.$$

The first term captures the static incentive (maximizing flow payoff) and the second term captures the dynamic incentives which comes from affecting the drift of public belief.

Given the linear-quadratic structure, it is natural to guess that the HJB equation has a quadratic functional form

$$rV(\theta,\mu,\hat{\mu},t)=v_0(t)+v_1(t)\theta^2+v_2(t)\theta\hat{\mu}+v_3(t)\hat{\mu}^2+v_4(t)\hat{\mu}\mu+v_5(t)\theta\mu, \tag{9}$$

where $v_k:\mathbb{R}\to\mathbb{R},\ k=0,\dots,5$ are continuously differentiable functions. Substituting this functional form into the first order condition, effort choice can be written as

$$a^*=\alpha_t^*\theta_i+\beta_t^*\hat{\mu}_t.$$

In equilibrium it must be that $\alpha^*=\alpha$ and $\beta^*=\beta$.

It is convenient to define an auxiliary game. Define the myopic coefficients as $\alpha_t^m=\frac{1}{2}\left(2-z_t\right),\ \ \beta_t^m=\frac{1}{2}z_t$. Myopic coefficients correspond to the case as if the game is sequence of static games and states evolves as in equation for $d\mu$.[27] $z_t=2$ corresponds to the perfect learning case where the outside observer knows the state $\Theta$ perfectly. In this case $\alpha_t=0$ and $\beta_t=1$. That is, agents assign zero weight to their initial private

[27] An alternative interpretation is that $r=\infty$.

information. However, learning is partial since outside observe can not distinguish between $\theta_1$ and $\theta_2$. The following Theorem gives necessary and sufficient conditions for $(\alpha, \beta)$ to be a symmetric linear Markov equilibrium.

**Theorem 1** *A symmetric linear Markov equilibrium exists and is unique for all parameters. The equilibrium strategy profile $(\alpha, \beta)$ is characterized as the solution to following system of non-linear differential equations:*

$$\dot{\alpha}_t = r\frac{\alpha_t}{\alpha_t^m}(\alpha_t - \alpha_t^m) + \lambda_t \beta_t \alpha_t \left(\frac{\alpha_t}{\alpha_t^m} - 2(1+\alpha_t)\right) \tag{10}$$

$$\dot{\beta}_t = r\frac{\alpha_t}{\alpha_t^m}(\beta_t - \beta_t^m) + \lambda_t \beta_t \left(\frac{\alpha_t}{\alpha_t^m}(\beta_t - \beta_t^m) + 2\alpha_t(1-\beta_t) - \beta_t\right) \tag{11}$$

$$\dot{\gamma}_t = -\left(\frac{\alpha_t \gamma_t}{\sigma}\right)^2 \tag{12}$$

*with the the boundary conditions $\alpha_T = \alpha^m(\gamma_T), \beta_T = \beta^m(\gamma_T), \gamma_0 = 2g_0$.*

There is no off-the-shelf result one can invoke to prove either existence or uniqueness of an equilibrium. Here we sketch the proof. Instead of focusing on the system above, consider the backwards system and write the problem as an initial value problem (IVP). In this way the problem reduces to guessing the terminal value of $\gamma_T$ which pins down $(\alpha_T^m, \beta_T^m)$, therefore one can trace back the $\alpha, \beta$ and $\gamma$ using the evolutions given above.[28] Then the goal becomes to find a guess $\gamma_T$ such that the corresponding $\gamma_0$ equals to $2g_0$. Observe that the solution is continuous with respect to initial values and observe that if $\gamma_T = 0$ there is a unique solution $\gamma_t = 0,\ \beta_t = 1,\ \alpha_t = 0\ \forall t \in [0, T]$. Then to prove that such a guess exists, it is sufficient to find a guess yielding $\gamma_0 > 2g$. Because, in this case, the intermediate value theorem implies that there must be such guess for which the corresponding $\gamma_0$ equals $2g_0$. Monotonicity of $\gamma(\dot{\gamma}_t < 0)$ implies that correct guess must be in the range $[0, 2g]$. Hence, the proof is completed by showing that, as we vary the guess in the range $[0, 2g]$ none of the equations $(\alpha, \beta)$ diverge and there are uniform bounds for $(\alpha, \beta)$ in the corresponding range which are independent of $(\sigma, r, T)$. In the Appendix we show that these uniform bounds exist, and the next section gives some economical interpretations for the bounds.

For the uniqueness part, the argument goes as follows: $\alpha$ is a *graph-monotone* func-

[28]Alternatively, one can guess the $\dot{\alpha}_0$ but it turns out backwards system is more tractable than the forward system.

tion with respect to the initial guess, which implies that $\gamma$ is also graph-monotone.[29] Therefore, there exists a one-to-one relationship between the initial guess and the corresponding value at time zero, hence the solution must be unique. Uniqueness is crucial in the comparison of different information structures. We now move to the properties of the equilibrium.

## 2.3 Properties of the Equilibrium

On the equilibrium path, strategies have the following equivalent representation,

$$a_{i,t} = \alpha_t \theta^i + \beta_t \mu_t = \tilde{\alpha}_t \theta + \tilde{\beta}_t m_t^i,$$

where $\tilde{\beta}_t := \frac{\beta_t}{z_t}$ and $\tilde{\alpha}_t := \alpha_t - \beta_t \frac{1-z_t}{z_t}$. Moreover, the following Lemma establishes $\tilde{\alpha}_t = \tilde{\beta}_t$.

**Lemma 4** *In the symmetric linear Markov equilibrium,* $\forall t \in [0, T]$ $\frac{\alpha_t}{\beta_t} = \frac{\alpha_t^m}{\beta_t^m}$.

If the game is static we must have $\tilde{\alpha} = \tilde{\beta} = \frac{1}{2}$. Interestingly, the equality of $\tilde{\alpha}$ and $\tilde{\beta}$ also holds in the dynamic game. This equivalence allows us to eliminate $\beta_t$ from system above. In this representation it is clear that agent $i$ assigns the same weight to his private information and to his belief about $\theta_j$ in his effort choices. Equivalently, agent $i$ only conditions on his private belief about $\Theta$ which is equal to $\theta_i + m_t^{i,j}$. The next Proposition derives the properties of linear Markov equilibrium,

**Proposition 1** *The unique linear Markov equilibrium has the following properties,*

1. $2 \geq \alpha_t \geq \alpha_t^m > 0,\ 2 \geq \beta_t \geq \beta_t^m > 0 \ \ \forall t \in [0, T]$ *(encouragement effect)*
2. $\alpha_t$ *is decreasing function of time*
3. *If* $r > 0$, $\beta_t$ *is increasing around 0 and decreasing around T*
4. *If* $r = 0$, $\beta_t$ *is weakly decreasing function of time*
5. $\beta_t > \alpha_t \ \forall t \in (0, T]$ *and* $\beta_t = \alpha_t$ *at* $t = 0$
6. $\frac{\alpha_t \beta_t \gamma_t}{2\sigma^2}$ *(volatility of total effort) is decreasing over time*

[29] Pick two guesses, call a function *graph-monotone* if $\gamma_1 > \gamma_2$ implies $\forall t \in [0, T]$ $\alpha_t(\gamma_1) \geq \alpha_t(\gamma_2)$ and such that equality is strict for some positive measure of time.

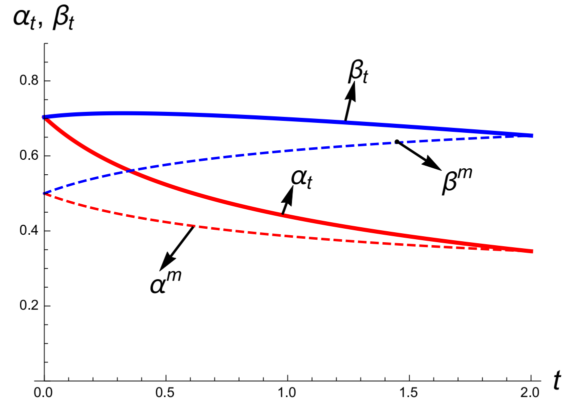



Figure 1: Equilibrium coefficients $\left(T = 2, r = 0.2, g = 1, \sigma = 1\right)$

The dynamics of $\alpha$ and $\beta$ are depicted in Figure 1. At every instant $t$ the equilibrium levels of $\alpha_t$ and $\beta_t$ are always above the corresponding myopic levels. Dashed lines in Figure 1 corresponds to the myopic levels. In equilibrium, agents consider the impact of their actions at time $t$ on the future behavior through $\mu_t$ which makes them exert more effort relative to the myopic level. This is the mutual *encouragement effect* mentioned in the Introduction.[30] It is intuitive that both $\alpha$ and $\beta$ decrease around the deadline. As the deadline approaches incentives coming from the *encouragement effect* vanish, therefore $\alpha$ and $\beta$ converge to their myopic levels.

The upper bound of 2 on the coefficients corresponds to the efficient level of effort when learning is complete ($\gamma_t = 0$). This bound would be $N$ in the $N$-person game. The intuition for why the weight on the public information is larger than the private information is as follows. As time passes, the outside observer has a better estimate of $\Theta$, which contains $\theta_i$. Therefore it is enough to condition on public information to have an estimate about $\Theta$. Perhaps surprisingly, $\beta_t$ is increasing initially if $r > 0$. The dynamics of $\alpha_t$ and $\beta_t$ has two components: the myopic part and the forward looking part. On equilibrium, the myopic part of $\beta_t$ is increasing and the forward looking part is decreasing over time. When $r > 0$, the myopic part increases faster relative to the decreasing forward looking part, resulting in the dynamics being increasing initially.

Figure 2 demonstrates three randomly generated equilibrium paths. Since $\mu_t$ is stochastic then, depending on realizations of the noise, aggregate effort might increase or decrease over time.

[30]The term *encouragement effect* is introduced by Bolton and Harris (1999).

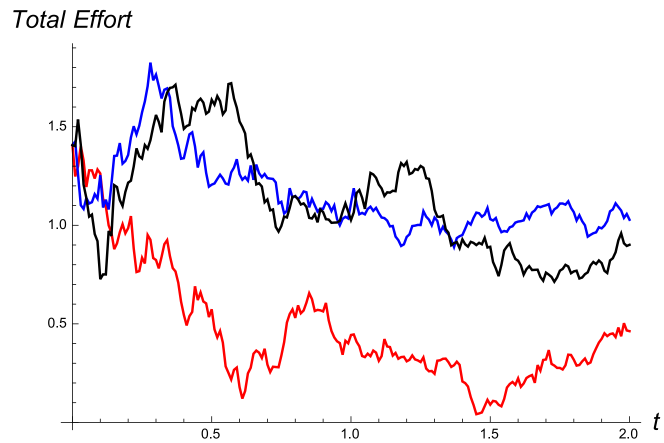



Figure 2: Equilibrium effort paths $(T = 2, r = 0.2, g = 1, \sigma = 1, \Theta = 2, \mu_0 = 1)$

## 2.4 Centralized Information

This section analyzes the model with a different information structure, while keeping the payoff structure. One of the agents (leader) has perfect information about the state $\Theta$, the other agent (follower) has a Gaussian prior $\left(\Theta \sim \mathcal{N}\left(2\mu_0, 2g_0\right)\right)$. The analogue of a linear Markov equilibrium in the previous section is that the leader uses the strategy $\alpha_t\Theta$ and the follower uses the strategy $\frac{1}{2}\mu_t^*$.[31] Under centralized information structure and prescribed linear strategies on the equilibrium path follower's belief is $\mu_t^*$ is common knowledge. Therefore, compared to decentralized information structure there is no higher order belief considerations. The derivation of the beliefs and the agent's problems are similar to the baseline model, hence they are delegated to appendix. The leader's problem is

$$\sup_{a_{i,t}\in L^2[0,T]} \mathbb{E}[\int_0^T \left(\frac{1}{2}e^{-rt}\Theta\left(a_{i,t}+\frac{1}{2}\mu_t^*\right)-\frac{1}{2}\left(a_{i,t}\right)^2\right)dt$$

subject to

$$d\mu_t^* \;:=\; \frac{\alpha_t\gamma_t}{\sigma^2}\left(a_t-\alpha_t\mu_t^*\right)+\frac{\alpha_t\gamma_t}{\sigma^2}dZ_t.$$

Again by following standard methods, the leader's HJB equation can be written as

$$\begin{aligned} rV(\theta,\mu^*,t) &= \sup_{a\in\mathbb{R}}\frac{1}{2}\theta\left(a+\frac{\mu_t^*}{2}\right)+V_t+V_{\mu^*}\frac{\alpha_t\gamma_t}{\sigma^2}\left(a-\alpha_t\mu^*\right)+\frac{1}{2}\frac{\alpha^2\gamma_t^2}{\sigma^2}V_{\mu^*\mu^*} \\ d(\mu_t^*) &= \frac{\alpha_t\gamma_t}{\sigma^2}\left(a_t-\alpha_t\mu_t^*\right). \end{aligned}$$

[31] But notice that here $\mu_t^*$ is the posterior mean of the follower, whereas in the decentralized information case of the previous section it referred to $\frac{1}{2}\Theta$.

For this case, the natural guess for the HJB equation is:

$$rV(\theta, \mu^*) = v_0(t)\theta^2 + v_1(t)\theta\mu^*, \tag{13}$$

where $v_k : \mathbb{R} \to \mathbb{R},\ k = 0, 1$ are continuously differentiable functions. The next proposition states the necessary and sufficient conditions for an equilibrium.

**Theorem 2** *A linear Markov equilibrium exists and is unique for all parameters. The equilibrium strategy profile $\alpha$ is characterized as the solution to following system of non-linear differential equations:*

$$\begin{aligned} \dot{\alpha}_t &= 2r\left(\alpha_t - \frac{1}{2}\right)\alpha_t - \frac{\alpha_t^2 \gamma_t}{2\sigma^2} \\ \dot{\gamma}_t &= -\left(\frac{\alpha_t \gamma_t}{\sigma}\right)^2 \end{aligned}$$

*with the boundary conditions $\alpha = \frac{1}{2}$ and $\gamma_0 = 2g$.*

The proof follows a similar approach to decentralized information case, based on guessing the terminal value of $\gamma_T$. In this case it also sufficient to find a guess of $\gamma_T$ such that corresponding $\gamma_0$ is larger than $2g$. The major difference here is that there is no uniform bound for $\alpha$. While checking the guesses in the range $(0, 2g]$, one of four things can happen: either $(i)$ $\gamma$ diverges, $(ii)$ $\alpha$ diverges, $(iii)$ both diverge or $(iv)$ none of them diverges. For each guess if we are at case $(iv)$ proof is complete since monotonicity of $\gamma$ implies there must be a guess which corresponding $\gamma_0$ equal to $2g$. If we are in $(i)$ then there must be an another guess which is close to initial guess in which solution does not diverge, and the corresponding $\gamma_0$ is above 2g. Thus, the proof is complete once we show that, for any guess, if $\alpha$ diverges, so does $\gamma$. This is established in the Appendix. The next Proposition states the properties of the equilibrium.

**Proposition 2** *In the unique linear Markov equilibrium*

1. *$\alpha_t \geq \frac{1}{2}$ $\forall t \in [0, T]$ (encouragement effect)*
2. *$\alpha_t$ is decreasing function over time*

The dynamics of $\alpha$ depicted in Figure 3. Line $1/2$ represents the followers equilibrium coefficient. Behavior of $\alpha$ is quite similar to the decentralized information case with one major difference: there is no uniform bound for $\alpha$ in this case. In Figure 3, $\alpha$ is

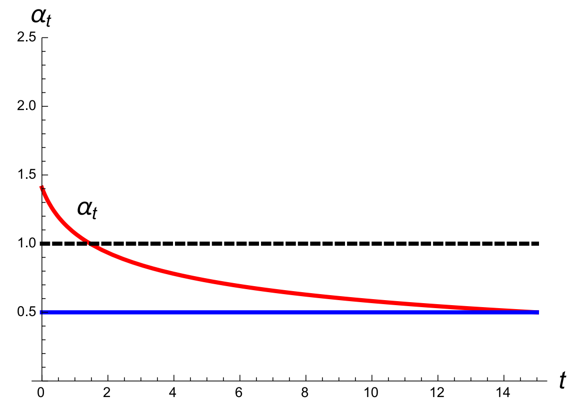



Figure 3: Equilibrium coefficients $\left(T = 15, r = 0, 2g = 2, \sigma = 2.5\right)$

above the efficient level, which is labeled by the dashed black line 1, in the beginning of the relationship. The *encouragement effect* causes $\alpha$ to be above the myopic value 1/2 all the time. One immediate implication of $\alpha$ being decreasing is total effort is also decreasing function over time. Moreover, decreasing $\alpha$ implies from the perspective of an outside observer volatility of total effort is decreasing over time. Figure 4 demonstrates

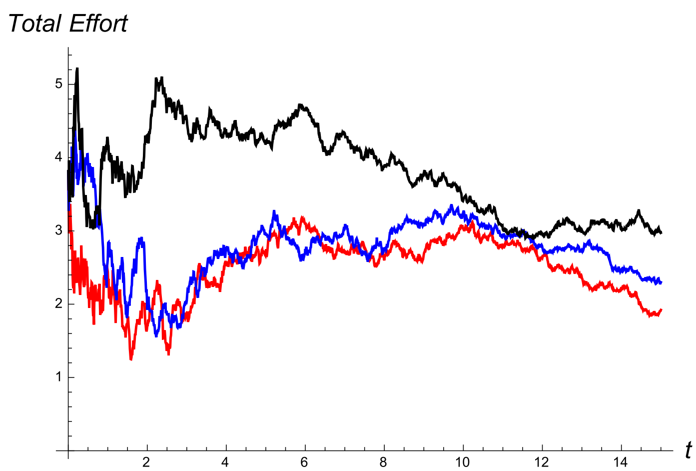



Figure 4: Equilibrium effort paths $\left(T = 15, r = 0, 2g = 2, \sigma = 2.5, \Theta = 2, \mu_0^* = 2\right)$

three randomly generated equilibrium paths. Since $\mu_t^*$ is stochastic then, depending on realizations of the noise, aggregate effort might increase or decrease over time. Note that under centralized information structure only the follower's effort is stochastic, expert's effort choices are deterministic.

## 2.5 The Infinite-Horizon Limit

This section analyzes the limit as $T \to \infty$ and shows that limiting equilibrium is an equilibrium of the game when $T = \infty$ for both centralized and decentralized information structures. First, we start with some observations about the long run behavior of the equilibrium. Proposition 9 below shows that if the game is long enough, arbitrarily precise learning happens in finite time in both centralized and decentralized information models. Let $\gamma_T^C$ denotes the terminal value of variance at time $T$ in the case of centralized information and $\gamma_T^D$ denotes the terminal value of variance at time $T$ in the case of decentralized information.

**Proposition 3** *In the unique (symmetric) linear Markov equilibrium*

1. *In both centralized and decentralized information case for every $\epsilon > 0, \exists t^*$ such that in the T-horizon game $\forall t,\ \ T \geq t \geq t^*,\ \gamma_t < \epsilon$.*

2. *For any $T$, in the equilibrium of T-horizon game $\gamma_T^D > \gamma_T^C$, where $\gamma_T^D$ denotes the terminal value of variance at time $T$ in the case of decentralized information and $\gamma_T^C$ denotes the terminal value of variance at time $T$ in the case of centralized information.*

Proposition 3 implies that the public belief converges to true the value of $\Theta$ as $T \to \infty$, but as mentioned in Section 2.2, learning is partial. Given that agents use symmetric strategies, the outside observer cannot distinguish between $\theta_1$ and $\theta_2$. From the perspective of agents common learning happens, each agent has perfect estimate about other agents private information.[32] Even though we do not have closed form expressions for the equilibrium, we can rank the information contents of the equilibrium outcomes. Interestingly, the outside observer will have a better estimate about $\Theta$ in centralized information.

We conclude this section by stating the convergence result for the infinite horizon limit.

**Proposition 4** *Any sequence of (symmetric) linear Markov equilibria contains a subsequence that converges pointwise to a (symmetric) linear Markov equilibrium of the infinite-horizon game.*

Under some restrictions on parameters it is possible to show that convergence is in fact uniform.[33]

[32] Note that this is only true for two agents, if $N > 2$ only the $\Theta = \sum_{i=1}^{N} \theta_i$ becomes common knowledge.

[33] Bonatti, Cisternas and Toikka (2017) obtained uniform convergence. But they did so under the restriction on parameters that they need to ensure existence of equilibrium. Here no such restriction is needed, as existence and uniqueness hold for all parameter values. Imposing similar restrictions on parameter values here ensures uniform convergence as well.

# 3 Comparing Information Structures

Optimizing over the space of information structures is not an easy task. Given an information structure one needs to solve for an equilibrium and typically equilibra do not have a closed form solution.[34] Instead, we will consider three specific information structures and compare them with respect to ex-ante total output and welfare. The information structures are: $(i)$ Full information disclosure, $(ii)$ Centralized information, and $(iii)$ Decentralized information.

## 3.1 Total Output

As a benchmark, let us begin with the static version of the model. In the static model, full information revelation dominates both the centralized and decentralized information structures.[35] In the dynamic model, the answer is not so clear due to the *encouragement effect*. In the full information disclosure, ex-ante expected output is

$$\int_0^T e^{-rt}\left(4\mu_0^2 + 2g\right) dt.$$

In the full information case, we assumed agents play the strategy $\frac{\Theta}{2}$ $\forall t \in [0,T]$. This strategy is the limit of the unique subgame perfect equilibrium of discrete time approximations. However, it is not clear that uniqueness is preserved in the continuous-time version.[36] If we change the model as described in Remark II and restrict attention to strategies to lie in a compact set $[0,\Theta]$, uniqueness follows from Iijima and Kasahara (2015). Ex-ante output is calculated by plugging equilibrium strategies to the evolution of the beliefs. In the centralized information case ex-ante total output is

$$\int_0^T e^{-rt}\left[\left(4\mu_0^2 + 2g\right)\left(\alpha_t^* + \frac{1}{2}\right) - \frac{1}{2}\gamma_t^*\right] dt \tag{14}$$

The term $\frac{1}{2}\gamma_t^*$ represents the cost of uninformed effort choices by the follower.

[34] Observe that the space of possible information structures is quite rich. For instance, the designer could send signals to the agents, and different levels of correlation of signals would correspond to different information structures.

[35] The ex-ante total output is: $(\theta_1+\theta_2)(\theta_1+\theta_2) = 4\mu_0^2 + 2g$, $(\theta_1+\theta_2)\left(\frac{1}{2}(\theta_1+\theta_2)+\mu_0\right) = 4\mu_0^2 + g$, and $(\theta_1+\theta_2)\left(\frac{1}{2}(\theta_1+\theta_2)+\mu_0\right) = 4\mu_0^2 + g$, for full information disclosure, decentralized, and centralized information structures, respectively.

[36] It is an open question in Linear Quadratic games whether the linear equilibrium is unique or not.

In the decentralized information case ex-ante output is

$$\int_0^T e^{-rt}\left[\left(4\mu_0^2+2g\right)\left(\alpha_t+\beta_t\right)-\beta_t\gamma_t\right]dt \tag{15}$$

The term $\beta_t\gamma_t$ represents the cost of uninformed effort choices by the follower. It is reasonable to expect that total output should be always higher in the decentralized information structure compared to centralized information structure since the *encouragement effect* is mutual in the former and in the latter it is one-sided. However, one immediate implication of Proposition 3 is that around the deadline ($T$) expected ex-ante output is higher in the centralized information model compared to decentralized model. The next Proposition shows that it is also possible that early in the relationship centralized information dominates decentralized information in terms of output.

**Proposition 5** *If $r=0$, then $\exists T^*$ such that $\forall T > T^*$ and for every such $T$ $\exists t^*$ such that $\forall t \in [0, t^*]$ ex-ante expected flow output in centralized information is higher than both decentralized and full information disclosure.*

This result and the above observation suggests that centralized information might dominate decentralized information at every point in time during the relationship. Numerical analysis shows that this is not true: depending on parameters, decentralized information structure can dominate centralized information structure and vice versa (in terms of total output).

## 3.2 Welfare

In this section, we compare information structures with respect to ex-ante total welfare. Similar to the total output case, as a benchmark start with the static version of the model. In the static model, not surprisingly full information disclosure dominates both the decentralized and centralized information structure.[37]

Maintaining the assumption of linear strategies, full information revelation gives the following ex-ante expected welfare

$$\frac{3}{4}\int_0^T e^{-rt}(4\mu_0^2+2g)dt.$$

[37]The ex-ante welfare is $(4\mu_0^2+2g)-\frac{1}{4}(4\mu_0^2+2g) = \frac{3}{4}\left(4\mu_0^2+2g\right)$, $(4\mu_0^2+g)-\frac{1}{4}\left(4\mu_0^2+g\right) = \frac{3}{4}\left(4\mu_0^2+g\right)$, and $(4\mu_0^2+g)-\frac{1}{4}\left(4\mu_0^2+g\right) = \frac{3}{4}\left(4\mu_0^2+g\right)$, for full information disclosure, decentralized, and centralized information structures, respectively.

By plugging equilibrium strategies to belief evaluations ex-ante welfare is calculated. In centralized information structure ex-ante welfare is

$$\int_0^T e^{-rt}\left(\left(4\mu_0^2+2g\right)\left(\alpha_t^*\left(1-\frac{1}{2}\alpha_t^*\right)+\frac{3}{8}\right)-\frac{3}{8}\gamma_t^*\right)dt. \tag{16}$$

Again, the term $\frac{3}{8}\gamma_t^*$ captures the uninformed effort choices of the followers.

In decentralized information structure ex-ante welfare is

$$\int_0^t e^{-rt}\left(\left(4\mu_0^2+2g\right)\left(\alpha_t+\beta_t(1-\frac{\alpha_t}{2})\right)-\beta_t\gamma_t(1-\frac{\alpha_t}{2})-\frac{\beta_t^2}{4}\left(4\mu_0^2+2g-\gamma_t\right)-\alpha_t^2(\mu_0^2+g)\right)dt. \tag{17}$$

In the case of welfare, decentralized information dominates full information and centralized information in terms of total welfare.

**Proposition 6** *For $r=0$ and given $\left(\mu,g_0,\sigma\right)$ as project length increases $(T\to\infty)$ the decentralized information structure gives the highest welfare among the three information structures considered.* [38]

It turns out that there is a stronger justification for decentralized information structure than stated above: *it approximates the first-best in the limit.* In the decentralized information model, ex-post efficient outcome is $\gamma_t=0$ and $\beta_t=2,\forall t\in[0,T]$.

**Proposition 7** *If $r=0$ and $T\to\infty$, decentralized information structure approximates ex-post efficient outcome. More precisely,*

1. *$\gamma_t$ stays close to 0 almost all the time. For every $\forall\epsilon>0$, and let $t_\gamma^\epsilon:=\inf\{t\in[0,T]\mid\gamma_t<\epsilon\}$, $\lim_{T\to\infty}\frac{T-t_\gamma^\epsilon}{T}\to 1$.*
2. *$\beta_t$ stays close to 2 almost all the time. For every $\forall\epsilon>0$, and let $t_\beta^\epsilon:=\sup\{t\in[0,T]\mid\beta_t>2-\epsilon\}$, $\lim_{T\to\infty}\frac{t_\beta^\epsilon}{T}\to 1$.*

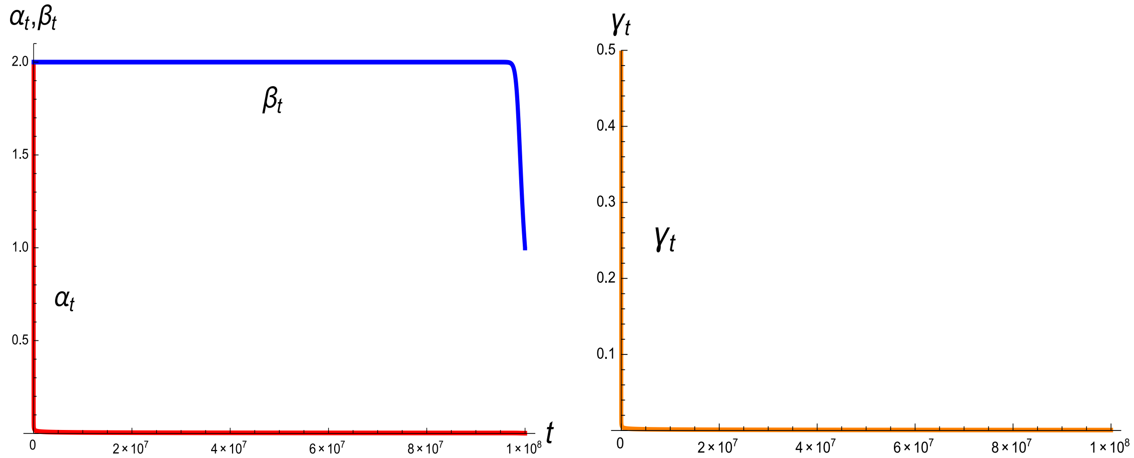
Figure 5 demonstrates the equilibrium behavior in the case of no-discounting and arbitrarily large $T$.[39] Here's the intuition. As $T\to\infty$, approximate learning happens

[38] The limit $T\to 0$ is easier to analyze, since in the static game full information revelation dominates every information structure. This result carries overs if $T$ is sufficiently small.

[39] Alternatively one can focus on the undiscounted game, with payoffs

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^{(a_1,a_2)}\left[\int_0^T\left(\frac{1}{2}\Theta_t\sum_i a_{i,s}-\frac{1}{2}a_{i,s}^2\right)ds\right]$$

and the same analysis carries over.




(a) Equilibrium coefficients $\alpha_t, \beta_t$ (b) Posterior variance $\gamma_t$

Figure 5: Dynamics of the equilibrium coefficients $(g = 0.25, T = 10^8, \sigma = 1)$

relatively fast compared to the length of the project. It turns out that if the discount $r$ is low enough, the *encouragement effect* only vanishes around the deadline, but as deadline goes to infinity this happens only at infinity. Therefore, perhaps surprisingly in the case of patient players and long horizon, there is a simple remedy to free-riding problem: decentralized information solves it.

Here is a heuristic argument for the efficiency result. Focus on the following approximate problem, in which almost perfect learning has happened. Meaning, $\gamma_t$ is arbitrarily low but non zero. We can assume $\gamma_t$ becomes flat, since $\dot{\gamma}_t \approx 0$. This observation suggests that actions are only a function $\mu_t$, alternatively $\alpha_t \approx 0$. Consider the deviation, $a_t^* = a_t + \epsilon$ over $[t, t + dt)$, for $\epsilon$ small. Intuitively, the deviation leads to an immediate response of public belief of size $\lambda_t \epsilon dt$. After the deviation beliefs of agents diverge. Let $\Delta$ denotes the difference between the deviating agent's belief and the other agent. Assume deviating agent continues to deviate until time $T$ (equivalently uses the strategy $a_t = \beta_t \hat{\mu}_t$). Then the dynamics of $\Delta$ can be written as,

$$d\Delta_t = \lambda_t \left( a_t \left( \hat{\mu}_t \right) - a_t(\mu_t) \right) dt.$$

Given our focus on linear strategies ,$\mu_t \beta_t$, we can rewrite the above as

$$d\Delta_t = -\lambda_t \beta_t \Delta_t dt.$$

Note that $\lambda_t$ has two effects: *i*) it measures the size of the initial impulse in the belief $\mu_t$ and *ii*) it determines the how fast the belief divergence decays. In equilibrium, agent must be indifferent between creating belief divergence or not.

$$\beta_t \mu_t = \mu_t \left( 1 + \lambda_t \int_0^T \beta_t e^{-\int_0^t \lambda_s \beta_s ds} dt \right)$$

First part in the right hand side is the flow benefit, second part of the right hand side captures benefit coming from future periods. In this approximate stationary environment ($\beta, \gamma, \lambda$ constant), deviation causes public belief to increase as $\lambda$ and this difference decays at the rate $e^{-t\lambda\beta}$. Therefore, $\beta_t e^{-t\lambda\beta}$ captures the future gain at time $t$. Observe that as $T \to \infty$, $\lambda \int_0^T \beta e^{-t\lambda\beta} dt \to 1$. This implies $\beta$ becomes 2 which is the ex-post efficient outcome.

# 4 Conclusion

This paper analyzed a stylized dynamic contribution game with asymmetric information and showed how different information structures change the equilibrium behavior. For tractability purposes we focused on linear Markov equilibrium and showed it exists and is unique. We identified a simple information structure – decentralized information – which approximates the first best in terms of efficiency.

There are several possible ways in which the analysis can be extended. In this section, we discuss some of the possible extensions.

**Evolving States** Throughout the paper, we focused on the case in which private information is one dimensional and does not change over time. It is natural to think that agents' private information changes over time. It is possible to extend the analysis for specific forms evolution, for instance if states evolve as

$$d\theta_t^i = \left( k + \alpha_t \theta_t^i \right) dt,$$

then a linear equilibrium is again characterized by system of non-linear differential equations, but this time it need not exist for all parameter values.[40]

**Risk Aversion** One possible extension is to allow agents to be risk-averse. Specific forms of risk aversion still keep the problem tractable. For instance, under CARA utilities, the linear-quadratic structure is preserved. Risk aversion brings an additional effect: as time passes, agents have more precise information about the true state, which pushes

[40] See Appendix A.5 for details

the agents to increase their actions. Due this effect, $\beta$ might be increasing near the deadline. Moreover, qualitative results such as the *encouragement effect* carry over to such environment.

**Experimentation** In this paper we focus on the effect of learning and signaling on free-riding. It is also interesting to explore the implications of experimentation on free-riding. Define the flow pay-off as

$$dP_t = \Theta(a_{1,t} + a_{2,t}) + \sigma dZ_t - \frac{1}{2}a_{1,t}^2$$

This model is similar to Bolton and Harris (1999) with the difference that actions are imperfectly monitored. In this model the posterior variance is not a deterministic function of time anymore, also the value function is no longer quadratic.

# 5 Appendix

## 5.1 Evolution of Beliefs

**Outside Observer's Belief**

Let $s_t := \mathbb{E}[\Theta|\mathcal{F}_t]$ be the posterior mean and $\gamma_t := \mathbb{E}[(\Theta - s_t)^2 |\mathcal{F}_t^Y]$ be the posterior variance of the outside observer. $s_t$ and $\gamma_t^o$ are the unique solution to following system of equations (By Theorem 12.1 of Liptser and Shiryaev (2013))

$$\begin{aligned} ds_t &= \frac{\alpha_t \gamma_t}{\sigma^2}\left(dY_t - s_t\beta_t - 2\int_0^t f_s^t dY_s\right) \quad s_0 = 2\mu_0, \\ \dot{\gamma}_t^o &= -\left(\frac{\alpha_t\gamma_t}{\sigma}\right)^2 \quad \gamma_0^o = 2g. \end{aligned}$$

Since $ds_t = 2d\mu_t$, then $d\mu_t$ evolves as

$$d\mu_t = \frac{\alpha_t\gamma_t}{2\sigma^2}\left(dY_t - 2\alpha_t\mu_t - 2\int_0^t f_s^t dY_s\right).$$

Let $\gamma_t^\mu$ denotes the posterior variance of $\frac{1}{2}\Theta$, therefore $\gamma_t^\mu = \frac{1}{4}\gamma_t^o$.

**Agent $i's$ Private Belief**

From the perspective of agent $i$ under the assumption of linear strategies, relevant signal is

$$dY_t^i = \alpha_t\theta^j + \sigma dZ_t.$$

Therefore, by Theorem 12.1 of Liptser and Shiryaev (2013) agent $i's$ first order belief and posterior variance evolves as

$$\begin{aligned} dm_t^{i,j} &= \frac{\alpha_t\gamma_t}{\sigma^2}\left(dY_t^i - \alpha_t m_t^{i,j} dt\right) \quad m_0^{i,j} = \mu_0, \\ \dot{\gamma}_t &= -\left(\frac{\alpha_t\gamma_t}{\sigma}\right)^2 \quad \gamma_0 = g. \end{aligned}$$

Moreover,

$$dZ_t^i = \frac{dY^i - \alpha_t m^{i,j}}{\sigma}$$

is a Brownian motion with respect to $\mathcal{F}_t^i$. Following Lemma allows for deterministic evolutions of the $\theta$, in the form $d\theta_{i,t} = \omega\theta_t$.

**Lemma 5** *If each agent uses symmetric linear strategies* $\left(\alpha_t^i = \alpha_t^j, \beta_t^i = \beta_t^j \ \forall t \in [0,T]\right)$*, then*

$$m_t^i = (1 - z_t)\theta_{i,t} + z_t \mu_t,$$

*where* $z_t = 2\frac{\gamma_t}{\gamma_t^o}$.

**Proof** Public variance is

$$\gamma_t^O = \frac{e^{2\omega t} 2 g_0}{1 + 2g_0 \int_0^t \left(\frac{e^{\omega s}\alpha_s}{\sigma}\right)^2 ds}, \ \ t \in [0,T].$$

Private variance is

$$\gamma_t = \frac{e^{2\omega t} g_0}{1 + g_0 \int_0^t \left(\frac{e^{\omega s}\alpha_s}{\sigma}\right)^2 ds}, \ \ t \in [0,T].$$

Note that private variance, could increase or decrease over time depending on the parameter $\omega$ and $\alpha$. If $\omega$, is zero it can only decrease. Evolution of the posterior mean is

$$d\mu_t = \omega \mu_t dt + \frac{\gamma_t \alpha_t}{\sigma^2}[dY_t - \alpha_t \mu_t dt].$$

Posterior public mean becomes

$$\mu_t = e^{\omega t} \left( \frac{\mu_0 + g_0 \int_0^t e^{\omega s} \left(\frac{\alpha_s \left(dY_s - \theta^i\right)}{\sigma^2}\right) ds}{1 + g_0 \int_0^t \left(\frac{e^{\omega s}\alpha_s}{\sigma}\right)^2 ds} \right).$$

Note that

$$\frac{e^{\omega t}\theta_t g_0 \int_0^t e^{\omega(s-t)} e^{\omega s} \frac{\alpha_s^2}{\sigma^2} ds}{1 + g_0 \int_0^t \left(\frac{e^{\omega s}\alpha_s}{\sigma}\right)^2 ds} = \theta_t(1 - z_t).$$

Therefore,

$$m_t^{i,j} = z_t \mu_t + (1 - z_t)\theta_t^i.$$

**Lemma 6**

**Proof** Define $\hat{\mu}_t^i$ as follows,

$$\hat{\mu}_t^i := \frac{m_t^{i,j} + (z_t - 1)\,\theta}{z_t}.$$

Recall that $z_t = \frac{1+2_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}{1+g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}$, and $\frac{m_t^{i,j}}{z_t} = \frac{\mu_0+g_0\int_0^t \frac{\alpha_s}{\sigma}dY_s^i}{1+g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}$. After plugging the $z_t$ in to above equation, we reach

$$\frac{\mu_0 + g_0\int_0^t \frac{\alpha_s}{\sigma}dY_s^i}{1+2g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds} + \theta_i = \hat{\mu}_t^i + \theta_i \frac{1+g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}{1+2g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}.$$

After simplification

$$\hat{\mu}_t^i = \frac{\mu_0 + g_0\int_0^t \frac{\alpha_s}{\sigma^2}\left(\sigma dY_s^i + \alpha_s\theta_i ds\right)}{1+2g_0\int_0^t \left(\frac{\alpha_s}{\sigma}\right)^2 ds}.$$

Take the time derivative of the both sides and rearrange to reach

$$d\hat{\mu}_t^i = \lambda_t\left(\alpha_t\left(\theta - \hat{\mu}_t\right) + \alpha_t(m_t^{i,j} - \hat{\mu}_t)\right) + \lambda_t\sigma dZ_t^i \quad \hat{\mu}_0 = \mu.$$

Below Lemma is derived by $BCT$, here I state it for completeness.

**Lemma 7** *A symmetric linear strategy profile is Markov if and only if there exists functions $(\alpha, \beta)$ such that*

$$a_{i,t} = \alpha_t\theta_i + \beta\mu_t + \delta_t.$$

**Proof** If all agents uses a symmetric linear strategy profile, then there is a one to one mapping between $(\theta_i, \mu_t, t)$ and firm $i's$ belief about $(\theta_1, \theta_2)$ and calendar time. Therefore when $a_{i,t} = \alpha_t\theta_i + \beta_t\mu_t + \delta_t, t \in [0, T]$ agent $i's$ effort is only a function of its belief and calendar time. Recall that from the belief evolution $\mu_t = \mu_0 h_t + \int_0^t k_s^t dY_s$. Therefore, form agrees with the linear strategy.

For the other direction, suppose symmetric linear strategy profile is only a function of beliefs and calendar time. Then again by Lemma 1 and Lemma 2, for each agent $i$ and for all $t$,

$$a_{i,t} = \psi_t(\theta_i, \mu_t)$$

for some function $\psi_t : \mathbb{R}^2 \to \mathbb{R}$.. Note that suppose $\alpha_t = 0$ for some $t$ public belief does not change. Suppose $\tau$ denotes the $\inf\{t|\alpha_t = 0\}$. Then $\forall t, 0 \leq t \leq \tau$ we have $m_t^{i,j} = \mu_0$. Then for each $t$,

$$a_{i,t} = \psi_t(\theta_i, \mu_t) = \psi_t\left(\theta_i, \mu_0 h_t + \int_0^t k_s^t dY_s\right) = \alpha_t\theta_i + \int_0 f_s^t dY_s + \delta_t$$

in order for this equality to hold $\forall t$, $\psi_t$ must be an affine function.

**Theorem 3** *Symmetric linear Markov equilibrium exists and unique.*

**Proof** Proof has two parts $i)$ showing that any solution to described system is a linear Markov equilibrium , $ii)$ any linear Markov equilibrium solves the above system. We start the proof with the following observation.

**Lemma 8** *In any symmetric linear Markov equilibrium,* $\frac{\alpha_t}{\beta_t} = \frac{\alpha_t^m}{\beta_t^m}$ *(equivalently* $\tilde{\alpha}_t = \tilde{\beta}_t$*).*

**Proof** By boundary conditions at time $T$, $\alpha_T^m = \alpha_T$ and $\beta_T = \beta_T^m$. Therefore, it is enough to show

$$z_t\left(\dot{\alpha}_t + \dot{\beta}_t\right) + \dot{z}_t(\alpha_t + \beta_t) = 2\dot{\beta}_t,\ \forall t \in [0, T).$$

Using $\frac{\alpha_t}{\alpha_t^m} = \frac{\beta_t}{\beta_t^m}$, we reach

$$\dot{\alpha}_t = \dot{\beta}_t\left(\frac{2 - z_t}{z_t}\right) - \alpha_t\frac{\dot{z}_t}{z_t}\left(\frac{2}{2 - z_t}\right).$$

Last term in the right equals to $-\alpha_t\frac{\dot{z}_t}{z_t}\left(\frac{2}{2-z_t}\right) = -\frac{\alpha_t^3\gamma_t}{\sigma^2} = \lambda_t\beta_t\alpha_t\left(2\frac{\alpha_t}{\beta_t}\right)$ and rewriting $\dot{\beta}$ as follows

$$\dot{\beta}_t = r\frac{\alpha_t}{\alpha^m}(\beta_t - \beta^m) + \lambda_t\beta_t\alpha_t\left(\frac{(\beta_t - \beta^m)}{\alpha_m} + 2(1 - \beta_t) - \frac{\beta_t}{\alpha_t}\right).$$

This leads us showing that if below equations hold, desired result is satisfied.

$$\begin{aligned} r\frac{\alpha_t}{\alpha^m}(\alpha_t - \alpha^m) &= r\frac{\alpha_t}{\alpha^m}(\beta_t - \beta^m)\left(\frac{\alpha_t^m}{\beta_t^m}\right) \\ \left(\frac{\alpha_t}{\alpha^m} - 2\left(1 + \alpha_t\right)\right) &= \left(\frac{(\beta_t - \beta^m)}{\alpha_t^m} + 2(1 - \beta_t) - \frac{\beta_t}{\alpha_t}\right)\left(\frac{\alpha_t^m}{\beta_t^m}\right) - 2\frac{\alpha_t}{\beta_t} \end{aligned}$$

Since $\beta_t\frac{\alpha_t^m}{\beta_t^m} = \alpha_t$ by assumption, we get the first equality. For the second equation, rearrange the right side of the equation to get

$$\left(\frac{\alpha_t}{\alpha_m} - 2\beta_t\frac{\alpha_t^m}{\beta_t^m} - 2\right).$$

This completes the proof, since $\beta_t\frac{\alpha_t^m}{\beta_t^m} = \alpha_t)$ which implies $-2\beta_t\frac{\alpha_t^m}{\beta_t^m} = -2\alpha_t$.

**Lemma 9** *If* $(\alpha, \beta)$ *is symmetric linear Markov equilibrium with posterior variance* $\gamma$*, then* $(\alpha, \beta, \gamma)$ *is a solution to Boundary Value problem.*

**Proof** Fix an equilibrium pair of strategies $(\alpha, \beta)$ with associated variance $\gamma$. Observe that agent $i's$ best response problem is stochastic-linear regulator. Proof closely follows the Chapter 6.4 of Yong and Zhou (1999).[41] Given the other agent uses linear strategies, write the best response problem with state $(\theta_i, \mu_t, \hat{\mu}_t, t)$ as a optimization problem in Hilbert space, where the control, effort choices, is an square integrable process $a_{i,t}$ on $[t, T]$. From here on $X_t$ refers to state at time $t$. Objective function has the following form:

$$\mathbb{E}^a \left[ \frac{1}{2} \int_t^T \langle Q_t X_t, X_t \rangle + 2 \langle S_t X(t), a_{i,t} \rangle + \langle R_t a_{i,t}, a_{i,t} \rangle \right]$$

where $Q, R, S$ are the matrices of appropriate size and $X$ evolve as in the best response problem stated in Section 2.2. Then by Proposition 4.1 of Chapter 6 in Yong and Zhou (1999) objective has the equivalent representation:

$$\frac{1}{2} \langle N_t a_i, a_i \rangle + \langle H_t(X_t), a \rangle + \langle M_t(X_t) \rangle,$$

where $N, H, M$ are some linear functionals, see Equation 4.18 (page 308) of Yong and Zhou (1999) for their form. Since we are at an equilibrium, value of the problem at any $X_t$ must be finite. Therefore, by Theorem 4.2 in Chapter 6 of Yong and Zhou (1999), $N_t \leq 0$. In the agents flow pay-off the term in front of the $a_t^2$ is $-\frac{1}{2}$, therefore is invertible. Then we can invoke Corollary 5.6 in Chapter 6 of Yong and Zhou (1999) which states linear optimal policy is exists if and only if forward-backward stochastic differential equation (FBSDE) defined as (5.17) in Chapter 6 of Yong and Zhou (1999) has a unique solution. In this case existence of unique solution to corresponding FBSDE guaranteed by Yong (2006). Observe that $V$ is twice continuously differentiable in $(\theta, \mu, \hat{\mu})$ and continuously differentiable in $t$. Each $v_i$ are differentiable since $(\alpha, \beta)$ are continuously differentiable by assumption. Then we conclude that $V$ solves the HJB equation. This implies a linear optimal policy $a = \alpha_t \theta + \beta_t \hat{\mu}_t$, where $\alpha_t$ and $\beta_t$ are the equilibrium conditions, which satisfies the first order condition. Therefore,

$$\alpha_t \theta_t + \beta_t \hat{\mu}_t = \left( \theta \left( 2 - z_t \right) + z_t \hat{\mu}_t \right) + \lambda_t \left( v_4(t) \hat{\mu} + v_5(t) \theta_i \right)$$

[41] See also Lemma A.2 of Bonatti, Cisternas and Toikka (2017).

is derived by using the fact that $V_\mu = v_4(t)\hat{\mu} + v_5(t)\theta_i$. By matching we coefficients reach

$$\begin{aligned}\alpha_t &= \frac{1}{2}(2-z_t) + \frac{\alpha_t\gamma_t}{2\sigma^2}v_5(t)\\ \beta_t &= \frac{1}{2}z_t + \frac{\alpha_t\gamma_t}{2\sigma^2}v_4(t).\end{aligned}$$

Recall that $v_k(T) = 0, k = 0, \ldots, 5$, then $\alpha_T = \frac{1}{2}(2-z_t)$ and $\beta_T = \frac{z_t}{2}$. Therefore, boundary conditions are satisfied since $z_T$ is only a function of $\gamma_T$. Boundary condition for $\gamma$ is satisfied since $\gamma_t = \frac{2g_0}{1+2g_0\int_0^t\left(\frac{\alpha_s}{\sigma}\right)^2 ds}$.

By application of envelope theorem to HJB equation,

$$rV_\mu = \frac{1}{2}\beta_t\left(\theta_i + m_t^{i,j}\right) + d(\mu)_\mu V_\mu + V_{\mu\hat{\mu}}d(\hat{\mu}) + V_{t\mu}$$

Then rewrite as above equation as

$$-v_4(t)\,\lambda_t\alpha_t\,(2-z_t)\left(\theta^i - \hat{\mu}_t\right) + V_\mu\left(r + \lambda_t\left(2\alpha_t + \beta_t\right)\right) = \frac{\beta_t\left(\theta^i + m_t^i\right)}{2} + \dot{v}_4(t)\hat{\mu} + \dot{v}_5(t)\theta^i,$$

using the fact that $m_t^{i,j} = (1-z_t)\theta + z_t\hat{\mu}_t$ and matching coefficients

$$v_4(t)\left(\alpha_t\lambda_t(2-z_t) + r + \lambda_t(2\alpha_t+\beta_t)\right) = \dot{v}_4(t) + \frac{1}{2}\beta_t z_t.$$

and

$$v_4(t)\alpha_t\lambda_t(2-z_t) + v_5(t)\left(r + \lambda_t(2\alpha_t+\beta_t)\right) = \dot{v}_5(t) + \frac{1}{2}\beta_t(2-z_t)$$

After taking the time derivatives of the first-order conditions and rearranging the above equations, evolutions of $(\alpha, \beta)$can be written as:

$$\begin{aligned}\dot{\alpha}_t &= r\frac{\alpha_t}{\alpha_t^m}(\alpha_t - \alpha_t^m) + \lambda_t\beta_t\alpha_t\left(\frac{\alpha_t}{\alpha_t^m} - 2\,(1+\alpha_t)\right)\\ \dot{\beta}_t &= r\frac{\alpha_t}{\alpha_t^m}(\beta_t - \beta_t^m) + \lambda_t\beta_t\left(\frac{\alpha_t}{\alpha_t^m}\left(\beta_t - \beta_t^m\right) + 2\alpha_t(1-\beta_t) - \beta_t\right)\\ \dot{\gamma}_t &= -\left(\frac{\alpha_t\gamma_t}{\sigma}\right)^2\end{aligned}$$

**Existence of Solution**

Proof is going to use "shooting method". Treat the backwards system as IVP (initial value problem) and guess the value of public variance at the boundary ($\gamma_T$) and solve the system backwards until the $\gamma_0$ hits the initial variance $2g_0$. System is locally Lipschitz continuous.[42] If $\gamma_T = 0$, unique solution is given by $\gamma_t = 0,\ \alpha_t = 0,\ \beta_t = 1, \forall t \in [0, T]$. By continuity, IVP has a maximal domain of solution $E := [0, \gamma_E)$. Another implication of continuity is that if $\exists \gamma_T$ such that $\gamma_0(\gamma_T) > 2g$, then $\exists \gamma_T^*$ such that $\gamma_0(\gamma_T^*) = 2g$ which is an implication of Intermediate Value theorem. Let $\bar{\gamma}_E = \sup E$ and assume $\gamma_0(\bar{\gamma}_E) < 2g_0, \forall \gamma_g \in E$. Monotonicity of $\gamma$, implies that $\bar{\gamma}_E \leq 2g_0$. Goal is to show that if the assumption is true ($\gamma_0(\bar{\gamma}_E) < 2g_0, \forall \gamma_g \in E$.), it will contradict the fact that $E$ is the maximal domain of solution. We are going to show, $(\alpha, \beta, \gamma)$ are bounded uniformly in $\gamma_E$ on $E$. Condition $\frac{\alpha_t}{\beta_t} = \frac{\alpha_t^m}{\beta_t^m}$ and our assumption implies that $\beta_t \geq \alpha_t$, $\forall t \in [0, T]$, since $\forall t$ and any guess $\forall \gamma_I \in E$ $\frac{\gamma_t}{2g_0} \leq 1$. Therefore it is enough to bound $\beta_t$. Rewrite $\dot{\beta}_t$ as follows

$$\dot{\beta}_t = r\frac{\beta_t}{\beta_t^m}(\beta_t - \beta_t^m) + \lambda_t \beta_t \left(\beta_t^2 - 2\beta_t\right)\left(1 - \frac{\alpha_t^m}{\beta_t^m}\right)$$

Observe that, $\beta_t$ is bounded by 2. Note that $\beta_T \in [\frac{1}{2}, 1]$ also $\beta_t^m$ is increasing (in the forward system) function and $\beta_t^m \in [\frac{1}{2}, 1]$ $\forall t \in [0, T]$. Moreover, $\beta_t^m > \alpha_t^m$ $\forall t \in (0, T]$ and $\alpha_0^m = \beta_0^m$. These facts implies, $\beta_t$ can touch 2 (in the backwards system) only from below. Suppose there exists such $t^*$ that $\beta_t$ touches 2. At such $t^*$, the term $r\frac{\beta_t}{\beta_t^m}(\beta_t - \beta_t^m) > 0$ and the the second term is equal to 0. Therefore, $\beta$ has to decrease at $t^*$ which implies $\beta$ can not cross 2. Hence it is uniformly bounded by 2. Therefore, $(\alpha, \beta, \gamma)$ are uniformly bounded in $\gamma_E$ on $E$. This contradicts the fact that $E$ is the maximal domain of solution which implies $\exists t^*$ such that at least one of the functions diverges if system starts with the guess $\gamma_E$.

**Uniqueness**

Suppose above system has more than one solution. Choose two arbitrary solutions and denote $\left(\alpha^1, \beta^1, \gamma^1\right)$ as the first solution and $\left(\alpha^2, \beta^2, \gamma^2\right)$ as the second solution. Note that by Picard-Lindelof theorem given the value $\gamma_T$, corresponding paths for $\alpha, \beta$ and $\gamma$ gamma are unique. Picard-Lindelof does not imply that value of $\gamma_T$ has to be unique. Without loss of generality assume $\gamma_T^1 > \gamma_T^2$. By boundary condition at time 0 it must be the case $\gamma_0^1 = \gamma_0^2 = 2g_0$. This implies $\int_0^t \left(\frac{\alpha_s^1}{\sigma}\right)^2 ds < \int_0^t \left(\frac{\alpha_s^2}{\sigma}\right)^2 ds$, which follows from evolution of $\gamma$. Notice that at the boundary $(T)$, $\alpha_T^1 > \alpha_T^2$. This follows from the fact that at the boundary $\alpha_T^1 = \alpha_T^{1,m}, \alpha_T^2 = \alpha_T^{2,m}$. In order for the integral condition to hold, in the backwards system there must be a point $t^*$ such that $\alpha^2$ crosses $\alpha^1$ and stays

[42]Notice that $\frac{\alpha_t}{\alpha_t^m}$ is equal to $\frac{\beta_t}{\beta_t^m}$ and $\frac{\beta_t}{\beta_t^m}$ is bounded by 4. This follows from the fact that $\beta_t$ is bounded by 2.

above $\alpha^1$ for a positive measure of time. Boundary condition and continuity of $\alpha$ implies $\forall t \in [t^*, T]$ $\alpha_t^1 > \alpha_t^2$. Evaluating the $\dot{\alpha}_{t^*}^1, \dot{\alpha}_{t^*}^2$ and rearranging (use the fact $\frac{\alpha_t}{\beta_t} = \frac{\alpha_t^m}{\beta_t^m}$).

$$\begin{aligned}
\dot{\alpha}_{t^*}^1 &= r\left(\frac{\alpha_{t^*}^1}{\alpha_{t^*}^{1,m}}\right)\left(\alpha_{t^*}^1 - \alpha_{t^*}^{1,m}\right) + \frac{\left(\alpha_{t^*}^1\right)^3 \gamma_{t^*}^1 \beta_{t^*}^{1,m}}{2\sigma^2 \alpha_{t^*}^{1,m}}\left(\frac{\alpha_{t^*}^1}{\alpha_{t^*}^{1,m}} - 2\left(1 + \alpha_{t^*}^1\right)\right) \\
&= r\left(\frac{\alpha_{t^*}^1}{\alpha_{t^*}^{1,m}}\right)\left(\alpha_{t^*}^1 - \alpha_{t^*}^{1,m}\right) + \frac{\left(\alpha_{t^*}^1\right)^3 g_0}{2\sigma^2}\left(\frac{2g_0}{\gamma_{t^*}^1}\alpha_{t^*}^1 - 2 - \alpha_{t^*}^1\right)
\end{aligned}$$

At $t^*$ by assumption following conditions are satisfied $\alpha_{t^*}^1 = \alpha_{t^*}^2$, $\alpha_{t^*}^{1,m} > \alpha_{t^*}^{2,m}$ and $\gamma_{t^*}^1 > \gamma_{t^*}^2$. Combining these facts with above equation implies that $-\dot{\alpha}_{t^*}^1 > -\dot{\alpha}_{t^*}^2$. Therefore $\alpha^2$ can not cross $\alpha^1$ at any time in $[0, T]$. Therefore the integral equation can not be satisfied, contradiction.

**Lemma 10** *If $(\alpha, \beta, \gamma)$ solves the boundary value problem, then $(\alpha, \beta)$ is a symmetric linear Markov equilibrium with posterior variance $\gamma$.*

**Proof** $(\alpha, \beta, \gamma)$ are bounded away from the zero, since $2 \geq \beta \geq \alpha$ and $T$ is finite. This implies $\gamma_T > 0$, therefore $\alpha_t > 0$, $\forall t \in [0, T]$. First order conditions pin down $v_4$ and $v_5$, then plugging the optimal policy to the HJB equation and matching coefficients $v_0, v_1, v_2, v_3$ are pinned down.

$$\begin{aligned}
\dot{v}_0(t) &= rv_0(t) - (\lambda_t\sigma)^2\left(v_3(t) + v_4(t)\right) \\
\dot{v}_1(t) &= rv_1(t) + \frac{1}{2}\alpha_t^2 - (\alpha_t^m)^2\alpha_t - 2\alpha_t\alpha_t^m\lambda_t\left(v_2(t) + v_5(t)\right) \\
\dot{v}_2(t) &= v_2(t)(r + \lambda_t\alpha_t 2\alpha_t^m) - \alpha_t^m\beta_t - \beta_t^m\alpha_t - 2\alpha_t\alpha_t^m\beta_t^m - v_5(t)\lambda_t(2\alpha_t\beta_t^m + \beta_t) \\
&- 2\lambda_t\alpha_t\alpha_t^m\left(v_3(t) + v_4(t)\right) \\
\dot{v}_3(t) &= v_3(t)\left(r + 4\lambda_t\alpha_t\alpha_t^m\right) - \beta_t^m\beta_t - 2\alpha_t(\beta_t^m)^2 + \frac{1}{2}\beta_t^2 - v_4(t)\lambda_t\left(2\alpha_t\beta_t^m + \beta_t\right) \\
\dot{v}_4(t) &= v_4(t)\left[r + \lambda_t\left(2\alpha_t + \beta_t\right) + \lambda_t\alpha_t 2\alpha_t^m\right] - \beta_t\beta_t^m \\
\dot{v}_5(t) &= v_5(t)\left[r + \lambda_t\left(2\alpha_t + \beta_t\right)\right] - \beta_t\alpha_t^m - v_4(t)\left[\lambda_t\alpha_t 2\alpha_t^m\right]
\end{aligned}$$

with the boundary conditions $v_0(T) = 0$, $v_1(T) = 0$, $v_2(T) = 0$, $v_3(T) = 0$, $v_4(T) = 0$, $v_5(T) = 0$. Given $(\alpha, \beta)$ this is a linear system therefore it has a unique solution. Therefore, $V$ solves the HJB equation. $V$ is twice continuously differentiable in $(\theta, \mu, \hat{\mu})$ and continuously differentiable in t. Since the form is quadratic, it satisfies the polynomial growth condition. Then, by Theorem 3.1 (Fleming and Soner (2006) Chapter IV), conclude that the conjectured policy and the HJB equation solves the agents' problem.

**Proposition 8** *Properties of the linear equilibrium*

1. $2 \geq \alpha_t \geq \alpha_t^m > 0,\ 2 \geq \beta_t \geq \beta_t^m > 0 \ \ \forall t \in [0,T]$
2. $\beta_t > \alpha_t\ \forall t \in (0,T]$ *and* $\beta_t = \alpha_t$ *at* $t=0$
3. $\alpha_t$ *is decreasing*
4. *If* $r>0$, $\beta_t$ *is increasing at time 0 and decreasing around* $T$.
5. *If* $r=0$, *from the perspective of outsider total effort is decreasing over time*
6. $\lambda_t\beta_t\alpha_t$ *(volatility) is decreasing over time*

**Proof** If at any point $t \in [0,T]$ $\alpha_t$ becomes zero then $\dot{\alpha}_t = 0$ for all $t \in [t,T]$. This observation combined with the fact $\alpha_T \geq 0$ implies $\alpha_t \geq 0\ \forall t \in [0,T]$. For $\beta_t \geq 0$, note that $\frac{\alpha_t}{\beta_t} = \frac{\alpha_t^m}{\beta_t^m}$ and $\alpha_t^m \geq 0, \beta_t^m \geq 0, \alpha_t \geq 0\ \forall t \in [0,T]$. Therefore, $\beta_t \geq 0\ \forall t \in [0,T]$.

Following result will be useful in the rest of the analysis, $\alpha_t + \beta_t \geq 1\ \ \forall t \in [0,T]$. In the boundary $\alpha_T + \beta_T = 1$ and $\dot{\alpha}_T + \dot{\beta}_T < 0$ by the boundary conditions. Focus on the backwards system and at the first time $t^*$ such that $\alpha_{t^*} + \beta_{t^*} = 1$. At $t^*$ we are going to show that $\dot{\alpha}_{t^*} + \dot{\beta}_{t^*} < 0$. Using equilibrium conditions,

$$\dot{\alpha}_t + \dot{\beta}_t = r\frac{\alpha_t}{\alpha_t^m}(\alpha_t + \beta_t - 1) + \lambda_t\beta_t\left((\alpha_t+\beta_t)\left(\frac{\alpha_t}{\alpha^m} - 2\alpha_t\right) - \frac{\alpha_t}{\alpha^m} + \alpha_t - \beta_t\right).$$

At $t^*, r\frac{\alpha_t}{\alpha_t^m}(\alpha_t+\beta_t-1) = 0$ and second term equals to $-\lambda_t\beta_t$, then $\dot{\alpha}_{t^*} + \dot{\beta}_{t^*} = -\lambda_{t^*}\beta_{t^*} < 0$. Therefore, $\alpha_t + \beta_t \geq 1\ \ \forall t \in [0,T]$.

At the boundary $\dot{\beta}_T^m > 0,\ \dot{\beta}_T < 0$ and $\beta_T^m = \beta_T$. Look at the first time (backwards) $\beta_t^m$ crosses $\beta_t$, at the intersection ($t^*$)

$$\dot{\beta}_{t^*} = \lambda_t\beta_t\left(2\alpha_t(1-\beta_t) - \beta_t\right).$$

Going to show that $2\alpha_t(1-\beta_t) - \beta_t \leq 0$. There are two cases: $i)$ $\alpha_t > \frac{1}{2}$ and $ii)$ $\alpha_t \leq \frac{1}{2}$. In case i), $\beta_t \geq \alpha_t \geq \frac{1}{2}$ implies $2\alpha_t(1-\beta_t) \leq \frac{1}{2}$, therefore $\dot{\beta}_{t^*} < 0$. In case ii), in any linear equilibrium $\alpha_t + \beta_t \geq 1$ therefore $\alpha_t \leq \frac{1}{2}$ implies $\beta_t \geq \frac{1}{2}$, therefore $\dot{\beta}_{t^*} < 0$. This shows that $\beta_t \geq \beta_t^m \ \forall t \in [0, T]$.

At the boundary, by inspection $-\dot{\alpha}_T > -\dot{\alpha}_T^m$. Then look at the first time $\alpha_t^m$ crosses $\alpha_t (\alpha_t = \alpha_t^m)$. Notice $\dot{\alpha}_t^m = -\frac{1}{2}\frac{\alpha^2 z_t \gamma_t \alpha_t^m}{2\sigma^2} = -\frac{\alpha^3 z_t \gamma_t}{2\sigma^2}$ and $\dot{\alpha}_t = \frac{\alpha_t^2 \beta_t \gamma_t}{2\sigma^2}(-1 - 2\alpha_t)$. There are two cases if $\beta_t^m > \beta_t$, but then $\alpha_t + \beta_t \geq 1$ this implies $\alpha_t > \alpha_t^m$ contradiction. Therefore at the point they cross $\beta_t^m \leq \beta_t$, then we have $-\dot{\alpha}_t > -\dot{\alpha}_t^m$. Therefore $\alpha_t \geq \dot{\alpha}_t^m, \forall t \in [0, T]$.
At the boundary, $\dot{\beta}_T < 0$. Any point besides the boundary, $\dot{\beta}_t$

$$
\begin{aligned}
&= \lambda_t \beta_t \left( \frac{\beta_t^2}{\beta_t^m} - 2\beta_t + 2\frac{\alpha_t^m}{\beta_t^m}\beta_t(1 - \beta_t) \right) \\
&= \lambda_t \beta_t \left( \beta_t^2 \left( 1 - \frac{\alpha_t^m}{\beta_t^m} \right) + 2\beta_t \left( \frac{\alpha_m^t}{\beta_t^m} - 1 \right) \right)
\end{aligned}
$$

sign of the equation is determined by $\beta_t^2 - 2\beta_t$, this is non positive since $\beta_t \leq 2$. Therefore $\beta_t$ is decreasing.

Follows from the boundary condition $\beta_T > \alpha_T$ and continuity. For $\dot{\beta}_T < 0$ note that $\alpha_T = 1 - \beta_T$ also $\beta_T = \beta_T^m \;\; 2(1 - \beta_T)^2 < \beta_T$. For $\alpha_T$ directly follows from the equation

For $\alpha_t$ decreasing, at the boundary $\dot{\alpha}_T < 0$. Again in the backwards equation, look at the first time $\dot{\alpha}_t = 0$, this implies

$$
r\frac{\alpha_{t^*}}{\alpha_{t^*}^m}\left(\alpha_{t^*} - \alpha_{t^*}^m\right) = \frac{\alpha_{t^*}^3 g_0}{\sigma^2}\left( \frac{\alpha_{t^*}}{\alpha_{t^*}^m} - 2\left(1 + \alpha_{t^*}\right) \right)
$$

take time derivative of the both sides and evaluate at $t^*$. Observe that derivative of the left side will be equal to $-r\frac{\alpha_{t^*}^2 \dot{\alpha}_{t^*}^m}{\left(\alpha_{t^*}^m\right)^2} > 0$ and derivative of the right hand side is $\frac{\alpha_{t^*}^3 g_0}{\sigma^2}\left( \frac{\alpha_{t^*}\dot{\alpha}_{t^*}^m}{\left(\alpha_{t^*}^m\right)^2} - 2 \right) < 0$. Therefore, $\alpha_t$ is decreasing.
At $t = 0, \alpha_0 = \beta_0, \; \alpha_0^m = \beta_0^m = \frac{1}{2}$, $\dot{\beta}_t$ becomes

$$
2r\beta_0(\beta_0 - \frac{1}{2}) + \lambda_0\beta_0\left[2\beta_0\left(\beta_t - \beta^m\right) + 2\beta_0 - 2\beta_0^2 - \beta_0\right]
$$

notice second term is equal to zero and $\beta_0 - \frac{1}{2} > 0$.

For the volatility part, volatility of total effort is given by $2\beta_t\lambda_t\sigma$, which is equivalent to $\frac{2g}{\sigma}\alpha_t^2$. Therefore it is decreasing over time, since $\alpha_t$ is decreasing over time.

## 5.2 Proofs for Centralized information

These section contains the proofs for the centralized information case. One sided incomplete information makes the analysis easier compared to the two sided case. HJB Equation for the leader can be written as:

$$rV(\theta,\mu,t) = \frac{1}{2}\theta\left(a + \frac{\mu_t}{2}\right) - \frac{1}{2}a^2 + V_t + V_\mu \frac{\alpha_t\gamma_t}{\sigma^2}\left(a - \alpha_t\mu_t\right) + \frac{1}{2}\frac{\alpha^2\gamma_t^2}{\sigma^2}V_{\mu\mu}.$$

This case natural guess for the functional form is:

$$rV(\theta,\mu) = v_1(t)\theta^2 + v_2(t)\theta\mu.$$

First order conditions implies

$$a_t = \frac{1}{2}\theta + \theta v_2(t)\frac{\alpha_t\gamma_t}{\sigma^2} \rightarrow \alpha_t = \frac{1}{2} + v_2\frac{\alpha_t\gamma_t}{\sigma^2}.$$

Envelope theorem applied to the HJB equation

$$(r + \frac{\alpha_t^2\gamma_t}{\sigma^2})v_2(t) = \frac{1}{4} + \dot{v}_2(t)$$

Rearranging above equation

$$\dot{\alpha}_t = \dot{v}_2(t)\frac{\alpha_t\gamma_t}{\sigma^2} + v_2(t)\frac{\dot{\alpha}_t\gamma_t + \dot{\gamma}_t\alpha_t}{\sigma^2},$$

and plugging $v_2(t), \dot{v}_2(t)$ we end up

$$\dot{\alpha}_t = r\left(\alpha_t - \frac{1}{2}\right)2\alpha_t - \frac{\alpha^2\gamma_t}{2\sigma^2}.$$

**Existence**

By continuity, there exist a region $E := [0, \gamma_E)$ in which initial value problem has a solution. Goal is to show that $\exists\gamma^o \in [0,\gamma_E)$ which initial variance $\gamma(\gamma^o)$ becomes the $2g_0$. Since system is continuous in the initial value it is enough to show that $\exists\gamma$ such that $\gamma(\gamma) > 2g_0$. To prove that, start with the $\gamma_E$, therefore $\exists t^* \in [0,T)$ at least one of the equations ($\alpha$, $\gamma$ or both diverges) diverges.

Case $i$), suppose only $\gamma$ diverges. Therefore, there exists $t^{**} < t^*, \gamma(\gamma_E)_{t^{**}} > 2g_0$. But then by continuity, $\exists \gamma^* \in [0, \gamma_E)$ such that $\gamma(\gamma^*)_{t^{**}} > 2g_0$. Since $\gamma$ is monotonic $\gamma(\gamma^*)_0 > \gamma(\gamma^*)_{t^{**}} > 2g_0$ .

Case $ii$) suppose only $\alpha$ diverges at $t^*$. $\alpha$ can be written as a function $\gamma$ rather than time as follows,

$$\frac{d\alpha}{d\gamma} = \frac{1}{2\gamma} - 2\frac{r(\alpha - \frac{1}{2})\sigma^2}{\alpha\gamma^2}$$

since ($\alpha - \frac{1}{2} > 0$) above function is always bounded by the following function $b : \mathbb{R} \to \mathbb{R}$.

$$\frac{db}{d\gamma} = \frac{1}{2\gamma}.$$

Above ODE has a unique solution $\beta(\gamma) = \frac{1}{2} + C + Log[\gamma]$. If $\alpha$ diverges at some variance $\gamma^*$, since the defined function $b$ is always above the $\alpha$, $b$ must be diverging at $\gamma^*$ or already diverged at $\gamma^b < \gamma^*$. Solution of $b$ implies that if $\beta(\gamma)$ is diverging, it must be the case $\gamma$ becomes arbitrarily high as well. Since the point which $\alpha(\gamma)$ diverges has to be larger than the point $b(\gamma)$ diverges in the variance space. Therefore, $\gamma^*$ has to be arbitrarily high number as well. Then, we can adopt the proof at *Case i*) since $\gamma$ is diverging as well. Both $(\alpha, \beta)$ diverges is a special case of *Case i*).

**Uniqueness**

Uniqueness follows from similar arguments as in the decentralized information case. Suppose there are two different solutions labeled as $\left(\gamma^1, \gamma^2\right)$, without loss of generality assume $\gamma^1_T > \gamma^2_T$. Working with the backwards system observe that, $-\dot{\alpha}_{1,T} > -\dot{\alpha}_{2,T}$, then therefore $\alpha^2$ can only cross $\alpha^1$ from below and suppose if there is such point call it as $t^*$. Recall the evolution of $\dot{\alpha}_t$,

$$\dot{\alpha}_{1,t} = 2r\left(\alpha_{1,t} - \frac{1}{2}\right)\alpha_{1,t} - \frac{\alpha^2_{1,t}\gamma^1_t}{2\sigma^2}.$$

At $t^*$, $\gamma^1_{t^*} > \gamma^2_{t^*}$ which implies $-\dot{\alpha}_{1,t} > -\dot{\alpha}_{2,t}$. Therefore, $\gamma_0(\gamma_1) > \gamma_0(\gamma_2)$ which implies the uniqueness of equilibrium.

**Properties of the linear equilibrium**

Observe that $\alpha_t \geq 0 \; \forall t \in [0, T]$. For the part $\alpha_t \geq \frac{1}{2}$ observe that at time $T$ $\alpha_T = \frac{1}{2}$ and

$\dot{\alpha}_T < 0$. In the backwards system, suppose there exists a $t^*$ such that $\alpha_t^* = \frac{1}{2}$ (observe that it can only cross from above) it is easy to see that $\dot{\alpha}_t^* < 0$. Therefore $\alpha$ is always above $\frac{1}{2}$.

For $\alpha_t$ is decreasing, suppose $\exists t^*$ such that $\dot{\alpha}_{t^*} = 0$ in the backwards system. Calculate the second time derivate of $\alpha_t$ at $t^*$ which is equal to

$$-2r\dot{\alpha}_{t^*}\left(\alpha_{t^*} - \frac{1}{2}\right) - 2r\alpha_{t^*}\dot{\alpha}_{t^*} + \frac{2\dot{\alpha}_{t^*}\alpha_{t^*}\gamma_{t^*} + \alpha_{t^*}\dot{\gamma}_{t^*}}{2\sigma^2}.$$

Since at $t^*$, $\dot{\alpha}_{t^*} = 0$, above expression is equal to $\frac{\gamma_t \alpha_t^*}{2\sigma^2}$ which is bigger than zero. Therefore, $\alpha_t$ is decreasing in $[0, T]$.

**Verification**

$(\alpha, \gamma)$ are bounded away from the zero. First order conditions pin down $v_2$, then plugging the optimal policy to the HJB equation and matching coefficients $v1$ is pinned down. By plugging the optimal policy to HJB equation and matching coefficients

$$\begin{aligned}
\dot{v}_1(t) &= rv_1(t) - \frac{\alpha_t}{2}(1-\alpha_t) - v_2(t)\frac{\alpha_t^2\gamma_t}{\sigma^2} \\
\dot{v}_2(t) &= v_2(t)(r - \frac{\alpha_t^2\gamma_t}{\sigma^2}) - \frac{1}{4}
\end{aligned}$$

with the boundary conditions $v_1(T) = 0,\ v_2(T) = 0$. Given $(\alpha, \gamma)$ this is a linear system therefore it has a unique solution. $V$ is twice continuously differentiable in $(\theta, \mu,)$ and continuously differentiable in t. Since the form is quadratic, it satisfies the polynomial growth condition. Then, by Theorem 3.1 (Fleming and Soner (2006) Chapter IV), we conclude that the conjectured policy and the HJB equation solves the agents' problem. Given $(\alpha, \beta)$ this is a linear system therefore it has a unique solution. Therefore, $V$ solves the HJB equation.

## 5.3 Proof for Section 2.5

**Proposition 9** *Under both information structures (centralized and decentralized) as T becomes large $\Theta$ is learned with arbitrarily high probability. For every $\epsilon > 0, \exists t^*$ such that $\forall t, T \geq t \geq t^*$ $\gamma_t < \epsilon$.*

**Proof** Start with the centralized information case.

**Centralized Information** In this case it is easy to see that learning never stops $\alpha_t \geq \frac{1}{2}, \forall t \in [0,T]$. Then $\int_0^T \left(\frac{\alpha_s}{\sigma}\right)^2 ds \geq \int_0^T \left(\frac{1}{2\sigma}\right)^2 ds = \frac{1}{4\sigma^2}T$ therefore $\gamma_t \to 0$, as $T \to \infty$.

**Decentralized Information** Observe that $\int_0^T \left(\frac{\alpha_s}{\sigma}\right)^2 ds \geq \int_0^T \left(\frac{\alpha_s^m}{\sigma}\right)^2 ds = \int_0^T \left(\frac{\gamma_s}{\sigma(2g_0+\gamma_s)}\right)^2 ds.$

For a contradiction, assume $\lim_{t\to\infty}\gamma_t \to k$ where $k > 0$, then $\int_0^T \left(\frac{\gamma_s}{\sigma(2g_0+\gamma_s)}\right)^2 ds \geq T\left(\frac{k}{\sigma(2g_0+k)}\right)^2 dt$ then $\int_0^T \left(\frac{\alpha_s}{\sigma}\right)^2 ds \to \infty$ which contradicts the fact that $\gamma_t \to k$ as $t \to \infty$.

**Lemma 11** *For any $T$, $\gamma_T^D > \gamma_T^C$, where $\gamma_T^D$denotes the case of decentralized information structure and $\gamma_T^C$ denotes the centralized information case.*

**Proof** Assume for a contradiction, $\gamma_T^E > \gamma_T^I$. Start the system backwards, under the assumption we are going to show that $\alpha_t^E \geq \alpha_t^S \ \ \forall t \in [0,T]$.This contradicts the fact that $\gamma_0^E = \gamma_0^S = 2g$. At T, $\alpha_T^I < \alpha_T^E = 1/2$. Therefore, it is enough to show if ever $\alpha_t^I$ touches $\alpha_t^E$ from below $-\dot{\alpha}_t^E > -\dot{\alpha}_t^I$.

$$\begin{aligned}
\dot{\alpha}_t &= r\frac{\alpha_t}{\alpha_t^m}(\alpha_t - \alpha_t^m) + \lambda_t\beta_t\alpha_t\left(\frac{\alpha_t}{\alpha^m} - 2\left(1+\alpha_t\right)\right) \\
&= r\frac{\alpha_t}{\alpha_t^m}(\alpha_t - \alpha_t^m) + \frac{\alpha_t^2\gamma_t}{2\sigma^2}\left(\frac{\beta_t^2 - 2\beta_t - 2\alpha_t\beta_t}{\beta_t^m}\right).
\end{aligned}$$

It is sufficient to show that (notice $\frac{\alpha_t^I}{\alpha_t^m} \geq 2\alpha_t$)

$$\left(\frac{\beta_t^2 - 2\beta_t - 2\alpha_t\beta_t}{\beta_t^m}\right) \geq -1.$$

Observe that $\beta_t^m \in [\frac{1}{2}, 1]$, therefore below inequality is sufficient

$$2\beta_t^2 - 4\beta_t - 4\alpha_t\beta_t \geq -1$$

which is always satisfied since $\beta_t \geq \frac{1}{2}$ and $\alpha_t \geq \beta_t$.

**Lemma 12** *Let $(g_0, r, \sigma)$ be given then, $\forall \epsilon > 0$, $\exists T$ and $t_\epsilon$ such that $\forall t, t_\epsilon \leq t \leq T$ $\|p_t^T - (0,1,0)\| \leq \epsilon$.*

**Proof** $\gamma$ is already proven in the asymptotic learning part. For $\alpha_t$ and $\beta_t$ part result follows from the fact that system is continuous and $\alpha_t^m(\gamma) \to 0$ and $\beta_t^m(\gamma) \to 1$ as $\gamma \to 0$. Moreover around $T$, both $\alpha_t$ and $\beta_t$ arbitrarily close to $\alpha_t^m$ and $\beta_t^m$.

**Lemma 13** *Let $(g_0, r, \sigma)$ be given then, $\forall \epsilon > 0, \exists T$ and $t_\epsilon$ such that $\forall t, t_\epsilon \le t \le T$,* $\mathbb{P}\left[\|A_t - \Theta\| < \epsilon\right] > 1 - \epsilon$

**Proof** By the Lemma 12, $\forall \epsilon > 0$ $\exists T$ and $t_\epsilon$ such that $\mathbb{P}\left[|\mu_t - \frac{1}{2}\Theta| < \eta\right] > 1 - \eta$, $\forall t$ such that $T \ge t \ge t_\epsilon$, then

$$|a_{1,t} - \frac{1}{2}\Theta| \le \alpha_t|\theta_i| + |\beta_t - 1|\,|\mu_t| + |\mu_t - \frac{1}{2}\Theta|.$$

Observe that each term in right hand side converges in distribution to zero. Therefore, result follows.

Following notation is useful for the rest of analysis. For a given $T$, define $p^T := \left(\alpha^T, \beta^T, \gamma^T\right)$ and extend the functions as follows

$$\begin{aligned} \alpha_t^T &= \alpha_T^T \ \forall t \in [T, \infty) \\ \beta_t^T &= \beta_T^T \ \forall t \in [T, \infty) \\ \gamma_t^T &= \gamma_T^T \ \forall t \in [T, \infty) \end{aligned}$$

also define $P$ such $P\left(p_t^T\right) = \dot{p}_t^T$ for every $t < T$, note that $P$ is continuous function-which is the right hand side of the Boundary value problem. Hence it is bounded on compact domain $[0, T]$. Therefore, $p^T$ is equi-lipschitz.

**Lemma 14** *Finite horizon limit converges to equilibrium of an infinite horizon*

**Proof** Proof has two parts, *i*) centralized information, *ii*) decentralized information. I am going to focus on decentralized information part, analogous arguments can be used for centralized information part.

**Decentralized Information** Recall that, $\alpha_t$ is monotonically deceasing function and sequence $\alpha^T$ is uniformly bounded since $\alpha_t \le \beta_t \le 2$ independent of $T$. By Helly's selection theorem sequence $\alpha_T$ converges to some $\alpha^*$ pointwise. For the $\beta^T$, sequence is uniformly bounded as well. Hence by Helly's selection theorem sequence $\beta_T$ converges to some $\beta^*$ pointwise. If $r > 0$, convergence is in fact uniform. First we need the following strengthening of Lemma 13,

**Lemma 15** *If $r > 0$, for all $\epsilon > 0$ then there exists $t_\epsilon < \infty$ such that for all $T \geq t \geq t_\epsilon$, $\|p_t^T - (0,1,0)\| \leq \epsilon$*

**Proof** First, start the analysis for $\gamma$. By Proposition 3, for every $\epsilon > 0$ exist $T^*$ such that for all $T \geq T^*$, and for every such $T$ $\exists t_\epsilon$ $\forall t \in [t_\epsilon, T]$ $\gamma_t < \epsilon$. Fix an $\epsilon > 0$ and choose the corresponding $T^*$ and $t_\epsilon$. We are going to show that, choice of $t_\epsilon$ is in fact uniform (independent of $T$). To see that choose a length $T$, such that $T > T^*$. Then for the above result it is sufficient to show that $\gamma_{t_\epsilon}^T < \gamma_{t_\epsilon}^{T^*}$. Assume towards a contradiction, $\gamma_{t_\epsilon}^T > \gamma_{t_\epsilon}^{T^*}$ this implies $\alpha_{T^*}^T > \alpha_{T^*}^{T^*}$. Then this implies that $\alpha_t^T \geq \alpha_t^{T^*}$ $\forall t \in [0, T^*]$ and inequality must be strict for positive measure of time. To see that suppose $\alpha_t^{T^*}$ crosses $\alpha_t^T$ at some $t^\delta \in [0, T^*]$. By continuity and boundary conditions, $\alpha_t^{T^*}$ must cross from below in the backward system. Then by inspection of $\dot{\alpha}$, it is easy to see that in the forward system $-\dot{\alpha}_{t^\delta}^{T^*} < -\dot{\alpha}_{t^\delta}^T$. The ranking in $\alpha$, contradicts the fact that $\gamma_0^T = \gamma_0^{T^*} = 2g$ since it must be the case $\gamma_0^T > \gamma_0^{T^*}$. Therefore, it must be the case $\gamma_{t_\epsilon}^T < \gamma_{t_\epsilon}^{T^*}$.

For the $\beta$ part observe that at $t_\epsilon$, $\beta_{t_\epsilon}^T \geq \frac{2g_0}{2g_0+\gamma_t} = 1 - \frac{\epsilon}{2g_0+\epsilon}$ for all $T \geq T^*$. This puts a lower bound on $\beta$. For the upper bound choose a $T > T^*$. Recall the evolution of the $\beta$, since $(\alpha, \beta, \gamma)$ are uniformly bounded (independent of $T$) non $r$ term in the evolution of $\beta$ is bounded by $\gamma_{t_\epsilon} K$ where $K \in \mathbb{R}$. Look at the evolution of $\beta$ in the range $[t^*, T]$. As usual focus on the backward system, $\beta$ is always below the following function $\dot{w}_t = r(w_t - 1) - \epsilon K$ and $w_T = 1$. This follows since, $r\frac{\beta_t}{\beta_t^m}(\beta_t - \beta_t^m) \geq r(\beta_t - \beta_t^m) \geq r(\beta_t - 1)$. First inequality follows from the fact that $\beta_t \geq \beta_t^m$ in the range and second one follows from the fact that $\beta_t^m \leq 1$. $w_t$ is bounded above by $\epsilon\frac{k}{r} + 1$. Then by choosing $\epsilon$ small enough one can guarantee $\beta$ is arbitrarily close to 1, since both lower and upper bound becomes arbitrarily close to 1. Analogous analysis holds for $\alpha$, hence it is skipped.

By armed with Lemma 14 we can show convergence is in fact uniform. Suppose for a contradiction that convergence is not uniform. This implies $\exists \epsilon > 0$ and a subsequence combined with points $\{T_l, t_l\}_{l \in \mathbb{N}}$ such that $\|p_{t_l}^T - p_{t_l}\| > \epsilon$ for every $l \in \mathbb{N}$. Lemma 14 implies that $\exists t^* < \infty$ such that for all $T_l \geq t \geq t_\epsilon$, $\|p_t^T - (0,1,0)\| \leq \frac{\epsilon}{2}$. By pointwise convergence of $p$, $\left(p_t^{t_n} \to p_t \text{ as } n \to \infty\right)$ $\|p_t^{T_n} - p_t\| < \epsilon$ for all $T_n \geq t \geq t_\epsilon$. Therefore, it must be the case that $t_l \in [0, t_\epsilon]$ for all $l$ greater than some $l^* < \infty$. These observation implies that there is no subsequence of $\{p_l^T\}$ converges uniformly on $[0, t_\epsilon]$. However, this contradicts the Arzela-Ascoli theorem. Note that conditions for Arzela-Ascoli theorem is satisfied $i)$ $[0, t_\epsilon]$ is compact interval, $ii)$ $\{p_l^T\}$ are uniformly bounded, equi-continuous and equi-Lipschitz. Therefore, we conclude that $p$ converges uniformly.

**Differentiability**

Uniform convergence $p^T \to p$ implies that for every compact interval $[0,t]$ $\dot{p}^T = P(p^T) \to P(p)$ uniformly, since $P$ is continuous on a compact domain it is uniformly continuous. Define, $h : \mathbb{R}_+ \to \mathbb{R}^4$ as

$$h_i(t) =: p_i(0) + \int_0^T P_i\left(p_s\right) ds.$$

Therefore we need to show that $h = p \in [0, \infty)$. For that it is sufficient to show, $p^{T_n} \to h$ pointwise. Uniform convergence theorem implies that for any $t$,

$$\lim_{n\to\infty} \int_0^t P_i\left(p_s^{T_n}\right) ds = \int_0^t P_i(p_s) ds.$$

Note that for $t = 0$, $h_i(0) = p_i(0)$ by definition. Fix an $\epsilon > 0$ and $t > 0$. Choose $N$ such that $\left|\int_0^t P_i\left(p_s^{T_n}\right) ds - \int_0^t P_i(p_s) ds\right| < \frac{\epsilon}{2}$ and $\left|p_{i,0}^{T_n} - h_{i,0}\right| < \frac{\epsilon}{2}$. Therefore,

$$\left|p_{i,t}^{T_n} - h_{i,t}\right| \leq \epsilon.$$

Thus $p_i = h_i$ and $\dot{p} = \dot{h} = P\left(p\right)$. Since choice of $\epsilon$ depends on $i$, choose the smallest such epsilon which works for all $i$. Also Lemma 14 allows us to look at the limit as $t \to \infty$. Now, we can conclude the proof by showing transversality conditions are satisfied in the infinite horizon problem. For that following condition is used, function $\phi\left(x, t\right)$ satisfies polynomial growth condition if for all $(t, x)$ in the domain

$$\phi(t, x) \leq K(1 + |x|)^m$$

for some constants $K$ and $m$.

**Transversality conditions for the Decentralization Information**

We first prove the following preliminary lemma which is crucial for checking the transversality conditions.

**Lemma 16** *For any admissible strategy* $(a_t)_{t\geq 0}$

$$\lim_{t\to\infty} e^{-rt} v_t \mathbb{E}^a[\mu_t] = \lim_{t\to\infty} e^{-rt} v_t \mathbb{E}^a[\hat{\mu}_t] = \lim_{t\to\infty} e^{-rt} v_t \mathbb{E}^a[\hat{\mu}_t^2] = 0$$

*for any function $v$ which satisfies the polynomial growth condition.*

**Proof** Recall the evolution $\hat{\mu}_t$, from Lemma 3

$$d\hat{\mu}_t = \alpha_t\lambda_t\left(2 - z_t\right)\left(\theta - \hat{\mu}_t\right)dt + \lambda_t\sigma dZ_t^i$$

which is a linear stochastic differential equation. It has a unique weak solution

$$\hat{\mu}_t = \hat{\mu}_0 e^{-\int_0^t L_s ds} + \theta\int_0^t L_s e^{-\int_s^t L_u du}ds + \int_0^t \sigma\lambda_t e^{-\int_s^t L_u du}dZ_s,$$

where $L_t = \alpha_t\lambda_t\left(2 - z_t\right)$. Above expression is equal to (evaluate the middle term)

$$\hat{\mu}_t = \hat{\mu}_0 e^{-\int_0^t L_s ds} + \theta\left(1 - e^{-\int_0^t L_s ds}\right) + \sigma\int_0^t \lambda_t e^{-\int_s^t L_u du}dZ_s.$$

Since last term is a stochastic integral, it has a zero expectation. Then we reach

$$\mathbb{E}[\hat{\mu}_t] = \hat{\mu}_0 e^{-\int_0^t L_s ds} + \theta\left(1 - e^{-\int_0^t L_s ds}\right).$$

Since $0 < e^{-\int_0^t L_s ds} < 1$, $\mathbb{E}[\hat{\mu}_t] < \max\{\theta, \hat{\mu}_0\}$. By Ito Isometry

$$\mathbb{E}\left[\left(\int_0^t \sigma\lambda_s e^{-\int_s^t L_u du}dZ_s\right)^2\right] = \int_0^t \sigma^2\lambda_s^2 e^{-2\int_s^t L_u du}ds \leq K_1 t$$

for some $K_1 > 0$.[43] Then $\mathbb{E}\left[\hat{\mu}_t^2\right] \leq d + Kt$ for some $d \in \mathbb{R}$, therefore

$$\lim_{t\to\infty} e^{-rt}v(t)\mathbb{E}\left[\hat{\mu}_t\right] = \lim_{t\to\infty} e^{-rt}v(t)\mathbb{E}\left[\hat{\mu}_t^2\right] = 0$$

where $v(t)$ is a function which satisfies polynomal growth condition. For the $\mu_t$, recall the evolution of $\mu_t$ is

$$d\mu_t = \lambda_t\left(a_t - \mu_t\left(2\alpha_t + \beta_t\right) + \alpha_t\left(\left(1 - z_t\right)\theta_i + z_t\hat{\mu}_t\right)\right)dt + \lambda_t\sigma dZ_t^i.$$

which is a linear stochastic differential equation. It has a unique weak solution

$$\mu_t = \mu_0 e^{-\int_0^t \tilde{L}_s ds} + \int_0^t e^{-\int_s^t \tilde{L}_u du}\lambda_s\left(a_s + \left(1 - z_s\right)\theta_i + z_s\hat{\mu}_s\right)ds + \int_0^t \sigma e^{-\int_s^t \tilde{L}_u du}\lambda_s dZ_s^i$$

[43] Uniform boundedness of $\alpha_t \leq \beta_t \leq 2$ and $\gamma_t$ implies expectation is finite and such $K$ exists.

where $\tilde{L}_t = \lambda_t \left(2\alpha_t + \beta_t\right)$. Define $\mathbb{E}[I_t] := \mathbb{E}\left[\int_0^t \lambda_s e^{-\int_s^t \tilde{L}_u du} a_s ds\right] = \int_0^t \lambda_s e^{-\int_s^t \tilde{L}_u du} \mathbb{E}[a_s] ds$. Then by Cauchy - Schwartz inequality

$$\mathbb{E}[I_t] \leq \left(\int_0^t e^{-2\int_0^t \tilde{L}_s ds}\right)^{\frac{1}{2}} \left(\int_0^t \mathbb{E}[a_s^2] ds\right)^{\frac{1}{2}} < K_2 t^{\frac{1}{2}} \left(\mathbb{E}[a_s^2 ds]\right)^{\frac{1}{2}}$$

note that bound $K_2$ exists because $\alpha_t$ is uniformly bounded. Therefore,

$$e^{-rt}\mathbb{E}[J_t] < e^{-rt/2} K t^{\frac{1}{2}} \left(e^{-rt}\mathbb{E}[\int_0^t a_s^2 ds]\right)^{\frac{1}{2}} < e^{-rt/2} K t^{\frac{1}{2}} \left(\mathbb{E}[\int_0^t e^{-rs} a_s^2 ds]\right)^{\frac{1}{2}}$$

then admissibility of $a$ implies that $e^{-rt}\mathbb{E}[J_t] \to 0$. Observe that last term in $\mu_t$ is stochastic integral and for the all other terms there are bounds which are linear in time (follows from the uniform boundedness of $\alpha$). Therefore, $e^{-rt} v(t) \mathbb{E}^a[\mu_t] \to 0$ for all $v$ satisfies polynomial growth condition.

Then we can show that transversality conditions holds. By Theorem 5.1 (Chapter IV) of Fleming and Soner (2006), we need to show $\lim_{t\to\infty} e^{-rt}\mathbb{E}\left[V\left(\theta, \mu^*, t\right)\right] \to 0$ in the limiting strategy $(\alpha, \beta, \gamma)$ and for any admissible strategy $\limsup_{t\to\infty} e^{-rt}\mathbb{E}\left[V\left(\theta, \mu^*, t\right)\right] \geq 0$. As a first step show that

$$\lim_{t\to\infty} e^{-rt} v_4(t) \mathbb{E}^a\left[\mu_t \hat{\mu}_t\right] \to 0.$$

Observe that $v_4(t)$ is linear ODE and define $\chi_t := \left[r + \lambda_t\left(2\alpha_t + \beta_t\right) + \lambda_t \alpha_t 2\alpha_t^m\right]$. Note that $\chi_t > r > 0 \quad \forall t \in [0, T]$. Then ODE has the following unique solution

$$v_4(t) = v_0 e^{\int_0^t \chi_u du} - \int_0^t e^{\int_u^t \chi_n dn} \beta_u \beta_u^m du.$$

Therefore, for any $s > t$

$$v_4(s) e^{-\int_0^s \chi_u du} - v_4(t) e^{-\int_0^t \chi_u du} = -\int_t^s e^{\int_0^u \chi_v dv} \beta_u \beta_u^m du.$$

Guess of a solution in the form as $v_4(s) e^{-\int_0^s \chi_u du} \to 0$, as $s \to \infty$. Then if such solution exists, it must be the case

$$v_4(t) = \int_t^\infty e^{-\int_s^t \chi_v dv} \beta_u \beta_u^m du.$$

Observe that $\beta\beta^m \leq 2, \chi > 0$, which implies right hand side is uniformly bounded. Limiting value of the right hand side is (by L'hopital Rule)

$$\lim_{t\to\infty} v_4(t) = \lim \frac{\beta_t \beta_t^m}{\chi_t} = \frac{1}{r}.$$

For the term $\mathbb{E}^a \left[\mu_t \hat{\mu}_t\right]$, by Cauchy-Schwarz inequality

$$e^{-rt} v_4(t) \mathbb{E}^a \left[\mu_t \hat{\mu}_t\right] \leq e^{-rt/2} v_4(t) \mathbb{E}[\hat{\mu}_t^2]^{\frac{1}{2}} e^{-rt/2} \mathbb{E}^a [\mu_t^2]^{\frac{1}{2}}.$$

Then one need to show $\limsup e^{-rt} \mathbb{E}[\hat{\mu}_t^2] < \infty$. By recalling the solution of $\mu_t$ it is enough to show $\limsup T_1(t) < \infty$ and $\limsup T_2(t) < \infty$ where,

$$T_1(t) := \left[\int_0^t e^{-\int_s^t \tilde{L}_u du} \lambda_s a_s ds\right]^2$$

and

$$T_2(t) := \limsup_{t\to\infty} \int_0^t e^{-\int_s^t \tilde{L}_u du} \lambda_s a_s ds \int_0^t \sigma e^{-\int_s^t \tilde{L}_u du} \lambda_s dZ_s^i < \infty.$$

Recall that $2\alpha_t + \beta_t \geq 1 \ \ \forall t \in [0, \infty)$. By Cauchy-Schwarz inequality

$$T_1 \leq \int_0^t e^{-2\int_s^t \tilde{L}_u du} \lambda_s ds \int_0^t \lambda_s a_s^2 ds \leq B \int_0^t e^{-2\int_s^t \lambda_u du} \lambda_s ds \int_0^t a_s^2 ds$$

where $B > 0$ is a constant. After evaluating term $\int_0^t e^{-2\int_s^t \lambda_u du} ds$, equation simplifies to

$$B\frac{1}{2}\left(1 - e^{-\int_0^t \lambda_t dt}\right) \int_0^t a_s^2 ds.$$

By admissibility of the strategy profile, conclude that $T_1$ is bounded. For $T_2$ following the same methods (by Cauchy-Schwarz)

$$\mathbb{E} T_2(t) \leq \mathbb{E}\left[\int_0^t \left(e^{-\int_s^t \tilde{L}_u du} \lambda_s a_s ds\right)^2\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^t \sigma e^{-\int_s^t \tilde{L}_u du} \lambda_s dZ_s^i\right)^2\right]^{\frac{1}{2}}.$$

Recall that second term is bounded by some $(K + K_1 t)^{\frac{1}{2}}$. For the first term, recalling the bounding procedure for $T_1$.

$$e^{-rt} T_1(t)^{\frac{1}{2}} \leq e^{-rt/2} \left[B\frac{1}{2}\left(1 - e^{-\int_0^t \lambda_t dt}\right)\right]^{\frac{1}{2}} \int_0^t e^{-rs} a_s^2 ds$$

By admissibility of strategy profile, conclude that $\limsup T_2(t) < \infty$. Thus we can conclude that $e^{-rt}v_4(t)\mathbb{E}\left[\hat{\mu}_t\mu_t\right] \to 0$ as $t \to \infty$. Since all terms converge to zero, we conclude transversality conditions are satisfied.

## 5.4 Proofs for Section 3

It is easier to compare information structure if we apply change of variables. Write the equilibrium strategies function of variance rather than time.[44] $\gamma_t$ is strictly decreasing function under both information structures, this implies there is one to one mapping between the public variance and the calendar time.

**Centralized Information Structure Ex-ante Total Output**

After the change of variables,

$$\frac{d\alpha^C}{d\gamma} = \frac{1}{2\gamma}, \;\; \alpha(\gamma_F^C) = \frac{1}{2}$$

where $\gamma_F^C$ stands for the value of variance at time $T$. Above ODE has a unique solution

$$\alpha^C(\gamma) = \frac{1}{2} + \frac{1}{2}\mathrm{Log}(\frac{\gamma}{\gamma_F^C}).$$

Ex ante expected flow output given variance level $\gamma$ is

$$(4\mu_0^2 + 2g)\left(1 + \frac{1}{2}\mathrm{Log}(\frac{\gamma}{\gamma_F^C})\right) - \frac{\gamma}{2}.$$

**Centralized Information Ex-ante Welfare**

Plugging the equilibrium strategies and belief evolutions, we can calculate ex-ante total welfare as

$$\int_0^t e^{-rt}\left(\left(4\mu_0^2 + 2g\right)\left(\alpha_t^*\left(1 - \frac{1}{2}\alpha_t^*\right) + \frac{3}{8}\right) - \frac{3}{8}\gamma_t^*\right) dt.$$

It is easy to see that flow welfare is bounded by $\frac{3}{8}\gamma_t^* \leq \frac{7}{8}\left(4\mu_0^2 + 2g\right)$. Note that

[44] Observe that $\frac{d\alpha}{d\gamma} = \frac{\frac{d\alpha}{dt}}{\frac{d\gamma}{dt}}$.

$\alpha_t^* \left(1 - \frac{1}{2}\alpha_t^*\right) \leq \frac{1}{2}$, which implies

$$\left(4\mu_0^2 + 2g\right)\left(\alpha_t^*\left(1 - \frac{1}{2}\alpha_t^*\right) + \frac{3}{8}\right) - \frac{3}{8}\gamma_t^* \leq \frac{7}{8}\left(4\mu_0^2 + 2g\right).$$

Ex-ante expected flow welfare given variance $\gamma$ is

$$\left(4\mu_0^2 + 2g\right)\left(\frac{1}{2}\left(1 + \mathrm{Log}(\frac{\gamma}{\gamma_F^C})\right)\left(1 - \frac{1}{4}\left(1 + \mathrm{Log}(\frac{\gamma}{\gamma_F^C})\right)\right) + \frac{3}{8}\right) - \frac{3}{8}\gamma.$$

**Decentralized Information Structure Ex-ante Total Output**

After the change of variables,

$$\frac{d\alpha}{d\gamma} = -\frac{\alpha g}{\gamma^3}\left(2g\alpha - \gamma\left(2 + \alpha\right)\right), \quad \alpha\left(\gamma_F^D\right) = \frac{\gamma_F^D}{2 + \gamma_F^D}$$

where $\gamma_F^D$ stands for the value of variance at time $T$. Above ODE has a unique real solution

$$\alpha(\gamma) = \frac{\gamma}{g + \frac{\gamma}{\gamma_F^D}(g + \gamma_F^D)e^{\left(\frac{2g}{\gamma} - \frac{2g}{\gamma_F^D}\right)}}$$

Ex-ante flow output (using the fact $\alpha_t = \beta_t \frac{\gamma}{2g_0}$) given variance level $\gamma$ is

$$(4\mu_0^2 + 2g)\frac{\gamma}{g + \frac{\gamma}{\gamma_F^D}(g + \gamma_F^D)e^{\left(\frac{2g}{\gamma} - \frac{2g}{\gamma_F^D}\right)}}(1 + \frac{2g}{\gamma}) - \frac{2g\gamma}{g + \frac{\gamma}{\gamma_F^D}(g + \gamma_F^D)e^{\left(\frac{2g}{\gamma} - \frac{2g}{\gamma_F^D}\right)}}.$$

Observe that ex-ante flow output is bounded by $4(4\mu_0^2 + 2g)$.

**Decentralized Information Welfare**

Plugging the equilibrium strategies and belief evolutions, we can calculate total welfare as

$$\int_0^t e^{-rt}\left(\left(4\mu_0^2 + 2g\right)\left(\alpha_t + \beta_t(1 - \frac{\alpha_t}{2})\right) - \beta_t\gamma_t(1 - \frac{\alpha_t}{2}) - \frac{\beta_t^2}{4}\left(4\mu_0^2 + 2g - \gamma_t\right) - \alpha_t^2(\mu_0^2 + g)\right)dt$$

As $\gamma_F^D \to 0$ function $\alpha(\gamma)$ point wise converges to the function $\frac{\gamma}{g}$ in the domain $[0, 2g]$. Also observe that as $\gamma_F^D \to 0$, welfare at the boundary is converging to $W(\gamma_F^D) =$

$\frac{3}{4}(4\mu_0^2+2g)$. By plugging $\alpha(\gamma)$ to flow welfare equation reach

$$W(\gamma,\gamma_F^D)=\frac{2\gamma_F^D e^{\frac{2g}{\gamma_F^D}}\left(g^2+2g\mu_0^2+\mu_0^2\gamma\right)\left(e^{\frac{2g}{\gamma_F^D}}\left(g\gamma_F^D-\frac{1}{2}\gamma_F^D\gamma\right)\right)}{\left(g\gamma_F^D e^{\frac{2g}{\gamma_F^D}}+e^{\frac{2g}{\gamma}}\left(g+\gamma_F^D\right)\gamma+2\gamma^{\frac{2g}{\gamma}}\left(g+\gamma_F^D\right)\right)}.$$

At the boundary, $W(\gamma_F^D,\gamma_F^D)=\frac{\frac{6}{4}g^2+\frac{3}{4}g4\mu_0^2+\frac{3}{8}\gamma_F^D4\mu_0^2}{g+\frac{1}{2}\gamma_F^D}$. Therefore the limit as $\gamma_F^D\to 0$, welfare becomes $\frac{3}{4}(4\mu_0^2+2g)$. Evaluating welfare at $\frac{g}{2}, W(\frac{g}{2})=\frac{3}{4}\left(4\mu_0^2+2g\right)$, $W(\gamma)>\frac{3}{4}\left(4\mu_0^2+2g\right), \forall(\gamma_I^E,\frac{g}{2})$ and $W(2g)=0, W(\gamma)\geq 0\ \ \forall\gamma\in[0,2g]$.

Now write the backwards ODE for $\gamma$, by plugging the limiting function for $\alpha$ [45]

$$\dot{\gamma}_t=\left(\frac{\gamma_t^2}{g\sigma}\right)^2\quad \gamma_0=\gamma_F^D.$$

Above ODE has a unique real solution,

$$\gamma_t=\frac{(g\sigma)^{\frac{2}{3}}}{\left(\frac{\left(-3\gamma_F^D\right)^3 t+\sigma^2 g^2}{\left(\gamma_F^D\right)^3}\right)^{\frac{1}{3}}}.$$

Then, we can find the time required (this will be the $T$) such that $\gamma_0=2g$. Therefore, given $\gamma_F^D, T$ becomes

$$T=\frac{-\sigma^2\left(\gamma_F^D\right)^3+8\sigma^2 g^3}{24\left(\gamma_F^D\right)^3 g}.$$

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[45] Note that in the backwards system $\dot{\gamma}_t=\left(\frac{\gamma_t^2}{\sigma\left(g+\frac{\gamma_t}{\gamma_F^D}(g+\gamma_F^D)e^{\left(\frac{2g}{\gamma_t}-\frac{2g}{\gamma_F^D}\right)}\right)}\right)^2$, $\gamma_0=\gamma_F^D$. Treat the boundary value and the part $\gamma_F^D$ inside the function as two different variables, $\gamma_t(\gamma_F^D,\gamma_F^D)$. Observe that $|\gamma_t(\gamma_F^D,\gamma_F^D)-\gamma_t(0,0)|\leq|\gamma_t(\gamma_F^D,\gamma_F^D)-\gamma_t(0,\gamma_F^D)|+|\gamma_t(0,\gamma_F^D)-\gamma_t(0,0)|$. Then, Theorem 2.8 of Teschl (2012) implies that solution is continuous respect to initial value and the Theorem 2.12 of Teschl (2012) implies that solution is continuous respect to the parameter $(\gamma_F^D)$. Fixing the $\gamma_F^D$ as initial value $\gamma_t(\gamma_F^D,\gamma_F^D)$ can be seen as a *regular perturbation problem*. Morever, $\gamma_t(\gamma_F^D,\gamma_F^D)$ can be approximated as $\gamma_t(0,\gamma_F^D)+\epsilon\phi_t+O(\epsilon)$ where $\phi$ is only a function of time.

Similarly, the amount of time required $\gamma$ to reach $g/2$ is

$$t^* = \frac{-8\sigma^2\left(\gamma_F^D\right)^3 + \sigma^2 g^3}{3\left(\gamma_F^D\right)^3 g} \text{ then } \frac{t^*}{T} = \frac{\left(64\gamma_F^D\right)^3 - 8g^3}{\left(\gamma_F^D\right)^3 - 8g^3}.$$

Clearly, $\lim_{\gamma_F^D \to 0} \frac{t^*}{T} \to 1$, which implies $W > \frac{3}{4}\left(4\mu^2 + 2g\right)$.

For the comparison with the centralized information structure, we require the following argument as $\gamma_F^D \to 0$, $W(n\gamma_F^D, \gamma_F^D)$ where $n > 1, n \in \mathbb{R}$.

$$W(n\gamma_F^D, \gamma_F^D) = \frac{2\gamma_F^D e^{\frac{2g}{\gamma_F^D}}\left(g^2 + 2g\mu_0^2 + n\mu_0^2\gamma_F^D\right)\left(e^{\frac{2g}{\gamma_F^D}}\left(g\gamma_F^D - \frac{1}{2}n\gamma_F^D\gamma_F^D\right)\right)}{\left(g\gamma_F^D e^{\frac{2g}{\gamma_F^D}} + e^{\frac{2g}{n\gamma_F^D}}\left(g + \gamma_F^D\right)n\gamma_F^D + 2n\gamma_F^D \frac{2g}{n\gamma_F^D}\left(g + \gamma_F^D\right)\right)}.$$

therefore limit (rearranging and dividing both denominator and numerator by $g^2 e^{\frac{4g}{\gamma_F^D}}$).

$$\lim_{\gamma_F^D \to 0} W(n\gamma_F^D, \gamma_F^D) = \frac{2g + gne^{\frac{2g(1-n)}{n\gamma_F^D}} + 4\mu_0^2 + 2\left(4\mu_0^2\right)ne^{\frac{2g(1-n)}{n\gamma_F^D}}}{1 + 2ne^{\frac{2g(1-n)}{n\gamma_F^D}} + e^{\frac{4g(1-n)}{n\gamma_F^D}}n^2} = 4\mu_0^2 + 2g$$

as long as $n > 1$. Time to reach $n\gamma_F^D$

$$\frac{\sigma^2\left(-1 + n^3\right)g^2}{3\left(n\gamma_F^D\right)^3}$$

Note that graph of welfare as $\gamma_F^D \to 0$ lies above the function $\left(4\mu_0^2 + 2g\right) - \gamma\frac{\mu_0^2}{2g+1}$ in the domain $[0, 2g]$. Using this observation, proof is completed by showing that choose an arbitrary $k \in \mathbb{R}$, such that welfare is arbitrarily close to $\left(4\mu_0^2 + 2g\right)$ which is the ex-post efficient outcome. Time to reach $\frac{2g}{k}$ is

$$\left(\frac{-\sigma^2\left(\gamma_F^D\right)^3 k^3 + 8\sigma^2 g^3}{24\left(\gamma_F^D\right)^3 g}\right)$$

then a lower bound on the approximate amount of time spent around $\left(4\mu_0^2 + 2g\right)$ is given as

$$\left(\frac{-\sigma^2\left(\gamma_F^D\right)^3k^3+8\sigma^2g^3}{24\left(\gamma_F^D\right)^3g}\right)-\frac{\sigma^2\left(-1+n^3\right)g^2}{3\left(n\gamma_F^D\right)^3}.$$

Therefore, the ratio between the length of the game and the time spent around the $\left(4\mu_o^2+2g\right)$ is

$$\left(\left(\frac{-\sigma^2\left(\gamma_F^D\right)^3k^3+\sigma^2g^3}{3\left(\gamma_F^D\right)^3g}\right)-\left(\frac{\sigma^2\left(-1+n^3\right)g^2}{3\left(n\gamma_F^D\right)^3}\right)\right)/\left(\frac{-\sigma^2\left(\gamma_F^D\right)^3+8\sigma^2g^3}{24\left(\gamma_F^D\right)^3g}\right).$$

That is equivalent to

$$=\frac{8\left(n^3\left(\gamma_F^D\right)^3k^3-g^3\right)}{n^3\left(\left(\gamma_F^D\right)^3-8g^3\right)}.$$

Clearly, above equation goes to 1 in the limit (given $k$ is finite) $\lim_{n\to1}\lim_{\gamma_F^D\to0}$.

**Proposition 10** *If $r=0$, then $\exists T^*$ such that $\forall T>T^*$, for each $T$ $\exists t^*$ such that $\forall t\in[0,t^*]$ ex-ante expected flow output in centralized information is higher than both decentralized and full information revelation.*

**Proof** Recall in the centralized information, flow output is given as

$$(4\mu_0^2+2g)\left(1+\frac{1}{2}\text{Log}(\frac{\gamma}{\gamma_F^C})\right)-\frac{\gamma}{2}.$$

As $\gamma_F^C\to0$, it is easy to flow output diverges to $\infty$. As $T\to\infty$, $\gamma_F^C\to0,\gamma_F^D\to0$, and recall that total output is bounded by finite number in decentralized information, specifically by $\left(4(4\mu_0^2+2g)\right)$. Therefore, there $\exists$ some $t^*$, in the range $[0,t^*]$ flow output from centralized information dominates the decentralized.

**Evolution of the Beliefs Decentralized Information**

In equilibrium, plugging $a_t=\alpha_t\theta+\beta_t\mu_t$ to belief equations true data generating is

$$d\mu_t=\lambda_t\alpha_t\left(\Theta-2\mu_t\right)dt+\lambda_t\sigma dZ_t.$$

Equivalently,

$$\mu_t = \mu_0 e^{-\int_0^t 2\lambda_s\alpha_s ds} + \Theta\int_0^t \lambda_s\alpha_s e^{-\int_s^t 2\lambda_u\alpha_u du} ds + \sigma\int_0^t \lambda_s e^{-\int_s^t 2\lambda_u\alpha_u du} dZ_s.$$

Then ex-ante expected public mean at time $t$ is,

$$\mathbb{E}[\mu_t|\Theta] = \mu_0 e^{-\int_0^t 2\lambda_s\alpha_s ds} + \Theta\int_0^t \lambda_s\alpha_s e^{-\int_s^t 2\lambda_u\alpha_u du} ds.$$

By Ito Isometry variance is,

$$Var[\mu_t|\Theta] = \sigma^2\int_0^t \lambda_s^2 e^{-\int_s^t 4\lambda_u\alpha_u du} ds.$$

Observe that $-2\alpha_t\lambda_t = \frac{\dot{\gamma}_t}{\gamma_t}$ therefore $e^{-\int_s^t 2\lambda_u\alpha_u} = \frac{\gamma_t}{\gamma_s}$. Using $\lambda_s\alpha_s = -\frac{1}{2}\frac{\dot{\gamma}_s}{\gamma_s}$ we conclude that

$$\mathbb{E}[\mu_t|\Theta] = \mu_0\frac{\gamma_t}{\gamma_0} - \frac{1}{2}\Theta\int_0^t \frac{\dot{\gamma}_s}{\gamma_s}\frac{\gamma_t}{\gamma_s} ds = \mu_0\frac{\gamma_t}{\gamma_0} - \frac{1}{2}\Theta\left(\frac{\gamma_t}{\gamma_0} - 1\right).$$

Similarly variance equation simplifies to

$$Var[\mu_t|\Theta] = \sigma^2\int_0^t \lambda_s^2\frac{\gamma_t^2}{\gamma_s^2} ds = -\frac{\gamma_t^2}{4}\int_0^t \frac{\dot{\gamma}_s}{\gamma_s^2} ds = \frac{1}{4}\left(\gamma_t - \frac{\gamma_t^2}{\gamma_0}\right).$$

**Evolution of the Beliefs Centralized Information**

In equilibrium true data generating process becomes, plugging $a_t = \alpha_t\theta_t$ to evolution of public belief

$$d\mu_t^* = \frac{\alpha_t^2\gamma_t}{\sigma^2}\left(\Theta - \mu_t^*\right) dt + \frac{\alpha_t\gamma_t}{\sigma} dZ_t$$

Equivalently,

$$\mu_t = 2\mu_0 e^{-\int_0^t \frac{\alpha_s^2\gamma_s}{\sigma^2} ds} + \Theta\int_0^t \frac{\alpha_s^2\gamma_s}{\sigma^2} e^{-\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma^2} du} ds + \sigma\int_0^t \frac{\alpha_s\gamma_s}{\sigma^2} e^{-\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma^2} du} dZ_s.$$

Then ex-ante expected public mean at time $t$ is,

$$\mathbb{E}[\mu_t|\Theta] = 2\mu_0 e^{-\int_0^t \frac{\alpha_s^2\gamma_s}{\sigma^2} ds} + \Theta\int_0^t \frac{\alpha_s^2\gamma_s}{\sigma^2} e^{-\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma^2} du} ds.$$

By Ito Isometry

$$Var[\mu_t|\Theta] = \sigma^2 \int_0^t \left(\frac{\alpha_s\gamma_s}{\sigma^2}\right)^2 e^{-2\int_s^t \frac{\alpha_u^2\gamma_u}{\sigma^2}du} ds.$$

Observe that $-\frac{\alpha_s^2\gamma_s}{\sigma^2} = \frac{\dot{\gamma}_s}{\gamma_s}$, therefore

$$\mathbb{E}[\mu_t|\Theta] = 2\mu_0\frac{\gamma_t}{\gamma_0} - \Theta\int_0^t \frac{\dot{\gamma}_s}{\gamma_s}\frac{\gamma_t}{\gamma_s}ds = 2\mu_0\frac{\gamma_t}{\gamma_0} - \Theta\left(\frac{\gamma_t}{\gamma_0} - 1\right)$$

Similarly variance becomes,

$$Var[\mu_t|\Theta] = \sigma^2 \int_0^t \left(\frac{\alpha_s\gamma_s}{\sigma^2}\right)^2 \left(\frac{\gamma_t}{\gamma_s}\right)^2 ds = -\gamma_t^2\int_0^t \frac{\dot{\gamma}_s}{\gamma_s^2}ds = \gamma_t - \frac{\gamma_t^2}{\gamma_0}.$$

## 5.5 Extensions

Suppose we have another state $\theta^u$, which evolves as Ito process in the form

$$d\theta_t^u = k_t^1 + k_t^2\theta_t^u + \sigma_t^u dZ_t^u.$$

In this case it is natural to focus on following class of strategies

$$a_t = \alpha_t\theta_i + \beta_t\mu_t + \Psi_t\theta_t^U.$$

In this case, all most the same analysis goes through and results are qualitatively the same. Moreover, evolution of $\Psi_t$ only depends on $v_4$ and $v_3$.

### 5.5.1 Changing states

I can allow specific forms for evolution of the private information (states). Suppose, evolution of the private information is the following form:

$$d\theta_{i,t} = \left(\omega\theta_{i,t}\right)dt$$

Then,

$$\theta_{i,s} = e^{\omega(s-t)}\theta_{i,t} \quad \forall s \leq t.$$

Agent $i's$ optimal control problem is,

$$\sup_{A^i \in L^2[0,T]} \mathbb{E}[e^{rT} \int_0^T \left( \frac{1}{2} e^{-rt} \left( \theta_i^t + m_t^i \right) \left( A_t^i + \alpha_t m_t^i + \beta_t \mu_t \right) - \frac{1}{2} \left( A_t^i \right)^2 \right) dt$$

$$\begin{aligned}
d\mu_t &= \frac{1}{2}\omega \left( m_t^{i,j} + \theta_t - 2\mu_t \right) + \lambda_t \left( \alpha_t \left( m_t^{i,j} - \mu_t \right) - \mu_t(\alpha_t + \beta_t) + a_t \right) + \lambda_t \sigma dZ_t^i \\
d\hat{\mu}_t^i &= \frac{1}{2}\omega \left( m_t^{i,j} + \theta_t - 2\hat{\mu}_t \right) + \lambda_t \left( \alpha_t \left( \theta_t - \hat{\mu}_t \right) + \alpha_t (m_t^{i,j} - \hat{\mu}_t) \right) + \lambda_t \sigma dZ_t^i \\
d\theta_t &= \omega_t \theta_t \\
m_t^i &= z_t \hat{\mu}_t + (1 - z_t)\theta_t^i \\
z_t &= \frac{4e^{\omega t} g_0}{2e^{\omega t} g_0 + \gamma_t}.
\end{aligned}$$

**HJB Equation** Following standard methods, HJB equation is derived as

$$\begin{aligned}
rV\left(\theta, \mu, \hat{\mu}, t\right) &= \sup_{a \in R} \frac{1}{2} \left( \left(2 - z_t\right) \theta_t + z_t \hat{\mu}_t \right) \left( a + \alpha_t \left( z_t \hat{\mu}_t + (1 - z_t)\theta_t \right) + \mu_t \beta_t \right) - \frac{1}{2} a^2 \\
&+ V_\mu d(\mu) + V_{\hat{\mu}} d(\hat{\mu}) + V_\theta d\left(\theta\right) + V_t + \frac{(\lambda_t \sigma_t)^2}{2} \left( V_{\mu\mu} + V_{\hat{\mu}\hat{\mu}} + 2V_{\hat{\mu}\mu} \right) \\
d(\mu) &:= \frac{1}{2}\omega (m_t^{i,j} + \theta_t - 2\mu_t) + \lambda_t \left( \alpha_t \left( m_t^{i,j} - \mu_t \right) - \mu_t(\alpha_t + \beta_t) + a_t \right) \\
d(\hat{\mu}) &:= \frac{1}{2}\omega (m_t^{i,j} + \theta_t - 2\hat{\mu}_t) + \lambda_t \left( \alpha_t \left( \theta_t - \hat{\mu}_t \right) + \alpha_t (m_t^{i,j} - \hat{\mu}_t) \right). \\
d(\theta) &:= \omega\theta
\end{aligned}$$

**Proposition 11** *$(\alpha, \beta)$ defines a symmetric equilibrium if and only if it solves the following system*

$$\begin{aligned}
\dot{\alpha} &= (r + \omega) \frac{\alpha_t}{\alpha^m} (\alpha_t - \alpha^m) + \lambda_t \beta_t \alpha_t \left( \frac{\alpha_t}{\alpha^m} - 2\left(1 + \alpha_t\right) \right) - \omega \left( \beta_t - \beta_t^m \right) \alpha_t^m \\
\dot{\beta} &= (r + \omega) \frac{\alpha_t}{\alpha^m} (\beta_t - \beta^m) + \lambda_t \beta_t \left[ \frac{\alpha_t}{\alpha_m} \left( \beta_t - \beta^m \right) + 2\alpha_t (1 - \beta_t) - \beta_t \right] + \omega \left( \beta_t - \beta_t^m \right) \left( 1 - \beta_t \right) \\
\dot{\gamma}_t &= 2\omega\gamma_t - \left( \frac{\alpha_t \gamma_t}{\sigma} \right)^2
\end{aligned}$$

*with the the boundary conditions $\alpha_T = \alpha^m(\gamma_T), \beta_T = \beta^m(\gamma_T), \gamma_t = 2g_0$.*

First order conditions:

$$
\begin{aligned}
a_t &= \frac{1}{2}\left(\theta_t\left(2-z_t\right)+z_t\hat{\mu}_t\right)+\lambda_t\left(v_4(t)\hat{\mu}+v_5(t)\theta_t\right)\\
\alpha_t &= \frac{1}{2}(2-z_t)+\lambda_t v_5(t)\\
\beta_t &= \frac{1}{2}z_t+\lambda_t v_4(t)
\end{aligned}
$$

Applying envelope theorem to HJB equation

$$
-\frac{\omega}{2}v_4(t)\left(\left(2-z_t\right)\theta_i+z_t\hat{\mu}_t\right)-v_5(t)\theta\omega-v_4\left(t\right)\lambda_t\alpha_t\left(2-z_t\right)\left(\theta^i-\hat{\mu}_t\right)V_\mu\left(r+\lambda_t\left(2\alpha_t+\beta_t\right)\right)
$$

$$
=\frac{\beta_t\left(\theta^i+m_t^i\right)}{2}+\dot{v}_4(t)\hat{\mu}+\dot{v}_5(t)\theta^i
$$

After rearranging and solving for $\dot{\alpha}_t$ and $\dot{\beta}_t$

$$
\begin{aligned}
\dot{\alpha} &= \left(r+\omega\right)\frac{\alpha_t}{\alpha^m}(\alpha_t-\alpha^m)+\lambda_t\beta_t\alpha_t\left(\frac{\alpha_t}{\alpha^m}-2\left(1+\alpha_t\right)\right)-\omega\left(\beta_t-\beta_t^m\right)\alpha_t^m,\\
\dot{\beta} &= \left(r+\omega\right)\frac{\alpha_t}{\alpha^m}(\beta_t-\beta^m)+\lambda_t\beta_t\left[\frac{\alpha_t}{\alpha_m}\left(\beta_t-\beta^m\right)+2\alpha_t(1-\beta_t)-\beta_t\right]+\omega\left(\beta_t-\beta_t^m\right)\left(1-\beta_t\right).
\end{aligned}
$$

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